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EXTREME COMPACT OPERATORS BETWEEN REAL SPACES OF COMPLEX CONTINUOUS FUNCTIONS

For a compact Hausdorff space and a homeomorphism $\sigma: X \to X$, $\sigma \circ \sigma = id$, $C(X,\sigma)$ is a real space of all complex continuous functions on fulfilling $f(\sigma x) = \overline{f(x)}$. It is shown that a compact operator between two spaces of this kind, which is extreme, is a nice operator.

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1. Introduction

An operator (a compact operator) acting between Banach spaces V and W is called *extreme* if it is an extreme point of the unit ball of the space of bounded (compact bounded) linear operators $L(V,W)$ ($K(V,W)$). The problem of characterizing extreme operators was investigated by several authors (see e.g. [1]). The sufficient condition for an operator to be extreme is that the adjoint operator T^* maps extreme functionals on W into extreme functionals on V. (An operator with this property is called a *nice operator*.) It is natural to ask whether an extreme operator must be nice.

In this paper we will answer this question for $V = C(Y,\tau)$, $W = C(X,\sigma)$ (for definitions see below) and for compact operators only. Since $C(X,\sigma)$-spaces can be treated as a generalization of $C(X)$-spaces, we recall now the well-known results (see [2]) about the latter spaces. Let us consider an operator $T: V \to C(X)$ and let $T^*: X \to V^*$ be defined by $T^*(x) = \delta_x \circ T$. The association $T \to T^*$ is an isometric isomorphism between $L(V,C(X))$ and the space of weak*-continuous functions $C_*(X,V^*)$; the same applies to the spaces $K(V,C(X))$ and the space of norm continuous functions $C(X,V^*)$. Hence, instead of an operator T (bounded or compact) we can consider its representing function (weak*-continuous or norm continuous). T is nice if and only if the representing function takes only extremal values.

In the special case of $V=C(Y)$ the representing function takes values in the space $M(Y)$ of all regular complex Borel measures on Y. The extreme points of the unit ball of $M(Y)$ are the point measures $\alpha\delta_y$, $|\alpha|=1$, $y\in Y$. So the operator $T:C(Y)\to C(X)$ is nice if and only if $(Tf)(x)=\alpha(x)(f\circ\phi)(x)$ with $\phi: X\to Y$ continuous, $\alpha\in C(X)$, $|\alpha(x)|=1$.

Even in this special case of operator T acting between $C(Y)$ and $C(X)$, the extreme operator need not to be nice. It is, however, true in many cases (one must impose a suitable condition on X, Y or T).

Now, recall that for a compact space X and a homeomorphism $\sigma: X\to X$ which is also an involution ($\sigma\circ\sigma=id_X$), we define ([6]):

$$C(x,\sigma)=\left\{f\in C(X)|f\circ\sigma=\overline{f}\right\},$$

where $\overline{f}(x)=\overline{f(x)}$ (complex conjugation). $C(x,\sigma)$ is a real Banach algebra (with a sup norm), and it shares many properties with $C(X)$. In fact, $C(X)\cong C(X\ X,\sigma)$, where $X\ X$ denotes the disjoint sum and σ is defined in the obvious way. Hence we can treat $C(X,\sigma)$ as a generalization of $C(X)$. This allows us to expect that repetition of the method used for operators $C(Y)\to C(X)$ will give us results for operators $C(Y,\tau)\to C(X,\sigma)$.This is really the case, but one difficulty must be overcome: while the mapping $x\mapsto\delta_x$ is a homeomorphism $X\to C(X)^*$ (weak*-topology), it is not in the case of $C(X,\sigma)^*$ (in general, $\delta_x\notin C(X,\sigma)^*$!). So we will consider the mapping $X\to L_R(C(X,\sigma),\mathrm{C})$ which maps X into the space of real linear operators $C(X,\sigma)\to\mathrm{C}$. With the uniform operator norm $L_R(C(X,\sigma),\mathrm{C})$ is a Banach space. The same applies to the space $L_R(V,\mathrm{C})$, where V is any real Banach space.

