FASCICULI MATHEMATICI

Nr 26 1996

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REAL-VALUED FUNCTIONS ALMOST PERIODIC IN p-VARIATION

The paper gives a definition and some properties of real-valued functions almost periodic in the sense of the p-th variation.

Key words and phrases: almost periodic function, p-variation.

1991 Mathematics Subject Classification: 42A75.

§ 1. Let X_0 be the set of functions defined on the whole real axis and taking finite real values.

For $x \in X_o$ the value

$$V_p(x;a,b) = \sup_P \left\{ \sum_{i=0}^{m-1} |x(u_{i+1}) - x(u_i)|^p \right\}^{\frac{1}{p}},$$

where P is a partition $a = u_o < u_1 < ... < u_m = b$ of the interval $< a, b >$, $p > 0$, will be called the p-th variation of the function x in $< a, b >$. As usual, we let

$$BV_p(a,b) = \{x \in X_o : V_p(x;a,b) < \infty\}.$$

It is known that the following properties of the set $BV_p(a,b)$ are true:

1°. *If* $x, y \in BV_p(a,b)$, *then* $x + y \in BV_p(a,b), \lambda x \in BV_p(a,b)$, *where* λ *is a constant, and*

$$V_p(x+y;a,b) \le V_p(x;a,b) + V_p(y;a,b) \quad \textit{for } p \ge 1,$$

$$(V_p(x+y;a,b))^p \le (V_p(x;a,b))^p + (V_p(y;a,b))^p \quad \textit{for } 0 < p < 1,$$

2°. *If* $a < c < b$ *and* $x \in BV_p(a,b)$, *then*

$$V_p(x;a,b) \le V_p(x;a,c) + V_p(x;c,b) \quad \textit{for } p \ge 1,$$

$$(V_p(x;a,b))^p \le (V_p(x;a,c))^p + (V_p(x;c,b))^p \quad \textit{for } 0 < p < 1.$$

3°. *If* $0 < p_1 < p_2$, *then*

$$V_{p_1}(x;a,b) \geq V_{p_2}(x;a,b).$$

§ 2. For $x \in X_o$ let us write

$$V_p(x) = \sup_{-\infty<t<\infty} \{|x(t)| + V_p(x;t)\}, \quad p > 0,$$

where $V_p(x;t) = V_p(x;t-1,t+1)$.

A sequence (x_n), where $x_n \in X_o$ for n = 1, 2, ..., is called V_p -convergent to $x_o \in X_o$ iff for an arbitrary $\varepsilon > 0$ there exists $N > 0$ such that for $n > N$, $V_p(x_n - x_o) \leq \varepsilon$. The limit x_0 of the sequence (x_n) which is V_p -convergent is uniquely defined.

We say that $x \in X_o$ satisfies the condition (+) iff for every $t \in (-\infty, \infty)$ $V_p(x;t) < \infty$.

Let us write

$\tilde{X}_o^p = \{x \in X_o : x$ is continuous and satisfies the condition (+)$\}$.

We say that $x \in \tilde{X}_o^p$ is a V_p -bounded function iff $V_p(x) < \infty$.

A set $E \subset (-\infty, \infty)$ is called relatively dense iff there is a positive number l such that in each open interval $(\alpha, \alpha + l)$, $\alpha \in (-\infty, \infty)$, there is *t* least one element of the set *E*.

Let $x \in \tilde{X}_o^p$. If for $\varepsilon > 0$ there is $V_p(x - x_\tau) \leq \varepsilon$, where $x(u) \equiv x(u + \tau)$, then the number $\tau \in (-\infty, \infty)$ is called V_p, ε-almost period of the function *x*. Let us denote the set of V_p, ε-almost periods of *x* by $E_v^p\{\varepsilon; x\}$.

The function $x \in \tilde{X}_o^p$ is called almost periodic in the *p*-th variation ($V_p -$ a.p.) iff for each $\varepsilon > 0$ the set $E_v^p\{\varepsilon; x\}$. is relatively dense. For $p = 1$ we obtain a $V_p -$ a.p. function (see[2]).

Every $V_p -$ a.p. function is a Bohr's a.p. function (see[1]).

