FASCICULI MATHEMATICI

Nr 26 1996

LASHA KOKILASHVILI

ON THE EXISTENCE OF PROPER SOLUTIONS OF HIGH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

The paper considered the existence on the existence of proper solutions of high order nonlinear differential equations.

Key words: proper solution, initial condition, oscillation.

1. Statement of the Problem and Formulation of the Main Results

Consider the n-th order ordinary differential equation

$$u^{(n)} + u^{(n-2)} = f(t,u,u',...,u^{(n-1)}) \tag{1.1}$$

where $n \geq 2$and $f:[0,+\infty] \times \Re^n \to \Re$ is a continuous function satisfying the local Lipschitz with respect to the last n-variables.

Definition 1.1. *A solution of equation* (1.1) *is called propper if it is defined on any interval* $[t_0,+\infty] \subset [a,+\infty]$. *Otherwise it is called singular, i.e., a solution* u *is singular if it is defined on any finite interval* $[t_0,t_1]$ *and the extension to the right of* t_1 *is impossible.*

Definition 1.2. *A proper solution of equation* (1.1) *is called oscillatory if there exists a sequence of zeros of this solution which converges to* $+\infty$, *and is called vanishing at infinity if* $\lim\limits_{t\to+\infty} u(t) = 0$.

The oscillation criteria of proper solutions of equation (1.1) were investigated by I. Kiguradze in [1] under the assumption that such a solution exists.

In the special case when an increase order f with respect to phase variables is less than 1, i.e., speaking exactly, when

$$|f(t,x_1,\dots,x_n)|\le h(t)\left(1+\sum_{i=1}^{n}|x_i|\right)$$

every solution of (1.1) is proper. However, if an increase order of f with respect to at least one phase variable is greater than 1, the problem of the existence of proper solutions become questionable.

For example, let us consider Emden-Fowler' s differential equation

(1.2) $$u^{(n)}+u^{(n-2)}=p(t)|u|^{\lambda}\operatorname{sign}u$$

where $\lambda\ge 1$ and $p:[a,+\infty]\to\Re$ is a continuous function not identical to zero in any neighbourhood of ∞. If $\lambda=1$, then (1.2) is a linear equation and every solution of (1.2) is proper. But if $\lambda>1$, then we have to clarify whether a proper solution exists.

In this paper we are concerned with thiss very problem. We shall establish the sufficient conditions for solutions of equation (1.1) to exist (in the specjal case (1.2)), as well as define the asymptotic properties of the obtained solutions for some cases.

Theorem 1.1. *Let* $n=2n_0+1$ *and there exist constans* $\delta\in[0,1]$ *and* $\gamma\in[1,+\infty]$ *and a continuous function* $F:[a,+\infty]\times\Re\to[0,+\infty]$ *such that the inequality*

(1.3) $$0\le(-1)^{n_0}f(t,x_1,\dots,x_n)\operatorname{sign}x_1\le F(t,x_1)$$

holds on $[0,+\infty]\times\Re^n$ *and the inequality*

(1.4) $$(-1)^{n_0}f(t,x_1,\dots,x_n)\operatorname{sign}x_1\ge\delta|x_1|^{\gamma},$$

holds on $[a,a+\delta]$.

Then for arbitrary constants $c_1,\ldots,c_{n0}$ *there exists at least one proper solution of equation* (1.1), *satisfying the initial condition*

$$u^{(i-1)}(a) = c_i \quad (i = 1,\ldots,n_0) \tag{1.5}$$

Corollary 1.1. *Let* $n = 2n_0 + 1$,

$$(-1)^{n0}\, p(t) \ge 0 \quad for \quad t \ge a$$

and

$$(-1)^{n0}\, p(a) > 0.$$

Then for arbitrary real constants $c_1,\ldots,c_{n0}$ *there exists at least one proper solution of equation* (1.2), *satisfying condition* (1.5).

Theorem 1.2. *Let* $n = 2n_0$ *and there exists a constant* $\gamma > 1$ *and continuous functions* $\delta : [a,+\infty] \to [0,+\infty]$ *and* $F : [a,+\infty] \times \Re \to [0,+\infty]$ *such that*

$$\int_0^{+\infty} \delta^{-\frac{2}{\gamma-1}}(t)dt < +\infty \tag{1.6}$$

and the inequality

$$\delta(t)|x_1|^\gamma \le (-1)^{n0-1} f(t,x_1,\ldots,x_n)\operatorname{sign} x_1 \le F(t,x_1), \tag{1.7}$$

holds on $[a,+\infty] \times \Re^n$. *Then for arbitrary constans* $c_1,\ldots,c_{n0}$ *there at least one proper solution of equation* (1.1), *satisfying condition* (1.5). *Moreover, if* n_0 *is odd, then all such solutions are oscillatory.*

Corollary 1.2. *Let* $\lambda > 1$, $n = 2n_0$,

$$(-1)^{n0-1} p(t) > 0 \quad for \quad t \ge a$$

and

$$\int_a^{+\infty} |p(t)|^{-\frac{2}{\lambda-1}} dt < +\infty,$$

Then for arbitrary constants $c_1,\ldots,c_{n_0}$ *there exists at least one proper solution of* (1.2), *satisfying condition* (1.5). *Moreover, if* n_0 *is odd, then all such solutions are oscillatory.*

2. Auxiliary Statements

2.1. Some Integral Inequalities

Lemma 2.1. *Let* $u:[a,b]\to\Re$ *be an m-times continuously differentiable function satysfying the condition*

(2.1) $$u(a) = u(b),\ldots,u^{(m\text{-}2)}(a)$$

Then inequqlities

(2.2) $$\int_a^b |u^{(i)}(t)|^2\,dt \le$$

$$\le \left(\int_a^b u^2(t)dt\right)^{\frac{m-1}{m}} \left(\int_a^b |u^{(m)}(t)|^2\,dt\right)^{\frac{i}{m}} \qquad (i=0,\ldots,n\text{-}1)$$

are fufiled.