2. The space $L_R(C(X,\sigma),\mathrm{C})$

Riesz theorem for $C(X,\sigma)$ ([3], Theorem 3.1) states that $C(X,\sigma)^*$ is linearly isometric to the space $M(X,\sigma)$ of all regular complex Borel measures on X fulfilling the condition $\mu=\overline{\mu}\circ\sigma$. If F is a linear real functional on $C(X,\sigma)$ and μ denotes the measure associated with it by Riesz theorem, then

$$F(f)=\int_X f\,d\mu=\int_X (\Re f)d(\Re\mu)-\int_X (\Im f)d(\Im\mu)$$

for $f\in C(X,\sigma)$, since it is easy to see that $\Im\left(\int_X f\,d\mu\right)=0$.

The complexification $M(X,\sigma)+i\,M(X,\sigma)$ is, as a set, equal to $M(X)$ (for any $\mu\in M(x)$, $\mu=(\mu+\overline{\mu}\circ\sigma)/2+i(\mu-\overline{\mu}\circ\sigma)/2i$). On the other hand, any $\mu=\mu_1+i\mu_2\in M(X,\sigma)+i\,M(X,\sigma)$ can be considered as an element from $L_R(C(X,\sigma),\mathrm{C})$:

$$\mu(f)=\int_X f\,d\mu_1+i\int_X f\,d\mu_2 .$$

The norm in the complexification is given by:

$$\begin{aligned}\|\mu\|_c&=\|\mu_1+i\,\mu_2\|_c\\ &=\sup\left\{(\mu_1(f)^2+\mu_2(f)^2)^{\frac{1}{2}}\mid f\in B_{C(X,\sigma)}\right\}\\ &=\sup\{|\mu(f)|: f\in B_{C(X,\sigma)}\}.\end{aligned}$$

Here the supremum has been counted over $B_{C(X,\sigma)}$ and not over $B_{C(X,\sigma)^{**}}$ since the first set is norming. Note also that μ_1 and μ_2 are complex, but $\mu_1(f)$ and $\mu_2(f)$ are real.

The equality above shows that the norm in the complexification is the usual norm in $L_R(C(X,\sigma),\mathrm{C})$, so we have

Proposition 1. The space $L_R(C(X,\sigma),\mathrm{C})$ is linearly isometric with the complexification $C(X,\sigma)^* + iC(X,\sigma)^*$.

The question arises what relation is between the complexification norm $\|\mu\|_c$ and the usual norm $\|\mu\| = |\mu|(X)$? By the definition we have:

$$\|\mu\|_c = \sup\left\{\left|\int_X f\, d(\mu_1 + i\mu_2)\right| \,\middle|\, f \in B_{C(X,\sigma)}\right\}$$
$$\leq \sup\left\{\left|\int_X f\, d(\mu_1 + i\mu_2)\right| : f \in B_{C(X)}\right\} = \|\mu\|.$$

3. Representing functions

It is well known ([2]) that bounded linear (compact) operators from a Banach space V into the space $C(X)$ are represented by w^*-continuous (norm continuous) mappings from X into V^*. Analogously, the operators from V into $C(X,\sigma)$ will be represented by mappings from X into $L_R(V,C)$. We need, however, the topology in $L_R(V,C)$ which will play the role of w^*-topology in V^*.

Definition 1. ([3]) The ω-topology in $L_R(V,C)$ is the weakest topology such that all the evaluations:

$$\delta_V : L_R(V,C) \to C, \delta_V(v^*) = v^*(v) \quad \textit{for} \quad v^* \in L_R(V,C)$$

are continuous.

It is easy to show that the unit ball $B_{L_R(V,C)}$ is ω-compact.

Proposition 2. ext $B_{L_R(C(X,\sigma),C)} = \{\alpha\,\delta_x \mid x \in X, \alpha \in C, |\alpha| = 1\}$.

