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THE FIRST THREE-PARABOLIC PROBLEM FOR THE TIME-SPATIAL CURVILINEAR TRAPEZIUM

The subject of the paper is the construction of the solution of the three-parabolic problem for general domain with boundary conditions of Lauricella type.

To construct the solution we shall apply the convenient heat potentials with unknown densities q_i. To determine q_i we use the system of six Volterra integral equations.

Key words: Three-parabolic problem, Three-parabolic potentials, the system of Volterra integral equations, the boundary conditions of Lauricella type.

1. The formulation of the problem

Let us consider the equation

(1) $P^3u(x,t) = f(x,t),\ P^3 = P(P^2),\ P^2 = P(P),\ P = D_x^2 - D_t$

in the domain

$$D = \{(x,t) \colon p_1(t) < x < p_2(t), t \in (0,T], T < \infty\}.$$

In the sequel we shall construct the solution $u \in C^{6,3}(D) \cap C^{2,2}(\overline{D})$ satisfying the initial conditions

(2) $D_t^i u(x,0) = f_i(x),\ i = 0,1,2$ for $x \in (p_1(0), p_2(0))$

and the boundary-value conditions

(3) $D_x^i u(p_1(t),t) = h_{i+1}(t),\ i = 0,1,2$ for $t \in (0,T]$,

(4) $D_x^i u(p_2(t),t) = h_{i+4}(t),\ i = 0,1,2$ for $t \in (0,T]$,

where $f,\ f_j;\ j = 0,1,2, h_k,\ k = 1,2,\dots,6$, are given functions, such that $f \in (K_5),\ f_j \in (K_3),\ j = 0,1,2,\ h_k \in (K_4),\ k = 1,2,\dots,6$. The class (K_3), (K_4) and (K_5) are defined in the sequel.

2. Motivation of the considered problem

In [1] and [2] the similar problems for bicaloric equation in the half-plane and in [2] the three-parabolic problem in *n*-dimensional half-space was treated. In the monograph [5] the similar problem for the equation $Pu = f$ with initial and boundary conditions was solved. In [13] the similar problem for the equation (1) with limit conditions of Lauricella type was treated. In the monograph [10], vol.II, p235 the biparabolic Cauchy problem was solved.

In [3] and [4] the similar biparabolic problems are treated. In [15] the nonlinear biparabolic problem was solved. In [6] and [7] the three-parabolic problem for three-dimensional cylinder with boundary conditions of Riquier type and for the strip with boundary conditions of Lauricella type respectively was solved.

In [8] and [9] the polyparabolic problems with boundary conditions of Lauricella type for the quart-time-plane and for the strip respectively was treated.

In [11] the Fourier problem for trapezium was solved.

3. Some definitions and denotations

Definition 1. Denote by (K_1) *the class of functions* $u\colon D \to \Re$, *such that* $u(x,t) \in C^{6,3}(D) \cap C^{2,2}(\overline{D})$.

Definition 2. Denote by (K_2) *the class of functions* $p_i : [0,T] \to \Re$, $i = 1,2$, *such that* $p_i \in C^2([0,T])$, $D_t^j p_i(t) \neq 0$, $j = 0,1,2$, $i = 1,2$ *for* $t \in [0,T]$ *and* $b > (p_i(t) - p_j(t)) > A > 0$, $i,j = 1,2$ *for* $t \in [0,T]$, *where* A,B *begin a positive constants.*

Definition 3. Denote by (K_3) *the class of functions* $f: D_1 \to \Re$, $D_1 = \{(x,0): x \in (p_1(0), p_2(0))\}$, *such that* $f \in C^8(D_1) \cap C^2(\overline{D_1})$ *and* $D_x^i f(p_i(0)) = 0$, $i = 0,1,2$, $j = 1,2$, *comp.supp* $D_x^6 f_k \subset D_1$, $k = 0,1,2$.

Definition 4. Denote by (K_4) *the class of functions* $h: [0,T] \to \Re$, *such that* $h \in C^3([0,T])$ *and* $D_t^i h(0) = 0$, $i = 0,1,2$.

Definition 5. Denote by (K_5) *the class of functions* $F: D \to \Re$, *such that* $F \in C^{2,1}(D)$ *and comp.supp* $F \subset D$.

4. Reduction of the initial conditions to the homogeneous initial conditions

Let us consider the function

$$w(x,t) = u(x,t) - r(x,t),$$

where

$$r(x,t) = f_0(x) + t\, f_1(x) + \frac{1}{2} t^2 f_2(x).$$

Lemma 1. Let $p_i \in (K_2)$, $i = 1,2$. *If* $f_i \in (K_3)$, $i = 0,1,2$, $h_{i+1}, h_{i+4} \in (K_4)$ $i = 0,1,2$ *and* $u \in (K_1)$ *is the solution of the problem (1) - (4), then* $w \in (K_1)$ *and satisfies the conditions:*

(1a) $$P^3 w(x,t) = F(x,t),$$
$$F(x,t) = f(x,t) - P^3 r(x,t), \qquad (x,t) \in D,$$

(2a) $$D_t^i\, w(x,0) = 0,\ i = 0,1,2, \qquad x \in (p_1(0), p_2(0)),$$

(3a) $$D_x^i w(p_1(0),t) = H_{i+1}(t),$$
$$H_{i+1}(t) = h_{i+1}(t) - D_x^i r(p_1(t),t), \qquad i = 0,1,2,$$

(4a) $$D_x^i w(p_2(0),t) = H_{i+4}(t),$$
$$H_{i+4}(t) = h_{i+4}(t) - D_x^i r(p_2(t),t), \qquad i = 0,1,2,$$
$$t \in (0,T].$$

Conversely, if $F \in (K_5)$, $H_{i+1}, H_{i+4} \in (K_4)$, $i = 0,1,2$ and $w \in (K_1)$ is a solution of problem (1a) - (4a), then function $w = (w + r) \in (K_1)$ and is a solution of the problem (1) - (4).

We omit the simple proof.

