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ON THE REVERSIBLE DIFFUSION PROCESS

The work shows that the diffusion equation in space of distributions has solutions of non-monotonic support. This results adjusts Newton's dynamics with Fourier's diffusion..

Key words: partial differential equations, diffusion process.

In [1] the diffusion problem was solved:

(1) $$\frac{\partial p}{\partial t} - \Delta p = 0 \quad \text{almost everywhere,}$$

$(x \neq 0,\ x \neq \lambda_i(t),\ i = 1,2,\ \lambda_1(t) = -\sqrt{t} < 0, \lambda_2(t) = \sqrt{t} > 0,\ t \in R^+)$
$p: R^1 \times R^+ \to R^+,\quad x \in R^1,\quad t \in R^+,\quad p \in C^{2,1}(D_i),\quad i = 1,2.,$
$D_1 = \{(x,t): \lambda_1(t) < x < 0, t > 0\},\quad D_2 = \{(x,t): 0 < x < \lambda_2(t),\ t > 0\}$
and $p \in C(\overline{D})$ (p satisfy the Lipschitz condition for $(x,t) \in D$, $D = \{(x,t): \lambda_1(t) < x < \lambda_2(t), t > 0\}$, with boundary conditions:

(2) $$p(x,0) = \delta_0$$

(3) $$p(\lambda_1(t),t) = 0,\ i = 1,2,\ t > 0$$

$$p(x,t) \equiv 0 \text{ for } x \leq \lambda_1(t) \text{ and } x \geq \lambda_2(t),\ t > 0,$$

(4) $$\int_{\lambda_1(t)}^{\lambda_2(t)} p(x,t)dx = \text{const},$$

in the form:

(5) $$p(x,t) = \frac{1}{\sqrt{t}} \sum_{j=0}^{\infty} a_j \left(1 - \frac{|x|}{\sqrt{t}}\right)^{j+1} \quad \text{where}$$

$$a_{j+2} = (a_{j+1} - a_j)\frac{1}{2(j+3)}$$

This solution eliminates the physical paradox of unbounded velocity of a diffusion impulse.

In [2] we solved (1) with boundary conditions:

(2a) $p(x,0)=\varphi(x),\ \varphi(x)=\frac{1}{\sqrt{r}}\sum_{j=0}^{\infty}a_j(1-\frac{|x|}{\sqrt{r}})^{j+1},\ r=\text{const}>0,$

(3a) $p(\lambda_i(t),t)=0,\ t\in(0,r)$

(4a) $\int_{\lambda_1(t)}^{\lambda_2(t)}p(x,t)dx=\text{const},\ \lambda_1(t)=-\sqrt{r-t},\ \lambda_2(t)=\sqrt{r-t}$, in the form:

(6) $p(x,t)=\frac{1}{\sqrt{r-t}}\sum_{j=0}^{\infty}a_j(1-\frac{|x|}{\sqrt{r-t}})^{j+1}$, where

$$a_{j+2}=(-a_{j+1}+a_j)\frac{1}{2(j+3)}$$

Solutions (5) and (6) reconcile properties of revesible process of Newton dynamics with properties of irreversible processes of Fourier diffusion. Z. Kowalski from Jagellonian University wrote that it is possible to construct the diffusion process starting from one point and come back to the same point, to describe the Newton's and Fourier's processes by solutions of the same form.

In this paper we present that process. The domain of this process is limited by non-monotonic boundaries $\lambda_1(t)$ and $\lambda_2(t)$. We consider the domains:

$$D_1=\{(x,t):\ \lambda_1(t)<x<0,\ -r<t<r\}$$

and $D_2=\{(x,t):\ 0<x<\lambda_2(t),\ -r<t<r\}$

r = const > *0*. Let $p\in C^{2,1}(D_i),\ i=1,2,\ p\in C^{L,L}(D_1\cup D_2)$. We solve the equation (1) with boundary conditions:

(2b) $p(x,-r)=\delta_0,$

(3b) $p(\lambda_i(t),t) = 0,\ i = 1,2,\ t \in (-r,r)$

$(p(x,t) \equiv 0$ for $x < \lambda_1(t)$ and $x > \lambda_2(t),\ t \in (-r,r))$

(4b) $$\int_{\lambda_1(t)}^{\lambda_2(t)} p(x,t)dx = \text{const} > 0.$$

We write the solution of the problem (1), (2b) - (4b) in the form:

(7) $$p(x,t) = \frac{1}{\sqrt{r-|t|}} \sum_{j=0}^{\infty} a_j (1 - \frac{|x|}{\sqrt{r-|t|}})^{j+1},$$ where

(7a) $$a_{j+2} = (a_{j+1} - a_j)\frac{1}{2(j+3)},$$ for $t > -r$

(7b) $$a_{j+2} = (-a_{j+1} + a_j)\frac{1}{2(j+3)},$$ for $t < r$.

We see that the series $\sum_{j=0}^{\infty} a_j k^j$ is uniformly convergent for every k, [1], [2] and the integral $\int_{\lambda_1(t)}^{\lambda_2(t)} p(x,t)dx = \text{const}$.

For $p(x,t) = 0$ we have: $\lambda_1(t) = -\sqrt{r-|t|}$ and $\lambda_2(t) = \sqrt{r-|t|}$.

By translation of the time we can make $p(x,0) = \delta_0$ instead of $p(x,-r) = \delta_0$. Then for the domain D we have $0 < t < 2r$. The solution is unique [4].

In this way we have the expansion process and the return process of Fourier compatible with the Newton's dynamical process by one formula and one initial condition.

Remarks. The process is symmetrical with t-axis and is non-symetrical with *x*-axis (see (7a) and (7b)). The solution satisfy the Lipschitz condition $p \in C^{L,L}$ (see(7)).

References

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