Proof. Suppose that a measure is extreme. We will show that it is a point measure. To this end let F denote the right-hand side of Proposition 2. We will write simply B instead of $B_{L_R(C(X,\sigma),C)}$. Clearly

$$\overline{\operatorname{conv} F}^{\omega} \subset B .$$

On the other hand, if $\phi \in \overline{\operatorname{conv} F}^{\omega}$ then there exists $f \in C(X,\sigma)$, $\| f \| \leq 1$ such that $\phi(f) > 1$. Hence $\phi \notin B$. It follows that

$$B \subset \overline{\operatorname{conv} F}^{\omega} .$$

Hence ([2], V.8.5) $B \subset F$. Conversely, we will show that for any $x \in X$, $|a| = 1$, $\alpha\delta_x$ is extreme. It is certainly extreme in the unit ball of the space $S_x = \{\alpha\delta_x \mid \alpha \in C\}$. From the equality:

$$\| \mu \| = \| \mu(\{x\})\delta_x \| + \| \mu - \mu(\{x\})\delta_x \| ,$$

(which in fact means that S_x is an L-summand of $M(X)$), it follows that the extreme point of the ball in S_x must also be an extreme point of $M(X)$.

We will need the following theorem and corollary.

Theorem 1. Let V be a real Banach space, (X,σ) a compact Hausdorff space with an involution and let $T: V \to C(X,\sigma)$ be a linear bounded operator. Then there exists ω-continuous mapping $t: X \to L_R(V,C)$, $t(\sigma x) = \overline{t(x)}$ such that:

1. $t(x) = \delta_x \circ T$ *(i.e., $t(x)(v) = (Tv)(x)$ for $x \in X, v \in V$);*

2. $\| T \| = \sup_{x \in X} \| t(x) \|$.

Conversely, if such a map t is given then the operator T defined by (1) is a bounded linear operator from V into $C(X,\sigma)$, with the norm given by (2). The operator T is compact if and only if it is continuous with the norm topology in $L_R(V,C)$.

The proof of this theorem can be found in [3].

Corollary 1. $T \in \text{ext}\, B_{L(V,C(X,\sigma))}$ $(T \in \text{ext}\, B_{K(V,C(X,\sigma))})$ *if and only if every ω-continuous (norm continuous) map :*

$$s: X \to L_R(V,C)$$

such that $\| t(x) \pm s(x) \| \le 1$ *for each* $x \in X$, *is identically zero.*

Remark. According to Theorem 1, $t(x)$ is an R-linear operator acting from V to C. When $V = C(Y, \tau)$ it can (and will) be treated as a complex measure on Y.

4. Compact operators

Proposition 3. Let $(T \in \text{ext}\ B_{K(C(Y,\tau),C(X,\sigma))})$. *Then for all* $x \in X$ *the value* $t(x)$ *of representing function is a point measure of norm one.*

Proof. From Theorem 1, $\| t(\cdot) \| = 1$, hence $\| t(x) \| \le 1$ for all $x \in X$. Let $s(x) = (1 - \| t(x) \|)\, \phi$, where $\phi \in C(Y, \tau)^*$, $\| \phi \| < 1$. Then $\| t(x) - s(x) \| \le 1$, hence $s(x) \equiv 0$ by Corlooary 1. Therefore $\| t(x) \| \equiv 1$.

Let us now consider the mapping

$$t_v : X \to M_R(Y), \qquad t_v(x) = |t(x)|,$$

where $|t(x)|$ denotes the variation of measure $t(x)|$. t_v is continuous with the norm topology in $M_R(Y)$ because

$$\| t_v(x) - t_v(x') \| = \| \, |t(x)| - |t(x')| \, \| \le \| t(x) - t(x') \| .$$

Also t_v has probability measures as values. We claim that t_v is extreme in the set:

$$\{ f : f \in C(X, M_R(Y)),\ \text{all}\ f(x)\ \text{are probability measures} \}.$$

To prove this statement suppose there exists a mapping

$$g: X \to M_R(Y)$$

which is continuous and such that $t_v(x) \pm g(x) \geq 0$, $(t_v(x) \pm g(x))(Y) = 1)$ for all $x \in X$, and $g(\sigma x) = g(x)$. Then $-t_v(x) \leq g(x) \leq t_v(x)$ and it follows that $g(x)$ is absolutely continuous with respect to $t_v(x)$. So, there exists a Borel function h_x with $g(x) = h_x \cdot t_v(x)$. It can be assumed that $-1 \leq h_x(y) \leq 1$ for all $y \in Y$. Let $\mu(x) := h_x \cdot t(x)$. Then $\mu(x) \in L_R(C(Y,\tau), C)$ and:

a) $x \mapsto \mu(x)$ is continuous (i.e., belongs to $C(X, L_R(C(Y,\tau), C))$,

b) $\|t(x) \pm \mu(x)\| \leq 1$.