5. Potentials w_i

Let us consider the potentials

$$w_1(x,t) = A\int_0^t q_1(s)(t-s)^{3/2} \exp(B(t,s)(x-p_1(s))^2)ds,$$

$$w_2(x,t) = A\int_0^t q_2(s)(t-s)^{1/2}(x-p_1(s)) \exp(B(t,s)(x-p_1(s))^2)ds,$$

$$w_3(x,t) = A\int_0^t q_3(s)(t-s)^{-1/2}(x-p_1(s))^2 \exp(B(t,s)(x-p_1(s))^2)ds,$$

$$w_4(x,t) = A\int_0^t q_4(s)(t-s)^{3/2} \exp(B(t,s)(x-p_2(s))^2)ds,$$

$$w_5(x,t) = A\int_0^t q_5(s)(t-s)^{1/2}(x-p_2(s)) \exp\left(B(t,s)(x-p_2(s))^2\right)ds,$$

$$w_6(x,t) = A\int_0^t q_6(s)(t-s)^{-1/2}(x-p_2(s))^2 \exp(B(t,s)(x-p_2(s))^2)ds,$$

$$w_7(x,t) = A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{3/2} \exp(B(t,s)(x-y)^2)dyds,$$

$$w_7^1(t) = w_7(p_1(t),t) = A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{3/2} \exp(B(t,s)(p_1(t)-y)^2)dyds$$

$$w_7^2(t) = w_7(p_2(t),t) = A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{3/2} \exp\left(B(t,s)(p_2(t)-y)^2\right)dyds$$

$$D_x w_7(p_1(t),t) = w_7^{1,1}(t) = \left(\frac{1}{2}\right) A \int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}(p_1(s)-y) \times$$
$$\times \exp(B(t,s)\,(p_1(t)-y)^2)dyds,$$

$$D_x w_7(p_2(t),t) = w_7^{1,2}(t) = \left(\frac{1}{2}\right) A \int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{1/2}(p_2(t)-y) \times$$
$$\times (B(t,s)(p_2(t)-y)^2)dyds,$$

$$D_x^2 w_7(p_1(t),t) = w_7^{2,1}(t) = 2A \int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{3/2} B(t,s) \times$$
$$\times (1-2B(t,s)(p_1(t)-y)^2)\exp(B(t,s)(p_1(t)-y)^2)dyds,$$

$$D_x^2 w_7(p_2(t),t) = w_7^{2,2}(t) = 2A \int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)^{3/2} B(t,s) \times$$
$$\times (1-2B(t,s)(p_2(t)-y)^2)\exp(B(t,s)(p_2(t)-y)^2)dyds,$$

where $A = (\pi)^{-1}$, $B(t,s) = (-4(t,s))^{-1}$, $q_i(s)$, $i = 1,2,\ldots,6$ are unknown functions continuous for $s \in [0,T]$.

6. Properties of the potential w_7

In the sequel by C, C_i, $i \in \aleph$ we shall denote a positive constants.

Lemma 2. If $f_i \in (K_3)$, $h_j \in (K_4)$, $i = 0,1,2$, $j = 1,2,\ldots,6$, $F \in (K_5)$ and if $F(p_1(s),s) = F(p_2(s),s) = 0$, $D_y F(p_1(s),s) = D_y F(p_2(s),s)$, $D_s F(y,0) = 0$, then:

1° $P^3 w_7(x,t) = F(x,t)$ *for* $(x,t) \in D$,

2° $D_t^i w_7(x,0) = 0$, $i = 0,1,2$, $x \in (p_1(0), p_2(0))$,

3° *there exist the boundary functions* w_7^1, w_7^2, $w_7^{1,1}$, $w_7^{1,2}$, $w_7^{2,1}$, $w_7^{2,2}$ *continuous for* $t \in [0,T]$.

Proof. Ad 1°. Let us consider the integrals

$$I_7^j(x,t)=A\int_0^t\int_{p_1(s)}^{p_2(s)}F(y,s)(t-s)^{3/2}D_x^j\exp(B(t,s)(x-y)^2)dyds,$$

$$j=0,1,2.$$

and let $\overline{F}(y,s=F(y,s))$ for $(y,s)\in D$

and $\overline{F}(y,s)=0$ for $(y,s)\in((-\infty,\infty)\times[0,T])\setminus D$.

For $j=0$ we have

$$|I_7^0(x,t)|\le A(\sup_{\overline{D}(T)}|F|)\int_0^t(t-s)^{3/2}ds=C_0t^{5/2}=C_0t^{5/2}\le C_0T^{5/2}.$$

Consequently the integral $I_7^0(x,t)$ is locally uniformly convergent for $x\in[p_1(s),p_2(s)]$, $s\in(0,T]$ and consequently continuous for $x=p_1(t)$ and $x=p_2(t)$, $t\in[0,T]$,

i. e. $I_7^0(p_1(t),t)=w_7^1(t)$, $I_3^0(p_2(t),t)=w_7^2(t)\in C([0,T])$.

For $j=1$ we have

$$I_7^1(x,t)=-A\int_0^t\int_{p_1(s)}^{p_2(s)}F(y,s)(t-s)^{3/2}(D_y\exp(B(t,s)(x-y)^2))dyds=$$

$$=-A\int_0^t\overline{F}(y,s)(t-s)^{3/2}\exp(B(t,s)(x-y)^2)\Big|_{y=-\infty}^{y=\infty}ds+$$

$$+A\int_0^t\int_{-\infty}^{\infty}(D_y\overline{F}(y,s))(t-s)^{3/2}\exp(B(t,s)(x-y)^2)dyds=$$

$$=A\int_0^t\int_{-\infty}^{\infty}(D_y\overline{F}(y,s))(t-s)^{3/2}\exp(B(t,s)(x-y)^2)dyds.$$

Consequently $I_7^1(x,t)$ is locally uniformly convergent for $x=p_i(t)$, $i=1,2.$ and $D_xw_7(p_1(t),t)=w_7^{1,1}(t)$, $D_xw_3(p_2(t),t)=w_7^{1,2}(t)$ are continuous for $t\in(0,T]$.