2.2. On the Unique Solvability of One Two-Point Linear Boundary Value Problem

Let us consider the homogeneous equation

(2.3) $$u^{(n)} + u^{(n-2)} = p(t)u$$

with the homogeneous boundary conditions

(2.4) $$\begin{aligned} &u^{(i\text{-}1)}(a) = 0 \qquad (i = 1,\ldots,n_0),\\ &u^{(j\text{-}1)}(b) = 0 \qquad (j = 1,\ldots,n - n_o) \end{aligned}$$

where n_0 is the integer part of n and $p:[a,b]\to\Re$ is a continuous function

Lemma 2.2. Let n be even (odd) and the inequality

(2.5)
$$(-1)^{n-n_0-1} p(t) > l$$
$$(-1)^{n_0} p(t) \geq 0, \quad (p(t) \not\equiv 0),$$

hold on [a,b], *where* $l = \dfrac{1}{n_0^{n_0} (n_0 - 1)^{n_0 - 1}}$. *Then problem* (2.3), (2.4) *has a trivial solution only.*

Proof. Assume that u is some solution of (2.3), (2.4). Multiplying (2.3) by $(-1)^{n-n_0} u(t)$ and integrating from a to b, we obtain

(2.6)
$$(-1)^{n-n_0} \int_a^b u^{(n)}(t)u(t)dt + (-1)^{n-n_0} \int_a^b u^{(n-2)}(t)u(t)dt =$$
$$= (-1)^{n-n_0} \int_a^b p(t)u^2(t)dt$$

Integration by parts for an even n_0 gives

$$\int_a^b u^{(n)}(t)u(t)dt = \int_a^b u(t)du^{(n-1)}(t) = u(t)u^{(n-1)}(t)\Big|_a^b - \int_a^b u^{(n-1)}(t)u'(t)dt =$$

(2.7)
$$= u(t)u^{(n-1)}(t)\Big|_a^b - u'(t)u^{(n-2)}(t)\Big|_a^b + \int_a^b u^{(n-2)}(t)u''(t)dt = \ldots =$$
$$= \sum_{i=0}^{n_0-1} (-1)^i u^{(i)}(t)u^{(n-i-1)}(t)\Big|_a^b + (-1)^{n_0} \int_a^b |u^{(n_0)}(t)|^2 dt.$$

By this conditions (2.4) we have

(2.8)
$$\int_a^b |p(t)|u^2(t)dt + \int_a^b |u^{(n_0)}(t)|^2 dt \leq \int_a^b |u^{(n_0-1)}(t)|^2 dt,$$

On the other hand, by virtue of Lemma 2.1

$$\int_a^b |u^{(n_0-1)}(t)|^2\, dt \le \left(\int_a^b u^2(t)dt\right)^{\frac{1}{n_0}} \left(\int_a^b |u^{(n_0)}(t)|^2\right)^{\frac{n_0-1}{n_0}} =$$

$$= \left(\int_a^b u^2(t)dt\right)^{\frac{1}{n_0}} \left(\frac{n_0}{n_0-1}\right)^{-\frac{n_0-1}{n_0}} \times \left(\frac{n_0}{n_0-1}\int_a^b |u^{(n_0)}(t)|^2\, dt\right)^{\frac{n_0-1}{n_0}} \le$$

$$\le \int_a^b |u^{(n_0)}(t)|^2\, dt + \left(\frac{n_0}{n_0-1}\right)^{1-n_0} n_0^{-1}\int_a^b u^2(t)dt = \int_a^b |u^{n_0}(t)|dt + l\int_a^b u^2(t)dt.$$

The latter inequalities imply

$$\int_a^b (|p(t)|-l)u^2(t))dt \le 0 \tag{2.9}$$

from which by condition (2.5) we find that $u(t) \equiv 0$.

Intergation by parts for an odd n gives

$$\int_a^b u^{(n)}(t)u(t)dt = \sum_{i=0}^{n-1}(-1)^i u^{(i)}(t)u^{(n-i-1)}(t)\Big|_a^b + (-1)^{n-n_0-1}\frac{|u^{(n_0)}(t)|^2}{2}\Big|_a^b$$

Moreover, if we consider condition (2.7) and equality (2.9), we shall have

$$\frac{1}{2}|u^{(n_0)}(a)|^2 + (-1)^{n_0}\int_a^b p(t)u^2(t)dt = 0,$$

but since $(-1)^{n_0} p(t) \ge 0$ and $p(t) \not\equiv 0$ the latter equality immediately implies that $u(t) \equiv 0$. ■

2.3. On Sufficient Conditions for the Solvability of Nonlinear Boundary Value Problem

Let us consider the boundary value problem

$$u^{(n)} + u^{(n-2)} = f(t, u, \ldots, u^{(n-1)}) \tag{2.10}$$

$$u^{(i-1)}(a) = c_i \qquad (i = 1,\dots,n_0),$$
(2.11)
$$u^{(i-1)}(a+m) = 0 \qquad (i = 1,\dots,n-n_0),$$

where $n > 2$, n_0 is an integer part of $\frac{n}{2}$, $f:[a,+\infty]\times\Re^n \to \Re$ is continuous function and m is any fixed natural number.