Properties of functions almost periodic in the p-th variation

1. *Every* V_p – a.p. *function x is a* V_p - *bounded function.*

Proof. For an arbitrary number $t \in(-\infty,\infty)$ there exists a $V_p, \varepsilon-$ almost period $\tau \in(-t,-t+l)$, where $l = l(\varepsilon)$ is a number which characte rizes the relative density of the set $E_v^p\{\varepsilon;x\}$. for $\varepsilon = 1$, such that for $p \geq 1$

$$V_p(x;t) \leq V_p(x-x_\tau;t) + \sup_{0<t<l} V_p(x;t)$$

and for $0 < p < 1$

$$V_p(x;t) \leq 2^{1/p-1}\left\{V_p(x-x_\tau;t) + \sup_{0<t<l} V_p(x;t)\right\}.$$

Consequently, for every $p > 0$ we obtain

$$V_p(x) \leq \max(1,2^{1/p-1})\left\{V_p(x-x_\tau) + \sup_{0<t<l} V_p(x;t) + \sup_{-\infty<t<\infty} |x(t)|\right\} \leq K,$$

where K is a constant, since any Bohr' s a.p. function x is bounded.

We say that $x \in X_o$ is a V_p-continuous function if for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in(-\infty,\infty)$, $|h| < \delta$, we have $V_p(x-x_h) \leq \varepsilon$. Similarly as in [2] we prove that if x is a V_p – a.p. function which satisfies the condition (V_p), i.e. for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in(-\infty,\infty)$, $|h| < \delta$, we have

$$\sup_{0<t<l} V_p(x-x_h\,;t) \leq \varepsilon,$$

where $l = l(\varepsilon)$ is a number which characterizes the relative density of the set $E_v^p\{\varepsilon;x\}$, then x is V_p – continuous.

2. *A linear combination of two* $V_p - a.p.$ *functions x, y which satisfy the* (V_p) – *condition is a* $V_p - a.p.$ *function.*

We prove Property 2 in the same way as for Stepanov' s a.p. functions (see [1] pp. 202-204) using the following:

Lemma. For a $V_p - a.p.$ function x which satisfies the $(V_p)-$ condition, where $p>0$, and for an arbitrary $\varepsilon>0$ there exist numbers $\delta>0$, $\omega>0$ such that for every $h\in(0,\delta>$ in every open interval of length ω there exists a $V_p,\varepsilon-$almost period τ of the function x *such that $\tau=kh$, where k is an integer.*

Example. Let us write

$$x(u)=x_1(u)+x_2(u) \text{ for } u\in(-\infty,\infty),$$

where

$$x_1(u)=\begin{cases}(u-2k)\cos\dfrac{\pi}{2(u-2k)} & \text{for } u\in(2k-1,2k+1>,\\ 0 & \text{for } u=2k,\end{cases} \qquad k=0,\pm1,+2,\dots,$$

$$x_2(u)=\cos(\sqrt{2}\pi u) \qquad \text{for } u\in(-\infty,\infty).$$

Then x is a Bohr' s a.p. function and $x\notin\tilde{X}_0^1$. Hence x is not $V-$a.p.

Because for $p>1$

$$V_p(x_1;0,1)\le w(p)=\left(\frac{1}{2^p}+\sum_{l=1}^{\infty}\frac{1}{l^p}\right)^{\frac{1}{p}}$$

and x_2 satisfies (+), so x satisfies the condition (+) for $p>1$.

From the estimation

$$V_p(x_1;-1,1)\le 2w(p), \quad p>1,$$

it follows that for an arbitrary $\varepsilon>0$ there exists $\delta'=\delta'(\varepsilon)>0$ such that

$$V_p(x_1;-\delta',\delta')\le\frac{\varepsilon}{5}.$$

Then for $|h|\le\delta'/2$ we have

$$V_p\left(x_{1h} - x_1; -\delta'/2, \delta'/2\right) \le \frac{2}{5}\varepsilon,$$

where $x_{1h}(u) \equiv x_1(u+h)$.