For $j=2$ we have

$$I_3^2(x,t) = A\int_0^t \int_{p_1(s)}^{p_2(s)} (D_y^2 F(y,s))(t-s)^{3/2} \exp(B(t,s)(x-y)^2)dyds.$$

Similarly as for $I_7^1(x,t)$ the integral $I_7^2(x,t)$ is locally uniformly convergent for $x \in [p_1(s), p_2(s)]$, $s \in (0,T]$ and consequently continuous for $x = p_i(t)$, $i = 1,2$, i. e.

$$D_x^2 w_7(p_i(t),t) = w_3^{2,i}(t) \in C([0,T]), \quad i = 1,2.$$

For $w_7(x,t) = I_7^0(x,t)$, we have

$$PI_7^0(x,t) = D_x^2 I_7^0(x,t) - D_t I_7^0(x,t) =$$

$$= A\int_0^t \int_{-\infty}^{+\infty} \Big\{\overline{F}(y,s)(t-s)^{3/2} D_x^2 \exp(B(t,s)(x-y)^2) - \overline{F}(y,s) \times$$

$$\times D_t\left[(t-s)^{3/2} \exp(B(t,s)(x-y)^2)\right]\Big\}dyds =$$

$$= \int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)\left[D_x^2\left[(t-s)^{3/2} \exp(B(t,s)(x-y)^2)\right] - \right.$$

$$\left. - D_t\left[(t-s)^{3/2} \exp(B(t,s)(x-y)^2)\right]\right]dyds =$$

$$= A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)\{D_x^2[(t-s)^2(t-s)^{1/2} \exp(B(t,s)(x-y)^2)] -$$

$$-D_t[(t-s)^2(t-s)^{1/2} \times \exp(B(t,s)(x-y)^2)]\}dyds =$$

$$= A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)\Big\{(x)^2 D_x^2[(t-s)^{1/2} \exp(B(t,s) \times (x-y)^2)] -$$

$$-(t-s)^2 D_t[(t-s)^{1/2} \exp(B(t,s)(x-y)^2)] -$$

$$-2(t-s)[(t-s)^{1/2} \times \exp(B(t,s)(x-y)^2)]\}dyds =$$

$$= -2A\int_0^t \int_{p_1(s)}^{p_2(s)} F(y,s)(t-s)v(x,y,t,s)dyds,$$

where

$$v(x,y,t,s)=(t-s)^{-1/2}\exp(B(t,s)(x-y)^2).$$

Similarly

$$PI_7^0(x,t)=-2A\int_0^t\int_{p_1(s)}^{p_2(s)}F(y,s)v(x,y,t,s)dyds$$

and by the Poisson theorem, [10], vol. I, p.522, we obtain

$$P^3I_7^0(x,t)=F(x,t) \text{ for } (x,t)\in D.$$

Ad 2° We have the estimations

$$(5)\qquad \begin{cases} |w_7(x,t)|\le C\int_0^t (t-s)^{3/2}ds = C_1t^{5/2}\le C_1T^{5/2}, \\ |D_tw_7(x,t)|\le C_2\int_0^t (t-s)^{1/2}ds = C_3t^{3/2}\le C_3T^{3/2}, \\ |D_t^2w_7(x,t)|\le C_4\int_0^t (t-s)^{-1/2}ds = C_5t^{1/2}\le C_5T^{1/2}. \end{cases}$$

By (5) we obtain the assertions 2° and 3°.

7. Properties of the potentials w_i for $i=1,2,3,...,6$

Lemma 3. If $q_i(s)\in C[0,T]$, $q_i(0)=0$, $i=1,2,...6$, *then:*

1° $P^3w_i(x,t)=0$ *for* $(x,t)\in D$, $i=1,2,...6$,

2° $D_t^kw_i(x,0)=0$ *for* $x\in(p_1(0),p_2(0))$, $k=0,1,2$, $i=1,2,...,6$,

3° $D_x^kw_i(p_j(t),t)\in C([0,T])$, $j=1,2$, $k=0,1,2$, $i=1,2,...,6$.

Proof. We shall give the proof only for w_i, $i=1,2,3$, because for w_i $i=4,5,6$ the proof is similar.

ad 1° For the function w_1 we have

$$Pw_1(x,t) = D_x^2 w_1(x,t) - D_t w_1(x,t),$$

$$D_x^2 w_1(x,t) = A\int_0^t q_1(s)(t-s)^{3/2} D_x^2 \exp(B(t,s)(x-p_1(s))^2)ds =$$

$$= A\int_0^t q_1(s)(t-s)^2 D_x^2[(t-s)^{-1/2} \exp(B(t,s)(x-p_1(s))^2)]ds = I_{1,1}(x,t),$$

$$D_t w_1(x,t) = \lim_{s\to t} q_1(s)(t-s)^{3/2} \exp\left(B(t,s)(x-p_1(s))^2\right) +$$

$$+ I_{1,1}(x,t) = I_{1,1}(x,t)$$

where

$$I_{1,1}(x,t) = A\int_0^t q_1(s) D_t[(t-s)^2 (t-s)^{-1/2} \exp(B(t,s)(x-p_1(s))^2)]ds =$$

$$= A\int_0^t q_1(s)\{[(t-s)^2 (t-s)^{-1/2} \exp(B(t,s)(x-p_1(s))^2)] +$$

$$+2(t-s)[(t-s)^{-1/2} \exp(B(t,s)(x-p_1(s))^2)]\}ds.$$

$$Pw_1(x,t) = A\int_0 q_1(s)[(t-s)^2[D_x^2 V(x,t,s) - D_t V(x,t,s)] +$$

$$+2(t-s)V(x,t,s)]ds = 2\int_0^t q_1(s)(t-s)V(x,t,s)ds.$$

$$P^2 w_1(x,t) =$$

$$= 2\int_0^t q_1(s)[(t-s)D_x^2 V(x,t,s) - (t-s)D_t V(x,t,s) + V(x,t,s)]ds =$$

$$= 2\int_0^t q_1(s)V(x,t,s)ds, \qquad \text{where}$$

$$V(x,t,s) = (t-s)^{-1/2} \exp(B(t-s)(x-p_1(s))^2).$$

By [10], vol. I, p.484, we obtain

$$P^3 w_1(x,t) = 2\int_0^t q_1(s) P_{x,t} V(x,t,s)ds = 0 .$$

For the function w_2 we have

$$w_2(x,t) = A\int_0^t q_2(s)(t-s)^2[((x-p_1(s))(t-s)^{-3/2})\exp(B(t,s)(x-p_1(s))^2)]ds$$

$$Pw_2(x,t) = A\int_0^t q_2(s)(t-s)^2[D_x^2 W(x,t,s)]ds - A\int_0^t q_2(s)[(t-s)^2 D_t W(x,t,s) +$$