Lemma 2.3. *Let* $n = 2n_0 + 1$ *and there exist constants* $\gamma \in [1,+\infty]$ *and* $\delta \in [0,1]$ *and a continuous function* $F:[a,+\infty]\times\Re^n \to \Re$ *such that the inequality*

(2.12) $$0 \le (-1)^{(n_0)} f(t,\dots,x_n)\,\mathrm{sign}\,x_1 \le F(t,x_1)$$

holds on $[0,\infty]\times\Re^n$ *and the inequality*

(2.13) $$(-1)^{n_0} f(t,x_1,\dots,x_n)\,\mathrm{sign}\,x_1 \ge \delta |x_1|^{\gamma},$$

holds on $[\mathrm{a}, \mathrm{a}+\delta]$. *Then for any arbitrary number m there exists a solution* u_m *of problem* (2.10), (2.11) *such that*

(2.14) $$\sum_{i=1}^{n} |u_m^{(i-1)}(t)| < r(t), \qquad \text{for} \quad a \le t \le a+m,$$

where $r:[a,+\infty] \to \Re$ *is a continuous function independent of m*;

Proof. Let us introduce the following notations

$$a_0 \le a + \frac{\pi}{6},$$

$$\alpha = 2(1+c_{\max})^2 + \frac{c_{n0}^2 - 1}{2},$$

where $$c_{\max} \overset{def}{=} \max\{(|c_i|+1) : i = 0,\dots,n_0 - 1\},$$

$$\beta = 8n\alpha,$$

$$F^*(t,y) = \max\{F(t,x) : |x| \le y\},$$

$$r_0 = \alpha\left[3(n+1)!(2n)^{n+2} + 8n\int_a^{a_0} F^*(\tau,\beta)dr + 2\right]^2$$

$$r_i(t) = r_0\sum_{j=i}^{n}\frac{(t-a)^{j-i}}{(j-i)!} + \frac{(t-a)}{(n-i)!}\left[2r_0 + 2\int_a^t F^*(s,1)ds + (\frac{r_0}{\alpha}-1)(t-a)\right],$$

$$\chi_i(t,x) = \begin{cases} x & \text{when } |x| \le r_i(t) \\ r_i(t)\,\mathrm{sign}x & \text{when } |x| > r_i(t), \end{cases}$$

Consider an auxiliary boundary value problem

$$(2.15) \qquad u^{(n)} + u^{(n-2)} = (-1)^{n_0}\,\delta u + f(t,\chi_1(tr,u),\ldots,\chi_n(t,u^{(n-1)})) - -(-1)^{n_0}\,\delta\chi_1(t,u)$$

and (2.11).

For any fixed natural m by Lemma 2.2 there exists only a trivial solution of the the homogeneous linear equation

$$(2.15) \qquad u^{(n)} + u^{(n-2)} = (-1)^{n_0}\,\delta u$$

with boundary condition (2.4).

Now the nonhomogeneous term of (2.15) satisfies the conditions of Conti's theorem and hence the latter theorem implies that exists a solution u_m of problem (2.15), (2.11).

It is obvious that

$$(2.16) \qquad \delta|x_1| = \gamma^{\frac{1}{\gamma}}\delta^{\frac{1}{\gamma}}|x_1|\delta^{-\frac{1}{\gamma}+1}\gamma^{-\frac{1}{\gamma}} \le \delta|x_1|^{\gamma} + \delta_{1,}$$

where $\delta_1 = \delta[\gamma]^{-\frac{1}{\gamma-1}}$, and thus by condition (2.13) we have

$$(-1)^{n_0} f(t,x_1,\ldots,x_n)\mathrm{sign}x_1 \ge \delta|x_1|^{\gamma} \ge \delta|x_1| - \delta_1$$

so that

$$|(u_m^{(n)}(t) + u_m^{(n-2)}(t))u_m(t)| + (-1)^{n_0-1}(u_m^{(n)}(t) + u_m^{(n-2)}(t))u_m(t) = 0$$

Integration from *a* to *a* + *m* gives us

$$\int_a^{a+m} |(u_m^{(n)}(t) + u_m^{(n-2)}(t))u_m(t)|dt =$$

$$= (-1)^{n_0} \int_a^{a+m} (u_m^{(n)}(t) + u_m^{(n-2)}(t))u_m(t)dt =$$

$$(2.17) \quad = \sum_{i=0}^{n_0-1} (-1)^{n_0-i} u_m^{(i)}(a) u^{(n-i-1)}(a) + \frac{|u_m^{(n_0)}(\tau)|}{2}\Bigg|_a^{a+m} +$$

$$+\sum_{i=0}^{n-2} (-1)^{n_0-i} u^{(i)}(a) u^{(n-i-1)}(a) + \frac{|u_m^{(n_0-1)}(\tau)|^2}{2}\Bigg|_a^{a+m} \leq$$

$$\leq 2(1+c_{\max})^2 \left(1 + \sum_{j=n_0}^{n-i} |u_m^{(j)}(a)|\right) + \frac{c_{n_0-1}^2}{2} \leq \alpha \left(1 + \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)|\right).$$

Denote

$$I_m = \{t{:}\, t \in [a, a_0] : |u_m(t)| < \beta\},$$

then

$$(2.18) \quad \int_a^{a_0} |u_m^{(n)}(t) + u_m^{(n-2)}(t)|dt \leq \int_{I_m} F^*(\tau,\beta)d\tau + \frac{1}{\beta} \int_{[a,a_0]\setminus I_m} |(u_m^{(n)}(t) +$$

$$+u_m^{n-2}(t))u_m(t)|dt \leq \int_{I_m} F^*(\tau,\beta)d\tau + \frac{\alpha}{\beta}\left(1 + \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)|\right).$$

Let $\vartheta(t) \overset{def}{=} u_m^{(n-2)}(t)$ than we an rewrite (2.15) in form

$$(2.19) \quad \upsilon''(t) + \upsilon(t) = \tilde{f}(t)$$

where

$$\tilde{f}(t) = (-1)^{n_0} \delta u_m(t) + f(t, \chi_1(t, u_m(t)), \ldots, \chi_n(t, u_m^{(n-1)}(t)) -$$

$$(2.20) \quad -(-1)^{n_0} \delta \chi_1(t, u_m(t))$$

If we solve (2.19) as a 2-th order differential equation, with constant coefficients, we obtain

$$(2.21)\qquad \upsilon(t)=\upsilon(b)\cos(t-b)+\upsilon'(b)\sin(t-b)+\int_b^t \sin(t-\tau)\widetilde{f}(\tau)d\tau$$