The derivative x_1' is uniformly continuous on intervals $\omega_1 =< -1-\delta'/4, -\delta'/4 >$ and $\omega_2 =< \delta'/4, 1+\delta'/4 >$, i.e. there exists $\delta'' = \delta''(\varepsilon) > 0$ such that for every $h \in (-\infty, \infty)$ such that $|h| < \delta''$ we have

$$|x_1'(u) - x_1'(u+h)| < \frac{\varepsilon}{5}$$

uniformly with respect to $u \in \omega_1 \cup \omega_2$. Hence

$$V_p(x_1 - x_{1h}; -1, -\delta'/2) \le \frac{\varepsilon}{5}, \quad V_p(x_1 - x_{1h}; \delta'/2, 1) \le \frac{\varepsilon}{5}.$$

By the uniform continuity of x_1 on the whole real axis it follows that for every *h*, where $|h| < \delta'''$, $\delta'''(\varepsilon) > 0$, we obtain

$$|x_1(t) - x_1(t+h)| < \frac{\varepsilon}{5}.$$

Therefore for $|h| < (\delta'/4, \delta'', \delta''')$ we have

$$V_p(x_1 - x_{1h}) \le \sup_{-\infty < t < \infty} |x_1(t) - x_{1h}(t)| + V_p(x_1 - x_{1h}; -1, 1) \le \varepsilon,$$

and so x_1 is V_p - continuous.

Because x_2' is uniformly continuous on $(-\infty, \infty)$, x_2 also V_p – continuous for $p > 1$.

By the Property 2 it follows that $x = x_1 + x_2$ is V_p – a.p. for $p > 1$.

3. *If the sequence* (x_n), *where* $x_n \in \tilde{X}_o^1$ *is a* V_{p_n} – *a.p. function for* $n = 1, 2, \ldots, p_n > 1$, $p_n \to 1$, *is* V_1*-convergent to a function* $x \in \tilde{X}_o^1$, *then is* V – *a.p.*

Proof. Because the sequence (x_n) is V_1-convergent to *x*, for an arbitrary $\varepsilon > 0$ there exists $N_1 = N_1(\varepsilon) > 0$ such that for $n_1 > N_1$ we have

$$V_1(x - x_{n_1}) \le \frac{\varepsilon}{6}.$$

For $\tau \in E_v^{w'}\{\varepsilon/6; x_{n_1}\}$, where $w' = p_{n_1}$, we obtain

$$V_{p_{n_1}}(x - x_\tau) \le 2V_1(x - x_{n_1}) + V_{p_{n_1}}(x_{n_1} - x_{n_1\tau}) \le \frac{\varepsilon}{2},$$

where $x_{n_1\tau}(u) \equiv x_{n_1}(u+\tau)$. Because for $n_2 > N_2 = N_2(\varepsilon) > 0$ the following inequality

$$\left| \left\{ \sum_{i=0}^{m-1} |x(u_{i+1}) - x(u_i)|^{p_{n_2}} \right\}^{\frac{1}{p_{n_2}}} - \sum_{i=0}^{m-1} |x(u_{i+1}) - x(u_i)| \right| \le \frac{\varepsilon}{2}$$

holds, so

$$V_1(x - x_\tau) \le \frac{\varepsilon}{2} + V_{p_{n_2}}(x - x_\tau).$$

Hence for $\tau \in E_v^{w''}\{\varepsilon/6; x_{n_o}\}$, where $w'' = p_{n_o}, n_o = \max(n_1, n_2)$, we have

$$V_1(x - x_\tau) \le \varepsilon,$$

i.e. $E_v^{w''}\left\{\varepsilon/6; x_{n_1}\right\} \subset E_v^1\left\{\varepsilon; x\right\}$ and so x is V-a.p.

We say that a V_p – a.p. function x, where $p \ge 1$, satisfies the condition (V_p^*) iff for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in (-\infty, \infty)$, $0 < |h| < \delta$, we have

$$\sup_{0<t<l} \frac{V_p(x - x_h; t)}{|h|^{\alpha(p)}} \le \varepsilon,$$

where $\alpha(p) = (p-1)/p$, $x_h \equiv x(u+h)$, $l = l(\varepsilon, p, h) > 0$ is the number which characterizes the relative density of the set of $V_p, |h|^{\alpha(p)}\ \varepsilon$–almost periods of x. If $p = 1$, we obtain the (V) - condition (see[2]).