$$+2(t-s)W(x,t,s)]ds = 2A\int_0^t q_2(s)(t-s)W(x,t,s)ds ,$$

where

$$W(x,t,s) = ((x-p_1(s))(t-s)^{-3/2})\exp(B(t,s)(x-p_1(s))^2).$$

$$P^2 w_2(x,t) = 2A\int_0^t q_2(s)(t-s)D_x^2 W(x,t,s)ds - 2A\int_0^t q_2(s)(t-s)D_t W(x,t,s)ds -$$

$$-2A\int_0^t q_2(s)W(x,t,s)ds = -2A\int_0^t q_2(s)W(x,t,s)ds ,$$

$$P^3 w_2(x,t) = -2A\int_0^t q_2(s)D_x^2 W(x,t,s)ds + 2A\int_0^t q_2(s)D_t W(x,t,s)ds =$$

$$= 2A\int_0^t q_2(s) \times (-D_x^2 W(x,t,s) + D_t W(x,t,s))ds = 0 .$$

For the function w_3 we have

$$w_3(x,t) = A\int_0^t q_3(s)(x-p_1(s))^2 V(x,t,s)ds$$

$$Pw_3(x,t) = A\int_0^t q_3(s)\Big[(x-p_1(s))^2 D_x^2 V(x,t,s) + 4(x-p_1(s))D_x V(x,t,s) +$$

$$+2V(x,t,s)]ds - A\int_0^t q_3(s)[(x-p_1(s))^2 D_xV(x,t,s)]ds =$$

$$= A\int_0^t q_3(s)4(x-p_1(s))D_xV(x,t,s)ds + 2A\int_0^t q_3(s)V(x,t,s)ds,$$

$$P^2w_3(x,t) = P(Pw_3(x,t))AP\left(\int_0^t q_3(s)V(x,t,s)ds\right) + A\int_0^t q_3(s)[8D_x^2V(x,t,s)+$$

$$+4(x-p_1(s))D_x^3V(x,t,s) - 4(x-p_1(s))D_x^2D_tV(x,t,s)]ds =$$

$$= 8A\int_0^t D_x^2V(x,t,s)ds$$

$$P^3w_1(x,t) = 8A\int_0^t P_{xt}D_x^2V(x,t,s)ds = 8A\int_0^t D_x^2P_{x,t}V(p_1(s),t,s)ds = 0.$$

For $i = 4,5,6$ the proof is similar.

Ad 2°. Obviously $w_1(x,0) = 0$, because

$$|w_1(x,t)| \le C_6\int_0^t (t-s)^{3/2}ds \le -(t-s)^{5/2}\Big|_0^t \le C_7t^{5/2} \le C_7T^{5/2},$$

$$|D_tw_1(x,t)| \le C_8\int_0^t (t-s)^{-1/2}ds \le C_9t^{1/2} \le C_9T^{1/2},$$

$$D_t^2w_1(x,t) = A\int_0^t q_1(s)[C_{10}(t-s)^{-1/2}\exp(B(t,s)(x-p_1(s))^2)+$$

$$+C_{11}(t-s)^{1/2}(x-p_1(s))^2(t-s)^{-2}\exp(B(t,s)(x-p_1(s))^2)+$$

$$+C_{12}(t-s)^{3/2}(x-p_1(s))^2(t-s)^{-3}\exp(B(t,s)(x-p_1(s))^2)+$$

$$+C_{13}(x-p_1(s))^4(t-s)^{-5/2}\exp(B(t,s)(x-p_1(s))^2)\Big]ds.$$

We have the estimations

$$\left|A\int_0^t q_1(s)\left(C_{10}(t-s)^{-1/2}\exp(B(t,s)(x-p_1(s))^2)\right)\right| \le$$

$$\leq C_{14}\int_0^t (t-s)^{-1/2}ds \leq C_{15}t^{1/2} \leq C_{15}T^{1/2},$$

$$\left| C_{11}A\int_0^t q_1(s)(x-p_1(s))^2(t-s)^{-1/2}(t-s)^{-2}\exp(B(t,s)(x-p_1(s))^2)ds \right| \leq$$

$$\leq C_{16}\int_0^t \frac{(x-p_1(s))^2}{(t-s)}(t-s)^{-1/2}\exp(B(t,s)(x-p_1(s))^2)ds \leq$$

$$\leq C_{17}\int_0^t (t-s)^{-1/2}ds \leq C_{18}t^{1/2} \leq C_{18}T^{1/2}.$$

Similarly

$$\left| C_{12}A\int_0^t q_1(s)(t-s)^{3/2}(x-p_1(s))^2(t-s)^{-3}\exp(B(t,s)(x-p_1(s))^2)ds \right| \leq$$

$$\leq C_{19}t^{1/2} \leq C_{19}T^{1/2}$$

and

$$\left| C_{13}A\int_0^t q_1(s)(x-p_1(s))^4(t-s)^{-5/2}\exp(B(t,s)(x-p_1(s))^2)ds \right| \leq$$

$$\leq C_{20}t^{1/2} \leq C_{20}T^{1/2}.$$

Similarly we obtain assertion 2° for w_i, $i = 2,3,\ldots,6$.

Ad 3°. Analogously as for 2°, in the proof of assertion 1°, we obtain the estimations

(6) $\quad |D_x^k w_i(x,t)| \leq C_{21}t^a \leq C_{21}T^a, \quad k = 0,1,2, \quad i = 1,2,\ldots,6,$

where a is a positive number.

By (6) we obtain the assertion 3°.

8. Integral equations compatible to the boundary conditions

We suppose that the function $w(x,t)=\sum_{i=1}^{7} w_i(x,t)$ is the solution (1a) - (4a) problem. By (3a) - (4a), we obtain the system of the integral equations for the functions q_i, $i=1,2,\dots,6$, of the form:

(I_1a) $$\sum_{i=1}^{6} I_{1,i}(t)=H^1(t),$$

(I_2a) $$\sum_{i=1}^{6} I_{2,i}(t)=H^2(t),$$

(I_3a) $$\sum_{i=1}^{6} I_{3,i}(t)=H^3(t),$$

(I_4a) $$\sum_{i=1}^{6} I_{4,i}(t)=H^4(t),$$

(I_5a) $$\sum_{i=1}^{6} I_{5,i}(t)=H^5(t),$$

(I_6a) $$\sum_{i=1}^{6} I_{6,i}(t)=H^6(t),$$

where

$$I_{k+1,i}(t)=D_x^k w_i(p_1(t),t),\quad k=0,1,2,\quad i=1,2,\dots,6$$

$$I_{k+4,i}(t)=D_x^k w_i(p_2(t),t),\quad k=0,1,2,\quad i=1,2,\dots,6$$

$$H^k(t)=H_k(t)-w_7^k(p_1(t),t),\qquad k=1,2,3,$$

$$H^{3+k}(t)=H_{k+3}(t)-w_7^{3+k}(p_2(t),t),\qquad k=1,2,3.$$

9. Abel transformation to the system $(I_i a)$, $i=1,2,\dots,6$

We shall prove the following

Lemma 4. If $0 \le K_1 \le |p_2(t) - p_1(t)| \le K_2 < \infty$, $p_i \in (K_2)$, $i = 1,2$, *and* $|p_2'(t) \le K_3|$, $t \in [0,T]$, K_i $i = 1,2,3$ *are positive constants, then*

1° $$\exp\left(-\frac{(p_1(t) - p_2(s))^2}{4(t-s)}\right) \to 0 \quad \text{as} \quad s \to 0,\ t \in [0,T].$$

2° $$(t-s)^{-n} \exp(B(t,s)(p_1(t) - p_2(s))^2) \to 0 \quad \text{as} \quad s \to t.$$

Proof. Ad 1°. We have

$$\exp\left(-\frac{(p_1(t) - p_2(s))^2}{4(t-s)}\right) = \exp\left(-\frac{(p_1(t) - p_2(t))^2}{4(t-s)}\right) \times$$

$$\times \exp\left(-\frac{2(p_1(t) - p_2(t))(p_2(t) - p_2(s))}{4(t-s)}\right) \times$$

$$\times \exp\left(-\frac{(p_2(t) - p_2(s))^2}{4(t-s)}\right) \le$$

$$\le \exp\left(-\frac{(p_1(t) - p_2(t))^2}{4(t-s)}\right) \exp\left(-\frac{2(p_1(t) - p_2(t))(p_2(t) - p_2(s))}{4(t-s)}\right) =$$

$$= \exp\left(-\frac{(p_1(t) - p_2(t))^2}{4(t-s)}\right) \times$$

$$\times \exp\left(-\frac{2(p_1(t) - p_2(t))(t-s)p_2'(s + \Theta(t-s))}{4(t-s)}\right) \le$$

$$\le \exp\left(-\frac{K_2^2}{4(t-s)}\right) \exp\left(-\frac{2K_2|p_2'|}{4(t-s)}\right) \le$$

$$\le C \exp\left(-\frac{K_2^2}{4(t-s)}\right) \to 0 \quad \text{as} \quad s \to t,\ t \in [0,T].$$

Ad 2°. By 1°, we have

$$(t-s)^{-n}\exp(T(t,s)(p_1(t)-p_2(s))^2)\le (t-s)^{-n}C\exp\left(-\frac{K_2^2}{4(t-s)}\right)=$$

$$=C(t-s)^{-n}\frac{1}{\exp\left(\frac{K_2^2}{4(t-s)}\right)}=$$

$$=C(t-s)^{-n}\frac{1}{1+\frac{1}{1!}\frac{K_2^2}{4(t-s)}+\frac{1}{2!}\frac{K_2^4}{16(t-s)^2}+\ldots}\le$$

$$\le C(t-s)^{-n}\frac{1}{\frac{1}{(n+1)!}\frac{K_2^{2n+2}}{2^{2n+2}(t-s)^{n+1}}}=C(t-s)^{-n}\frac{(n+1)!(t-s)^{n+1}}{K_2^{2n+2}}\le$$

$$C(t-s)\to 0 \quad \text{as} \quad s\to t,\ t\in[0,T].$$

Applying the Abel transformation, [14], Vol. I, p.16, to the system $(\mathrm{I}_i\mathrm{a})$, $i=1,2,\ldots,6$, by Lemma 4, we obtain

Lemma 5. If $q_i\in C([0,T])$, $H^i\in C^3([0,T])$, $D_t^2H^i(0)=0$, $i=1,2,\ldots,6$, *then the system* $(\mathrm{I}_i\mathrm{a})$, $i=1,2,\ldots,6$ *is equivalent to the following Volterra system of integral equations of the second kind.*

$(\mathrm{I}_1\mathrm{b})$ $\quad A_{11}(t)q_1(t)+A_{12}(t)q_2(t)+$

$$+A_{13}(t)q_3(t)+\sum_{j=1}^{6}\int_0^t D_tS_{1,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^1(t),$$

$(\mathrm{I}_2\mathrm{b})$ $\quad A_{22}(t)q_2(t)+A_{23}(t)q_3(t)+\sum_{j=1}^{6}\int_0^t D_tS_{2,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^2(t),$

$(\mathrm{I}_3\mathrm{b})$ $\quad A_{33}(t)q_3(t)+\sum_{j=1}^{6}\int_0^t D_tS_{3,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^3(t),$

$$(\mathrm{I}_4\mathrm{b})\quad A_{44}(t)q_4(t)+A_{45}(t)q_5(t)+$$

$$+A_{46}(t)q_6(t)+\sum_{j=1}^{6}\int_0^t D_t S_{4,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^4(t),$$

$$(\mathrm{I}_5\mathrm{b})\quad A_{55}(t)q_5(t)+A_{56}(t)q_6(t)+\sum_{j=1}^{6}\int_0^t D_t S_{5,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^5(t),$$

$$(\mathrm{I}_6\mathrm{b})\quad A_{66}(t)q_{61}(t)+\sum_{j=1}^{6}\int_0^t D_t S_{6,j}(t,s_1)q_j(s_1)ds_1=\overline{H}^6(t),$$

where

$$A_{11}(t)=\frac{3}{4}A\beta\left(\frac{1}{2},\frac{1}{2}\right)(1+(p_1'(t)))^2,$$

$$A_{12}(t)=\frac{1}{4}A\beta\left(\frac{1}{2},\frac{1}{2}\right)(p_1'(t))^2,$$

$$A_{13}(t)=A\beta\left(\frac{1}{2},\frac{1}{2}\right)\left(-4p_1'(t)p_1''(t)+\frac{7}{4}(p_1'(t))^2\right),$$