Denote $\rho_{im}\overset{def}{=}\min\left\{|u_m^{(i)}(t)|:t\in[a,a_0]\right\}$ $(i=n_0,\dots,n-1)$ when we put $b=t_{n-2}$, where t_{n-2} is a point of minimum of function $u_m^{(n-2)}$ on the segment $[a,\ a_0]$ we have

$$|u_m^{(n-2)}(t)|\le \rho_{n-2}+\rho_{n-1}+|\sin(t-t_{n-2})|\int_a^{a_0}|u_m^{(n)}(\tau)|d\tau+\int_a^{a_0}|\widetilde{f}(\tau)|d\tau\le$$

$$\le \rho_{n-2}+\rho_{n-1}+\frac{1}{2}\int_a^{a_0}|u_m^{(n)}(\tau)|d\tau+\int_{I_m}F^*(\tau,\beta)d\tau+\frac{\alpha}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right),$$

since $a_0\le a+\dfrac{\pi}{6}$. By definition of β and the last inequality we have

$$\int_a^{a_0}|u_m^{(n)}(\tau)|d\tau\le\int_a^{a_0}|u_m^{(n)}(\tau)+u_m^{(n-2)}(\tau)|d\tau+\int_a^{a_0}|u_m^{n-2}(\tau)|d\tau\le$$

$$\le\frac{2\alpha}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+2\int_{I_m}F^*(\tau,\beta)d\tau+\rho_{n-2}+\rho_{n-1}+\frac{1}{2}\int_a^{a_0}|u_m^{(n)}(\tau)|d\tau,$$

i.e.

$$(2.22)\qquad \int_a^{a_0}|u_m^{(n)}(\tau)|d\tau\le\frac{4\alpha}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+$$
$$+4\int_{I_m}F^*(\tau,\beta)d\tau+2(\rho_{n-2}+\rho_{n-1})\,.$$

According to Lemma 5.1 from [2], we have

$$(2.23)\qquad \rho_{im}\le n!(2n)^{n+1}\left(\int_a^{a_0}|u_m(t)|^2\,dt\right)^{\frac{1}{2}}$$

by condition (2.13) and inequality (2.17) we obtain

$$\delta\int_a^{a_0}|u_m(t)|^2\,dt \le \int_a^{a_0}|(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)|dt+$$

$$+\int_a^{a_0}|u_m(t)|^2\,dt+\frac{\delta_1^2}{2\delta}\int_a^{a_0}dt \le \alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+$$

$$+\frac{\delta}{2}\int_a^{a_0}|u_m(t)|^2\,dt+\frac{\delta_1^2}{2\delta}(a_0-a)$$

Now by definition $a_0 \le a+\frac{\pi}{6}$ and so

$$\int_a^{a_0}|u_m(t)|^2 \le \frac{2\alpha}{\delta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+\frac{\delta_1^2}{2\delta^2} \tag{2.24}$$

substitution of (2.24) in (2.23) and the following equality by definition of δ_1

$$\frac{\delta_1^2}{2\delta^2}=\frac{1}{2}\gamma^{-\frac{2}{\gamma-1}}$$

gives us

$$\rho_{im} \le n!(2n)^{n+1}\left[\frac{2\alpha}{\delta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+\frac{1}{2}\gamma^{-\frac{2}{\gamma-1}}\right]^{\frac{1}{2}} \le \tag{2.25}$$

$$\le n!(2n)^{n+1}\alpha_1\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)^{\frac{1}{2}}$$

by (2.22) and (2.23) we have

$$\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)| \le n\sum_{i=n_0}^{n-1}\rho_{im}+n\int_a^{a_0}|u_m^{(n)}(\tau)|d\tau \le$$

$$\le n\sum_{i=n_0}^{n-1}\rho_{im}+\frac{4\alpha n}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+4n\int_{I_m}F^*(\tau,\beta)d\tau+$$

$$+2(\rho_{n-2}+\rho_{n-1}) \le 3n\sum_{j=n_0}^{n-1}\rho_{im} + \frac{4\alpha n}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+$$

$$+4n\int_{I_m} F^*(\tau,\beta)d\tau \le 3n^2 n!(2n)^{n+1}\alpha_1\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)^{\frac{1}{2}}+$$

$$+\frac{1}{2}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+4n\int_{I_m}F^*(\tau,\beta)d\tau$$

hence

$$1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)| \le [6n^2 n!(2n)^{n+1}\alpha_1 + 4n\int_{I_m}F^*(\tau,\beta)d\tau+1]^2$$

and so

$$\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)| \le \frac{r_0}{\alpha}-1 \tag{2.26}$$

by (2.18) we have and (2.26)

$$\int_a^t |u_m^{(n)}(t)+u_m^{(n-2)}(t)|dt \le \int_a^t |(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)|dt + \tag{2.27}$$

$$+\int_a^t F^*(s,1)ds \le r_0 + \int_a^t F^*(s,1)ds.$$

If we return to (2.21) and put $b=a$, than we obtain

$$u_m^{(n-2)}(t) = u_m^{(n-2)}(a)\cos(t-a)+$$

$$+u_m^{(n-1)}(a)\sin(t-a)+\int_a^t \sin(t-\tau)\tilde{f}(\tau)dr,$$

and so

$$\int_a^t |u_m^{(n-2)}(\tau)|d\tau \le \int_a^t\left(\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)d\tau + \int_a^t\left(\int_a^\tau|\tilde{f}(s)|ds\right)d\tau. \tag{2.28}$$