Proof. of a):

$$\begin{aligned}\|\mu(x) - \mu(x')\| &= \|h_x \cdot t(x) - h_{x'} \cdot t(x')\| \\ &\leq \|h_x \cdot t(x) - h_{x'} \cdot t(x')\| + \|h_x \cdot t(x') - h_{x'} \cdot t(x')\| \\ &= \|h_x(t(x) - t(x'))\| + \|(h_x - h_{x'})\, t(x')\|.\end{aligned}$$

We have ([7] 6.13):

$$\begin{aligned}\|h_x(t(x) - t(x'))\| &= \int_Y |h_x|\, d|t(x) - t(x')| \leq \\ &\leq \int_Y d|t(x) - t(x')| = \|t(x) - t(x')\|\end{aligned}$$

Also

$$\|(h_x - h_{x'})t(x')\| = \int_Y |h_x - h_{x'}|\, dt_v(x') = \int_A \ldots + \int_{Y\setminus A} \ldots ,$$

where $A = \{y \in Y : h_x(y) \geq h_{x'}(y)\}$. Then

$$\begin{aligned}\int_A |h_x - h_{x'}|\, dt_v(x') &= \int_A h_x\, dt_v(x') - \int_A h_{x'}\, dt_v(x') \\ &= \int_A h_x\, dt_v(x) - \int_A h_{x'}\, dt_v(x) + \int_A h_x\, d(t_v(x') - t_v(x)) \\ &\leq |g(x) - g(x')|(A) + |t_v(x) - t_v(x')|(A).\end{aligned}$$

For $Y \setminus A$ the analogous inequality holds true, therefore:

$$\|\mu(x) - \mu(x')\| \leq \|t(x) - t(x')\| + \|g(x) - g(x')\| + \|t(x) - t(x')\|.$$

Hence $x \to \mu(x)$ is continuous, and that is statement a).

Proof of b):

$$\|t(x)\pm\mu(x)\|=\|(1\pm h_x)t(x)\|\leq\int_Y|1\pm h_x|\,d|t(x)|$$

$$=\int_Y(1\pm h_x)\,dt(x)=\int_Y dt(x)\pm\int_Y h_x\,dt_v(x)$$

$$=t(x)(Y)\pm g(x)(Y),$$

and this is equal (by assumption) to 1.

Now, from a), b), and Theorem 1, μ defines the operator M such that $\|T\pm M\|\leq 1$. T is extreme, so $M=0$ and successively μ, h_x and g are all equal to zero. This proves our claim.

To finish the proof, let us note that $t_v : X\to M(Y)$ defines an operator $S: C(Y)\to C(X)$ by the formula:

$$(Sf)(x)=\int_Y f\,dt_v(x)\ .$$

Since t_v is positive and extreme, S is also positive and extreme. It is known (see e.g. [1]) that S is nice, that is $t_v(x)=S^*(\delta_x)$ is a point measure, and so is $t(x)$. This completes the proof.

Remark. The first four lines of the proof are valid for all bounded (not only compact) operators. Hence

Corollary 2. If $T\in$ ext $B_{L_R(C(Y,\tau),C(X,\sigma))}$, *then the values of its representing function* $t(x)$ *are of norm one.*

From Proposition 2 and Proposition 3 follows:

Theorem 2. Every compact operator T which is extreme in the unit ball in $K(C(Y,\tau),C(X,\sigma)$, *is a nice operator.*

Proof. If T is extreme then the values of its representing function are point measures of norm one, hence they are extreme $B_{L_R(C(Y,\tau),C(X,\sigma))}$.

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