$$A_{1i}(t)=0\qquad\text{for}\quad i=4,5,6,$$

$$A_{22}(t)=\frac{1}{2}A\beta\left(\frac{1}{2},\frac{1}{2}\right),$$

$$A_{23}(t)=\frac{1}{3}A(-p_1'(t)),$$

$$A_{2i}(t)=0\qquad\text{for}\quad i=1,4,5,6,$$

$$A_{33}(t)=-\frac{2}{3}A\beta\left(\frac{1}{2},\frac{1}{2}\right),$$

$$A_{3i}(t)=0\qquad\text{for}\quad i=1,2,4,5,6,$$

$$A_{44}(t)=\frac{3}{4}A\beta\left(\frac{1}{2},\frac{1}{2}\right)(1+(p_2'(t)))^2,$$

$$A_{45}(t)=\frac{1}{4}A\beta\left(\frac{1}{2},\frac{1}{2}\right)(p_2'(t))^2,$$

$$A_{46}(t) = A\beta\left(\frac{1}{2},\frac{1}{2}\right)\left(-4p_2'(t)p_2''(t) + \frac{7}{4}(p_2'(t))^2\right),$$

$$A_{4i}(t) = 0 \qquad \text{for} \quad i = 1,2,3,$$

$$A_{55}(t) = \frac{1}{2}A\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

$$A_{56}(t) = \frac{1}{3}A\beta\left(\frac{1}{2},\frac{1}{2}\right)(-p_2'(t))$$

$$A_{5i}(t) = 0 \qquad \text{for} \quad i = 1,2,3,4,$$

$$A_{66}(t) = -\frac{2}{3}A\beta\left(\frac{1}{2},\frac{1}{2}\right),$$

$$A_{6i}(t) = 0 \qquad \text{for} \quad i = 1,2,3,4,5,$$

$$\overline{H}^k(t) = D_t H^k(t), \qquad k = 1,2,\ldots,6.$$

$$S_{k,j}(t,s_1) = \int_0^t D_t^{3-i}\Big[(t-s)^{(1/2)+3-j}(p_m(t)-p_n(s))^{j-1} \times \times \exp(B(t,s)(p_m(t)-p_n(s))^2)\Big]\Bigg|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du,$$

where

$m=1, \quad n=1 \quad$ for $\quad i,j=1,2,3,$

$m=1, \quad n=2 \quad$ for $\quad i=1,2,3, \quad j=4,5,6,$

$m=2, \quad n=1 \quad$ for $\quad i=4,5,6, \quad j=1,2,3,$

$m=2, \quad n=2 \quad$ for $\quad i,j=4,5,6.$

Proof. For $I_{1,1}$, we have

$$I_{1,1} = w_1(p_1(t),t) = A\int_0^t q_1(s)(t-s)^{3/2}\exp(B(t,s)(p_1(t)-p_1(s))^2)ds.$$

Differentiating $I_{1,1}(t)$ we obtain

$$D_t^2 I_{1,1}(t) = A\int_0^t (t-s)^{-1/2}\left\{\frac{3}{4}+\left[-\frac{1}{4}(-(t-s)^{-2}(p_1(t)-p_1(s))^2 +\right.\right.$$

$$\left.+2(t-s)^{-1}(p_1(t)-p_1(s))p_1'(t))\right] + (t-s)\left[-\frac{1}{4}\left[2(t-s)^{-3}(p_1(t)-p_1(s))^2 -\right.\right.$$

$$-2(t-s)^{-2}(p_1(t)-p_1(s))p_1'(t) - 2(t-s)^{-2}(p_1(t)-$$

$$-p_1(s))p_1'(t) - 2(t-s)^{-1}(p_1'(t))^2 + 2(t-s)^{-1}(p_1(t)-p_1(s))p_1''(t)+$$

$$\left.\left.\left.+\frac{1}{4}(t-s)^{-2}(p_1(t)-p_1(s))^2 + 2(t-s)^{-1}(p_1(t)-p_1(s))(p_1'(t))\right]^2\right]\right\}\times$$

$$\times \exp(B(t,s)(p_1(t)-p_1(s))^2)q_1(s)ds.$$

Applying to the $D_t^2 I_{1,1}(t)$ the Abel transformation, [14], Vol. I, P.16, we obtain the equivalent

$$A\int_0^t (t-s)^{-1/2}\int_0^s (s-s_1)^{-1/2}\left\{\frac{3}{4}+\left[-\frac{1}{4}(-(s-s_1)^{-2}(p_1(s)-\right.\right.$$

$$\left.-p_1(s_1))^2 + 2(s-s_1)^{-1}(p_1(s)-p_1(s_1))p_1'(s))\right] +$$

$$+(s-s_1)\left[-\frac{1}{4}\left[2(s-s_1)^{-3}(p_1(s)-p_1(s_1))^2 -\right.\right.$$

$$-2(s-s_1)^{-2}(p_1(s)-p_1(s_1))p_1'(s) - 2(s-s_1)^{-2}(p_1(s)-p_1(s_1))p_1'(s)-$$

$$-2(s-s_1)^{-1}(p_1'(s))^2 + 2(s-s_1)^{-1}(p_1(s)-p_1(s_1))p_1''(s)+$$

$$\left.\left.\left.+\frac{1}{4}(s-s_1)^{-2}(p_1(s)-p_1(s_1))^2 + 2(s-s_1)^{-1}(p_1(s)-p_1(s_1))(p_1'(s))\right]^2\right]\right\}\times$$

$$\times \exp(B(s,s_1)(p_1(s)-p_1(s_1))^2)q_1(s_1)ds_1.$$

Interchanging the order of the integration in the last formula and applying the change variable of the integral variable $s = s_1 + (t-s_1)u$, $ds = (t-s_1)du$, $u \in (0,1)$, we obtain

$$A\int_0^t \left\{ \int_0^s \frac{1}{(1-u)^{1/2}u^{1/2}} \left\{ \frac{3}{4} + \left[\frac{3}{4} + \left[-\frac{1}{4}((t-s_1)u)^{-2}(p_1(s_1+(t-s_1)u)-p_1(s_1))^2 + \right.\right.\right.\right.$$

$$+2((t-s_1)u)^{-1}(p_1(s_1+(t-s_1)u)-p_1(s_1))(p_1'(s_1+(t-s_1)u))+$$

$$+((t-s_1)u)\left[-\frac{1}{4}\left[2((t-s_1)u)^{-3}(p_1(s_1+(t-s_1)u)-p_1(s_1))^2 - \right.\right.$$