Now in view of (2.27) and (2.28)

$$\int_a^t |u_m^{(n)}(\tau)|d\tau \le \int_a^t |u_m^{(n)}(\tau)+u_m^{(n-2)}(\tau)|d\tau + \int_a^t |u_m^{(n-2)}(\tau)|d\tau \le \tag{2.29}$$

$$\le r_0 + \int_a^t F^*(s,1)ds + (t-a)(\frac{r_0}{\alpha}-1)+$$

$$+(t-a)r_0 + \int_a^t \left(\int_a^s F^*(\tau,1)d\tau\right)ds$$

and by the definition of $r_i(t)$ we have

$$\sum_{i=0}^{n-1} |u^{(i)}(t)| \le r(t) \overset{def}{=} \sum_{i=0}^{n-1} r_i(t). \ ■$$

Lemma 2.4. *Let $n = 2n_0$ and there exist a constant γ and continuous functions $\delta:[a,+\infty]\to[0,+\infty]$ and $F:[a,+\infty]\times\Re\to[0,\infty]$ such that*

$$\int_a^{+\infty} \delta^{-\frac{2}{\gamma-1}}(t)dt < +\infty \tag{2.30}$$

and the inequality

(2.31) $\delta(t)|x_1|^\gamma \le (-1)^{n_0-1} f(t,x_1,\ldots,x_n)\,\mathrm{sign}\,x_1 \le F(t,x_1)$,

holds on $[a,+\infty]\times\Re^n$. Then for arbitrary constants $c_1,\ldots,c_{n_0}$ there exists a solution u_m of problem (2.10), (2.11) *such that*

$$\int_a^{a+m} u_m^2(t)dt + \int_a^{a+m} |u_m^{(n_o)}(t)|^2\, dt < \bar{r}_0 \tag{2.32}$$

and

$$\sum_{i=1}^{n} |u_m^{(i-1)}(t)| < r(t), \quad a \le t \le a+m \tag{2.33}$$

where $\bar{r}_0 \in [0,+\infty]$ and $r \in C([a,+\infty])$ are independent of m.

Proof. Let us introduce the following notations

$$a_0 \le a + \frac{\pi}{6}$$

$$l > \max\left\{ \left[\frac{n_0}{n_0 - 1} \left(\frac{4}{n_0} \right)^{\frac{n_{0-1}}{n_0^2}} \right]^{\frac{n_0^2}{n_0 - 1}}, l_0 \right\} \text{ where } l_0 \text{ is from Lemma 2.2}$$

$$\rho = \int_a^{\infty} |\delta(t)|^{-\frac{2}{\gamma - 1}} dt < +\infty,$$

$$\alpha = 2 \max\left\{ (1 + |c_1|)^2 : 0 \le i \le n_0 - 1 \right\},$$

$$\alpha_0 = \frac{n_0}{n_0 - 1} \left(\frac{\ln_0}{4} \right)^{-\frac{1}{n_0} \frac{n_0}{n_0}},$$

$$\alpha_1 = \frac{1}{n_0} \left[\frac{1}{2} \frac{n_0 - 1}{n_0} \right]^{-\frac{n_0^2}{n_0 - 1}},$$

$$\alpha_2 = 2\alpha + \frac{4\alpha_1 \rho}{1} + \frac{\alpha}{1},$$

$$\beta = 8n\alpha,$$

$$r_0 = \alpha_2 [6n! n (2n)^{n+1} \alpha_2^{\frac{1}{2}} + 4n \int_{I_m} F^*(s, \beta) ds + 1]^{\frac{1}{2}}$$

$$\bar{r}_0 = \rho + r_0 + \frac{r_0}{\alpha_2} - 1$$

$$r_i(t) = r_0 \sum_{j=i}^{n} \frac{(t-a)^{j-i}}{(j-i)!} + \frac{(t-a)^{n-i}}{(n-i)!} [r_0 + (\frac{r_0}{\alpha_2} - 1 + r_0)(t-a) +$$

$$+\int_a^t F^*(s,1)ds+\int_a^t\left(\int_a^\tau F^*(s,1)ds\right)d\tau],$$

$$\chi_i(t,x)=\begin{cases} x & \text{when } |x|\le r_i(t) \\ r_i(t)\,\mathrm{sign}x, & \text{when } |x|>r_i(t) \end{cases},$$

Consider an auxiliary boundary value problem

$$u^{(n)}+u^{(n-2)}=(-1)^{n_0-1}lu+f(t,\chi_1(t,u),\dots,\chi_n(t,u^{(n-1)}))-\\ -(-1)^{n_0-1}l\chi_1(t,u) \tag{2.34}$$

and (2.11).

For any fixed natural m by Lemma 2.2 there exists only a trivial solution of the homogeneous linear equation

$$u^{(n)}+u^{(n-2)}=(-1)^{n_0-1}lu \tag{2.34_0}$$

with the boundary condition (2.4).

Now the nonhomogeneous term of equation (2.34) satisfies the conditions of Conti' s theorem and hence the latter theorem implies that there exists a solution u_m of problem (2.34), (2.11).

It is obvious that

$$l|x_1|=l|\delta(t)|^{-\frac{1}{\gamma}}|\delta(t)|^{\frac{1}{\gamma}}|x_1|\le|\delta(t)||x_1|^{\alpha}+l^{\frac{\gamma}{\gamma-1}}|\delta(t)|^{-\frac{1}{\gamma-1}}$$

end so we have

$$|\delta(t)||x_1|^{\gamma}\ge l|x_1|-g(t) \tag{2.35}$$

where

$$g(t)\overset{def}{=}l^{\frac{\gamma}{\gamma-1}}|\delta(t)|^{-\frac{1}{\gamma-1}},$$