4. *If x is a $V_p - a.p.$ function, where $p \geq 1$, which satisfies the $(V_p^*)-$condition, $S_x(\cdot;h)$ is the Steklov function of x of the form*

$$S_x(u;h) = \frac{1}{2h}\int_{u-h}^{u+h} x(s)ds,$$

where $h > 0, u \in (-\infty, \infty)$, then

$$\lim_{h\to 0} V_p(x - S_x(\cdot;h)) = 0.$$

Proof. It is known (see[3]) that the Steklov function $S_x(\cdot;h)$ of x is L-a.p.

Using the Hölder inequality, we obtain for every $t \in (-\infty, \infty)$

$$V_p(x - S_x(\cdot;h);t) \leq$$

$$\leq \sup_P \left\{ \sum_{i=0}^{m-1} \frac{1}{2h} \int_{-h}^{h} \left\| \left[x(u_{i+1}) - x(s + u_{i+1})\right] - \left[x(u_i) - x(s+u_i)\right] \right\|^p ds \right\}^{\frac{1}{p}},$$

where P is an arbitrary partition of the interval $< t-1, t+1 >$, and hence

$$V_p(x - S_x(\cdot;h)) \leq \sup_{-\infty<t<\infty} \frac{1}{2h} \int_{-h}^{h} |x(s+t) - x(t)| ds +$$

$$+ \sup_{-\infty<t<\infty} \frac{1}{2h} \left\{ \int_{-h}^{h} \left[V_p(x - x_s;t)\right]^p ds \right\}^{\frac{1}{p}} = I_1 + I_2.$$

Because x is a Bohr's a.p. function, for an arbitrary $\varepsilon > 0$ there exists $\delta' = \delta'(\varepsilon) > 0$ such that for every $h \in (-\infty, \infty)$, $0 < h < \delta'$, we have $I_1 \leq \varepsilon / 2$.

By the $(V_p^*)-$condition it follows that for $\varepsilon > 0$ there exists $\delta'' \in (0, \delta' >$ such that for every $s \in (-\infty, \infty)$, $|s| < \delta''$, there holds the following inequality

$$\sup_{0<t<l} V_p(x - x_s;t) \leq \frac{1}{4}|2s|^{\alpha(p)},$$

where $l = l(\varepsilon, p, s) > 0$ is the number which characterizes the relative density of the set $E_v^p = E_v^p\left\{(1/8)|2s|^{\alpha(p)}\varepsilon; x\right\}$, $\alpha(p) = (p-1)/p$. For an arbitrary $t \in (-\infty, \infty)$ and for $\tau \in E_v^p$, where $\tau \in (-t, -t+l)$, we have for $|s| < \delta''$

$$V_p(x - x_s; t) \le V_p(x - x_\tau; t) + \sup_{0<t'<l} V_p(x - x_s; t') + V_p(x_{\tau s} - x_s; t).$$

Hence for $0 < h < \delta''$ we obtain

$$I_2 \le \frac{1}{2h}\left\{\int_{-h}^{h}\left[2 \sup_{-\infty<t<\infty} V_p(x - x_\tau; t) + \sup_{0<t<1} V_p(x - x_s; t)\right]^p ds\right\}^{\frac{1}{p}} \le \frac{\varepsilon}{2}.$$

Therefore for $0 < h < \delta''$ we have

$$V_p(x - S_x(\cdot\,; h)) \le \varepsilon$$

and hence we obtain the result.

References

[1] B.M. Levitan, Almost Periodic Functions, Moscow 1953 (in Russian).

[2] S. Stoiński, Real-valued Functions Almost Periodic in Variation, *Functiones et Approximatio* XXII (1993), 141-148.

[3] S. Stoiński, Some Remarks on Bohr's Almost Periodic Functions and Stepanov's Almost Periodic Functions, *Functiones et Approximatio* (in press).

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Received on 26.9.1994 and, in revised form, on 24.2.1995.