$$-2((t-s_1)u)^{-2}(p_1(s_1+(t-s_1)u)-p_1(s_1))(p_1'(s_1+(t-s_1)u))-$$

$$-2((t-s_1)u)^{-2}(p_1(s_1+(t-s_1)u)-p_1(s_1))(p_1'(s_1+(t-s_1)u))+$$

$$+2((t-s_1)u)^{-1}(p_1'(s_1+(t-s_1)u))^2 +$$

$$+2((t-s_1)u)^{-1}(p_1(s_1+(t-s_1)u)-p_1(s_1))(p_1''(s_1+(t-s_1)u))+$$

$$+\left[\frac{1}{4}((t-s_1)u)^{-2}(p_1(s_1+(t-s_1))p_1(s_1))^2 + \right.$$

$$\left.\left.\left. +2((t-s_1)u)^{-1}(p_1(s_1+(t-s_1)u)-p_1(s_1))(p_1'(s_1+(t-s_1)u))\right]^2 \right]\right\} \times$$

$$\left. \times \exp((-4(t-s_1)u)^{-1}(p_1(s_1+(t-s_1)u)-p_1(s_1))^2)du \right\} q_1(s_1)ds_1 .$$

Applying the mean value theorem and differentiating with respect to t, we obtain

$$q_1(t)\int_0^s \frac{1}{(1-u)^{1/2}u^{1/2}} \left\{ \frac{3}{4}\left[-\frac{1}{4}(t-s_1)^{-2}((t-s_1)^2u^2(p_1'(s_1+\Theta(t-s_1)))^2) \right] + \right.$$

$$+2(t-s_1)^{-1}u^{-1}(t-s_1)u(p_1'(s_1+\Theta(t-s_1)))p_1'(s_1+(t-s_1)u)+$$

$$+(t-s_1)u(t-s_1)^{-3}u^{-3}((t-s_1)^2u^2(p_1'(s_1+\Theta(t-s_1)))^2)-$$

$$-2(t-s_1)^{-2}u^{-2}(t-s_1)up_1'(s_1+\Theta(t-s_1))p_1'(s_1+(t-s_1)u)+$$

$$+2(t-s_1)^{-1}u^{-1}(p_1'(s_1+(t-s_1)u))^2 +$$

$$+2(t-s_1)^{-1}u^{-1}(t-s_1)up_1'(s_1+\Theta(t-s_1))p_1''(s_1+(t-s_1)u)+$$

$$+\left[\frac{1}{4}(t-s_1)^{-2}u^{-2}((t-s_1)up_1'(s_1+\Theta(t-s_1)))^2 + \right.$$

$$+2(t-s_1)^{-1}u^{-1}((t-s_1)up_1'(s_1+\Theta(t-s_1)))p_1'(s_1+(t-s_1)u)\Big]^2\Big\}\times$$

$$\times\exp((-4(t-s_1)u)^{-1}((t-s_1)up_1'(s_1+\Theta(t-s_1)))^2)\Big|_{s_1=t}\,du+$$

$$+\int_0^t D_tS_{1,1}(t,s_1)q_1(s_1)ds_1,$$

where

$$S_{1,1}(t,s_1)=$$

$$=\int_0^t D_t^2\Big[(t-s)^{3/2}\exp(B(t,s)(p_1(t)-p_1(t))^2)\Big]\Big|_{\substack{s:=s_1\\ t:=s_1+(t-s_1)u}}du.$$

By last formula we obtain

$$q_1(t)\frac{3}{4}\beta\left(\frac{1}{2},\frac{1}{2}\right)(1+(p_1'(t))^2)+\int_0^t D_tS_{1,1}(t,s_1)q_1(s_1)ds_1,$$

where β denote beta Euler function.

Denoting $A_{11}(t)=\frac{3}{4}\beta\left(\frac{1}{2},\frac{1}{2}\right)(1+(p_1'(t))^2)$, we obtain

$$A_{11}(t)q_1(t)+\int_0^t D_tS_{1,1}(t,s_1)q_1(s_1)ds_1.$$

Similarly for $I_{2,i}(t)=w_i(p_1(t),t)$, $i=2,3$, we obtain

$$A_{12}(t)q_2(t)+\int_0^t D_tS_{1,2}(t,s_1)q_2(s_1)ds_1,$$

where

$$A_{1,2}(t) = \frac{1}{2} A\beta\left(\frac{1}{2},\frac{1}{2}\right)(p_1'(t))^2,$$

$$S_{1,2}(t,s_1) = \int_0^t D_t^2\left[(t-s)^{1/2} \exp(B(t,s)(p_1(t)-p_1(s))^2)\right]\Bigg|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du$$

and

$$A_{13}(t)q_3(t) + \int_0^t D_t S_{1,3}(t,s_1)ds_1,$$

where

$$A_{1,3}(t) = A\beta\left(\frac{1}{2},\frac{1}{2}\right)\left(-4p_1'(t)p_1''(t) - \frac{7}{4}(p_1'(t))^2\right),$$

$$S_{1,3}(t,s_1) =$$

$$= \int_0^t D_t^2\left[(t-s)^{-1/2} \exp(B(t,s)(p_1(t)p_1(t)-p_1(s))^2)\right]\Bigg|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du$$

For $I_{1,4}(t) = w_4(p_1(t),t)$, we have

$$D_t^2 I_{1,4}(t) = A\int_0^t (t-s)^{-1/2}\left\{\left[\frac{3}{4} + \left[-\frac{1}{4}(t-s)^{-2}(p_1(t)-p_2(s))^2 + \right.\right.\right.$$

$$\left.+(t-s)^{-1}(p_1(t)-p_2(s))p_1'(t)\right] + (t-s)\left[-\frac{1}{4}(2(t-s)^{-3}(p_1(t)-p_2(s))^2 - \right.$$

$$-2(t-s)^{-2}(p_1(t)-p_2(s))p_1'(t) - 2(t-s)^{-2}(p_1(t)-p_2(s))p_1'(t) +$$

$$+2(t-s)^{-1}p_1'(t))^2 + 2(t-s)^{-1}(p_1(t)-p_2(s))p_1''(t) +$$

$$+\left[\frac{1}{4}(t-s)^{-2}(p_1(t)-p_2(s))^2+2(t-s)^{-1}(p_1(t)-p_2(s))(p_1'(t))\right]^2\Bigg]\Bigg]\times$$

$$\times\exp\left(B(t,s)\left(\frac{(p_1(t)-p_2(s))^2}{4(t-s)}\right)\right)q_4(s)\Bigg\}ds.$$