By condition (2.31) we then obtain

$$(-1)^{n_0-1}f(t,x_1,\dots,x_n)\,\mathrm{sign}x_1\ge l|x_1|-g(t)$$

and thus the equation

$$\left|(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)\right|+(-1)^{n_0}(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)=0$$

is true. Integration from a to $a+m$ gives us

$$(2.36)\qquad \int_a^{a+m} \left|(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)\right|dt+\int_a^{a+m}\left|u_m^{(n_0)}(t)\right|^2 dt\le$$

$$\le 2\max\left\{(1+|c_i|)^2 : 0\le i\le n_0-1\right\}(1+\sum_{j=n_0}^{n-2}|u_m^{(j)}(a)|)+$$

$$+\int_a^{a+m}|u_m^{(n_0-1)}(t)|^2\,dt$$

By Lemma 2.1 we have

$$(2.37)\qquad \int_a^{a+m}|u_m^{n_0-1}(t)|^2\,dt\le\left(\int_a^{a+m}u_m^2(t)\right)^{\frac{1}{n_0}}\left(\int_a^{a+m}|u_m^{(n_0)}(t)|^2\,dt\right)^{\frac{n_0-1}{n_0}}\le$$

$$\le\omega^{-n_0}\frac{1}{n_0}\int_a^{a+m}u_m^2(t)dt+\frac{n_0}{n_0-1}\omega^{\frac{n_0-1}{n_0}}\int_a^{a+m}|u_m^{(n_0)}(t)|^2\,dt$$

when $\omega=\left[\dfrac{1}{2}\dfrac{n_0-1}{n_0}\right]^{\frac{n_0}{n_0-1}}$ by (2.36) and (2.37) we obtain

$$(2.38)\qquad \int_a^{a+m}|(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)|dt+\int_a^{a+m}|u_a^{(n_0)}(t)|^2\le$$

$$\le\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+\alpha_1\int_a^{a+m}u_m^2(t)dt+\frac{1}{2}\int_a^{a+m}|u_m^{(n_0)}(t)|^2\,dt$$

From (2.35) it is clear that the inequality

$$(-1)^{n_0}(u_m^{(n)}(t)+u_m^{(n-2)}(t)u_m(t)+lu_m^2(t)\le g(t)u_m(t)$$

is true.

Integrating by parts the left-hand side of the latter inequality and using the Young inequality

$$|g(t)||u_m(t)|\le\frac{l}{4}u_m^2(t)+\frac{1}{l}|g(t)|^2$$

we obtain

$$\int_a^{a+m} |u_m^{(n_0)}(t)|^2\,dt + l\int_a^{a+m} u^2(t)dt \le \int_a^{a+m} u_m^{(n_0-1)}(t)dt +$$

$$+\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+\frac{l}{4}\int_a^{a+m} u_m^2(t)dt+\int_a^{a+m}|g(t)|^2\,dt$$

Returning to (2.37) and putting there $\omega=\left(\frac{l}{4}n_0\right)^{-\frac{1}{n_0}}$, we have

$$(2.39)\quad \int_a^{a+m} |u_m^{(n_0)}(t)|^2\,dt + l\int_a^{a+m} u_m^2(t)dt \le$$

$$\le\frac{l}{4}\int_a^{a+m} u_m^2(t)dt+\frac{l}{a}\int_a^{a+m} u_m^2(t)dt+\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+$$

$$+\frac{n_0}{n_0-1}\left(\frac{l n_0}{4}\right)^{-\frac{1}{n_0}\frac{n_0-1}{n_0}}\int_a^{a+m}|u_m^{(n_0)}(t)|^2\,dt+\int_a^{a+m}|g(t)^2dt$$

An appropriate choice of l gives $\frac{n_0}{n_0-1}\left(\frac{4}{l n_0}\right)^{\frac{n_0-1}{n_0^2}}<1$ so that by (2.39) we have

$$(1-\alpha_0)\int_a^{a+m} |u_m^{(n_0)}(t)|^2\,dt+\frac{l}{2}\int_a^{a+m} u_m^2(t)\le\rho+\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)$$

Condition (2.30) cleary implies that $p<+\infty$. From this and the above inequality it is easy to show that (2.38) implies

$$(2.40)\qquad \int_a^{a+m}(u_m^{(n)}(t)+u_m^{(n-2)}(t))u_m(t)|dt+\int_a^{a+m}|u_m^{(n_0)}(t)|^2\,dt\le$$

$$\le 2\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)+\frac{2\alpha_1}{l}\left[2\rho+\alpha\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)\right]\le$$

$$\leq (2\alpha + \frac{4\alpha_1 \rho}{l} + \frac{\alpha}{l})\left(1 + \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)|\right).$$

Introducing the notation $I_m = \{t \in [a, a_0] : |u_m(t)| < \beta\}$, we obtain

$$\text{(2.41)} \quad \int_a^{a_0} |u_m^{(n)}(t) + u_m^{(n-2)}(t)| dt \leq \int_{I_m} F^*(s,\beta) ds + \frac{1}{\beta} \int_{[a,a_0]\setminus I_m} |(u_m^{(n)}(t) +$$

$$+ u_m^{(n-2)}(t)) u_m(t)| dt \leq \int_{I_m} F^*(s,1) ds + \frac{\alpha_2}{\beta}\left(1 + \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)|\right).$$

If we denote $\upsilon(t) = u_m^{(n-2)}(t)$, then (3.34) can be rewriten as

$$\text{(2.42)} \qquad \upsilon''(t) + \upsilon(t) = \tilde{f}(t)$$

where

$$\tilde{f}(t) = (-1)^{n_0 - 1} l \chi_1(t, u_m(t)) +$$
$$+ f(t, \chi_1(t, u_m(t)), \ldots, \chi_n(t, u_m^{(n-1)}(t))) -$$
$$-(-1)^{n_0-1} l \chi_1(t, u_m(t)).$$

Solving (2.42) as a second order differential equation with constant coefficients, we obtain

$$\text{(2.43)} \qquad \upsilon(t) = \upsilon(b)\cos(t-b) + \upsilon'(b)\sin(t-b) + \int_b^t \sin(t-\tau)\tilde{f}(\tau) d\tau$$

Denote $\rho_{im} \overset{def}{=} \min\left\{|u_m^{(i)}(t)| : t \in [a, a_0]\right\}$ $(i = n_o + 1, \ldots, n-1)$. If we now set $b = t_{n-2}$, where t_{n-2} is the point of minimum of the function $u_m^{(n-2)}$ on $[a, a_0]$, i.e.,

$$|u_m^{(n-2)}(t)| \leq |u_m^{(n-2)}(t_{n-2})| + |u_m^{(n-1)}(t_{n-1})| + \frac{1}{2}\int_a^{a_0} |u_m^{(n)}(t)| dt + \int_a^{a_0} |\tilde{f}(t)| dt$$

since $a_0 - a \leq \frac{\pi}{6}$, then by integration from a to a_0 we shall have

$$(2.44)\qquad \int_a^{a_0} u_m^{(n-2)}(t)dt \le \int_a^{a_0} |u_m^{(n)}(t)+u_m^{(n-2)}(t)|dt + \frac{1}{2}\int_a^{a_0} |u_m^{(n)}(t)|dt + +\rho_{n-2}+\rho_{n-1},$$

and therefore

$$(2.45)\qquad \int_a^{a_0} |u_m^{(n)}(t)| \le \int_a^{a_0} |u_m^{(n)}(t)+u_m^{(n-2)}(t)|dt + \int_a^{a+m} |u_m^{(n-2)}(t)|dt \le$$

$$\le 2\int_a^{a_0} |u_m^{(n)}(t)+u_m^{(n-2)}(t)|dt + \frac{1}{2}\int_a^{a_0} |u_m^{(n)}(t)|dt + \rho_{n-2}+\rho_{n-1}.$$

On account of (2.45) and (2.41)

$$(2.46)\qquad \int_a^{a_0} |u_m^{(n)}(t)|dt \le 2(\rho_{n-2m}+\rho_{n-1m}) +$$

$$+4\int_{I_m} F^*(s,b)ds + \frac{4a_2}{b}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)$$

Now by (2.46)

$$(2.47)\qquad \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)| \le n\sum_{i=n_0}^{n-1} \rho_{im} + n\int_a^{a_0} |u_m^{(n)}(t)|dt \le$$

$$\le n\sum_{i=n_0}^{n-1}\rho_{im} + 2n(\rho_{n-2m}+\rho_{n-1m}) +$$

$$+4n\int F^*(s,\beta)ds \frac{4\alpha_2 n}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right) \le$$

$$\le 3n\sum_{i=n_0}^{n-1} \rho_{im} + 4n\int_{I_m} F^*(s,\beta)ds + \frac{4\alpha_2 n}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right).$$

Due Lemma 5.1 from [2] we obtain

$$\rho_{im} \le n!(2n)^{n+1}\left(\int_a^{a_0} |u_m^{(n_0)}(t)|^2\, dt\right)^{\frac{1}{2}}$$

This inequality and (2.47) give us

$$(2.48) \qquad \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)| \le 3nn!(2n)^{n+1}\alpha_2^{\frac{1}{2}}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)^{\frac{1}{2}} +$$

$$+4n\int_{I_m} F^*(s,\beta)ds + \frac{4\alpha_2 n}{\beta}\left(1+\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right).$$

Since $\beta = 8\alpha_2 n$, we have

$$1+\sum_{j=n_0}^{n-1} |u_m^{(j)}(a)| \le [6nn!(2n)^{n+1}\alpha_2^{\frac{1}{2}} + 4n\int_{I_m} F^*(s,\beta)ds + 1]^2$$

From the definition of r_0 it follows that

$$(2.49) \qquad \sum_{j=n_0}^{n-1} |u_m^{(j)}(a)| \le \frac{r_0}{\alpha_2} - 1.$$

By (2.40) and (2.49)

$$(2.50) \qquad \int_a^t |u_m^{(n)}(\tau) + u_m^{(n-2)}(\tau)|d\tau \le \int_a^t \left|\left(u_m^{(n)}(\tau) + u_m^{(n-2)}(\tau)\right)u_m(\tau)\right|d\tau +$$

$$+\int_a^t F^*(s,1)ds \le r_0 + \int_a^t F^*(s,1)ds.$$

Returning to (2.43) and putting there $b = a$, we obtain

$$\int_a^t |u_m^{(n-2)}(\tau)|d\tau \le \int_a^t \left(\sum_{j=n_0}^{n-1}|u_m^{(j)}(a)|\right)d\tau + \int_a^t\left(\int_a^\tau |\tilde{f}(s)|ds\right)d\tau \le$$

$$\le (\frac{r_0}{\alpha_2} - 1)(t-a) + \int_a^t [r_0 + \int_a^\tau F^*(s,1)ds]d\tau.$$

Now

$$\int_a^t |u_m^{(n)}(\tau)|d\tau \le \int_a^t |u_m^{(n)}(\tau)+u_m^{(n-2)}(\tau)|d\tau + \int_a^t |u_m^{(n-2)}(\tau)|d\tau \le$$

$$\le r_0 + \int_a^t F^*(s,1)ds + (\frac{r_0}{\alpha_2} - 1 + r_0)(t-a) + \int_a^t \left(\int_a^\tau F^*(s,1)ds \right) d\tau .$$

The definition of $r_i(t)$ shows that

$$\sum_{i=0}^{n-1} |u_m^{(i)}(t)| \le r(t) \stackrel{def}{=} \sum_{i=0}^{n-1} r_i(t)$$

From (2.40) it is clear that

$$\int_a^{a+m} |u_m^{(n_0)}(\tau)|^2 d\tau \le r_0 \quad \text{and} \quad \int_a^{a+m} |u_m^2(t)|dt \le \rho + \frac{r_0}{\alpha_2} - 1. \ \blacksquare$$

3. Proof of the Main Results

Proof of Theorem 1.1. For arbitrarily fixed constants $c_1,\dots,c_{n_0}$ and a natural m by Lemma 2.3 there exists a solution of (1.1) satisfying the boundary condition (2.11) and

$$\sum_{i=1}^{n} |u_m^{(i-1)}(t)| < r(t)$$

where r is a continuous function independent of m. It can be easily verified that the sequence $\{u_m(t)\}_{m\ge1}$ is uniformly bounded and equicontinuous. Consequently, by the Ascoli-Arzela lemma from this sequence we can obtain a subsequence $\{u_{m_k}(t)\}_{k\ge1}$, such that sequences $\{u_{m_k}^{(i-1)}(t)\}_{k\ge1}$, where $i=1,\dots,n_0$, will converge uniformly on every finite interval $[a,a+m]$. It is obvious that

$$u(t) \stackrel{def}{=} \lim_{k\to\infty} u_{m_k}(t)$$

is a solution of equation (1.1). Passage to the limit in (2.11) implies that

$$u^{(i-1)}(a) = c_i \quad (i = 1, \dots, n_0)$$

From the fact that the right-hand side of (1.1) satisfies the local Lipschitz condition we conclude that the obtained is proper. ■

Proof of Theorem 1.2. For arbitrary fixed constants $c_1, \dots, c_{n_0}$ and a natural m by Lemma 2.4 there exists a solution of (1.1) satisfying the boundary condition (2.11), the condition

$$\sum_{i=1}^{n} |u_m^{(i-1)}(t)| < r(t) \tag{3.1}$$

and

$$\int_a^{a+m} u_m^2(t)dt + \int_a^{a+m} |u_m^{(n_0)}(t)|^2\, dt \le \bar{r}_0 \tag{3.2}$$

where r_0 and $r(t)$ are independent of m. By condition (3.1) there exists a sequence $\{u_{m_k}(t)\}_{k\ge1}$ such that sequences $\{u_{m_k}^{(i-1)}(t)\}_{k\ge1}$, where $i = 1, \dots, n_0$, will converge uniformly on every finite interval $[a, a+m]$ and

$$u(t) \overset{def}{=} \lim_{k\to\infty} u_{m_k}(t)$$

is a solution of equation of (1.1). Passage to the limit in (2.11) implies that

$$u^{(i-1)}(a) = c_i \quad (i = 1, \dots, n_0)$$

Since the righnt-hand side of (1.1) satisfies the local Lipschitz condition, the obtained solution is proper.

Applying Lemma 4.5 from [2] and condition (3.2), we obtain

$$\lim_{t\to+\infty} u^{(i-1)}(t) = 0 \quad (i = 1, \dots, n_0)$$

By Theorem 1.1 from [1] in order that everyone of such solutions be oscillatory it is sufficient to show that every proper solution of the equation

$$\upsilon^{(n-2)}(t) = f(t, \eta_1(t)u, \dots, \eta_n(t)u^{(n-1)}) \tag{3.3}$$

is oscillatory; here $\eta_1 \dots, \eta_n$ are arbitrary continuous functions satisfying the conditions

(3.4) $$\lim_{t\to+\infty} \eta_1(t) = 1, \quad \lim_{t\to+\infty} \eta_i(t) = 0 \ (i = 2,\dots,n).$$

Let the opposite be true, i.e., there exists a nonoscillatory proper solution of (3.3). It can be assumed without loss of generality that $\upsilon(t) > 0$, $\eta_1(t) > \frac{1}{2}$ for $t > t_0$ and n_0 is an odd number. On the other hand, since n_0 is odd, by condition (1.7) we have

$$\upsilon^{(n)} \le -\delta(t)|\upsilon(t)|^{\gamma} \le 0, \ t \ge t_0.$$

By Lemma 2.1 from [1] there exists a number $l \in \{1,\dots,n-1\}$ such that

$$\upsilon^{(i)}(t) \ge 0 \ \ (i = 1,\dots,l), \ \ (-1)^{i-1}\upsilon^{(i)}(t) \ge 0 \ \ (i = l+1,\dots,n),.$$

Therefore

$$\upsilon(t) \ge \upsilon(t_0) > 0 \quad \text{for} \quad t \ge t_0$$

and so

$$\upsilon^{(n)}(t) \le -\delta(t)c_0, \quad \text{where} \quad c_0 = \upsilon^{\gamma}(t_0),$$

Integrating the latter inequality from t_0 to t, obtain

$$\upsilon^{(n-1)}(t) \le \upsilon^{(n-1)}(t_0) - c_0\int_{t_0}^{t}\delta(s)ds \quad \text{for} \quad t \ge t_0$$

It is obvious that

$$1 = \delta^{\frac{2}{\gamma+1}}(t)\delta^{-\frac{2}{\gamma-1}}(t) \le \delta(t) + \delta^{-\frac{2}{\gamma-1}}(t)$$

or

$$\delta(t) \ge 1 - \delta^{-\frac{2}{\gamma-1}}(t).$$

Now

$$\int_{t_0}^{t}\delta(s)ds \ge t - t_0 - \int_{t_0}^{t}\delta^{-\frac{2}{\gamma-1}}(s)ds \to +\infty,$$

and therefore $\upsilon^{(n-1)}(t) \to -\infty$, which contradicts the conditions of Lemma 2.1 from [1]. ■

References

[1] I.T.Kiguradze, Criteria of Oscillation for a Certain Class of Ordinary Differential Equations. Differents. *Uravneniya.* 28(1992), 207-219 (Russian).

[2] I.T. Kiguradze, T.A. Chanturia, Asymptotic Properties of Solution of Solutions of Nonautonomous Ordinary Differential Equations. Kluwer Academic Publishers, Dodrecht, Bostom, London, 1993.

(A.Razmadze Mathematical Institute, Georgian Academy of Sciences, Tbilisi.
Tadeusz Kotarbinsky Pedagogical University, Zielona Góra).
Received on 26.10.1994.