Applying to the integral $D_t^2 I_{1,4}(t)$ the Abel transformation, interchanging the order of the integration and applying the change of the integral variable $s=s_1+(t-s_1)u$, $ds=(t-s_1)du$, $u\in(0,1)$, similarly to $I_{1,1}(t)$, we obtain

$$A\int_0^t\Bigg\{\int_0^1\frac{1}{(1-u)^{1/2}u^{1/2}}\Bigg\{\frac{3}{4}+\Bigg[-\frac{1}{4}\Big[(t-s)^{-2}u^{-2}(p_1(s_1+(t-s_1)u)-p_2(s_1))^2+$$

$$+2(t-s_1)^{-1}u^{-1}(p_1(s_1+(t-s_1)u)-p_2(s_1))p_1'(s_1+(t-s_1)u)\Big]+$$

$$+(t-s_1)u\Bigg[-\frac{1}{4}\Big[2(t-s_1)^{-3}u^{-3}(p_1((s_1)+(t-s_1)u)-p_2(s_1))^2-$$

$$-2(t-s_1)^{-2}u^{-2}(p_1(s_1+(t-s_1)u)-p_2(s_1))p_1'(s_1+(t-s_1)u)-$$

$$-2(t-s_1)^{-2}u^{-2}\big(p_1\big(s_1+(t-s_1)u\big)-p_2\big(s_1\big)\big)p_1'\big(s_1+(t-s_1)u\big)+$$

$$+2(t-s_1)^{-1}u^{-1}(p_1'(s_1+(t-s_1)u))^2+2(t-s_1)^{-1}u^{-1}\times$$

$$\times(p_1(s_1+(t-s_1)u)-p_2(s_1))p_1''(s_1+(t-s_1)u)+$$

$$+\frac{1}{4}(t-s_1)^{-2}u^{-2}(p_1(s_1+(t-s_1))p_2(s_2(s_1)))^2+$$

$$+2(t-s_1)^{-1}u^{-1}(p_1(s_1+(t-s_1)u)-p_2(s_1))p_1'(s_1+(t-s_1)u)\Big]^2\Bigg]\Bigg\}\times$$

$$\times\exp\frac{(p_1(s_1+(t-s_1)u)-p_2(s_1))^2}{4(t-s_1)u}du\Bigg\}q_4(s_1)ds_1.$$

Differentiating the last formula with respect to *t*, by Lemma 4, we obtain

$$\int_0^t S_{1,4}(t,s_1)q_4(s_1)ds_1 ,$$

where

$S_{1,4}(t,s_1) =$

$$= \int_0^1 D_t^2\left[(t-s)^{3/2}\exp(B(t,s)(p_1(t)-p_2(s))^2)\right]\Bigg|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du.$$

Similarly to $I_{1,4}(t)$ for $I_{1,i}(t)$, $i=5,6$, we obtain respectively

$$\int_0^t S_{1,5}(t,s_1)q_5(s_1)ds_1 ,$$

where

$S_{1,5}(t,s_1) =$

$$= \int_0^1 D_t^2\left[(t-s)^{1/2}\exp(B(t,s)(p_1(t)-p_2(s))^2)\right]\Bigg|_{\substack{s:=s_1 \\ t:=s_1+(t-s_1)u}} du$$

and

$$\int_0^t S_{1,6}(t,s_1)q_6(s_1)ds_1$$

where

$$S_{1,6}(t,s_1)=$$

$$=\int_0^1 D_t^2\left[(t-s)^{-1/2}\exp(B(t,s)(p_1(t)-p_2(s))^2)\right]\Bigg|_{\substack{s:=s_1\\ t:=s_1+(t-s_1)u}} du.$$

Differentiating $H^1(t)$ 2-times, substituting t by s, s by s_1, multiplying by $(t-s)^{-1/2}$ and integrating in the interval $[0,t]$, we obtain

$$\int_0^t \frac{1}{(t-s_1)^{1/2}} D_{s_1}^2 H^1(s_1)ds_1 .$$

Integrating the last formula by parts, we obtain

$$2(t-s)^{1/2} D_{s_1}^2 H^1(s_1)\Big|_0^t - 2\int_0^t (t-s_1)^{1/2} D_{s_1}^2 H^1(s_1)ds_1 .$$

Differentiating the last formula with respect to t, we obtain

$$-2\int_0^t (t-s_1)^{1/2} D_{s_1}^3 H^1(s_1)ds_1$$

because by assumptions is known that $D_s^2 H^1(0)=0$.

Denote by

$$\overline{H^1(t)} = -2\int_0^t (t-s_1)^{1/2} D_{s_1}^3 H^1(s_1)ds_1$$

Consequently by (I_1a) we obtain the equivalent equation (I_1b) in the following form

$$A_{11}(t)q_1(t)+A_{12}(t)q_2(t)+$$

$$+A_{13}(t)q_3(t)+\sum_{j=1}^{6}\int_0^t D_t S_{1,j}(t,s_1)q_j(s_1)ds_1 = \overline{H^1(t)} .$$

Similarly by the equation (I_4a), we obtain equivalent equation (I_4b).

Similarly differentiating the equations (I_ja), (I_{3+j}a), $j = 2,3$, 3-j-times, substituting t by s and s by s_1, multiplying both sides by $(t-s)^{-1/2}$ and integrating in the interval $[0,t]$, interchanging the order of the integration, applying the change variable $s = s_1 + (t-s_1)u$, $u \in (0,1)$ in the integrals, applying the mean value theorem $p_i(s_1 + (t-s_1)u) - p_i(s_1) = = (t-s_1)up_i'(s_1 + \Theta(t-s_1))$, $\Theta \in (0,1)$ or applying Lemma 4 and differentiating with respect to *t*, finally we obtain the equivalent equations (I_ja), (I_{3+j}a), $j = 2,3$.

Thus the proof of Lemma 5 is Finished.

10. Solution of system (I_ib), *i*=1,2,...,6

Lemma 6. If the assumptions of the Lemma 5 are satisfied, then the solution of system (I_ib), $i = 1,2,..,6$ *is of the form*

(I_ic) $$q_i(t) = F^i(t) + \sum_{j=1}^{6}\int_0^t q_j(s)N_{ij}(t,s)ds, \quad i = 1,2,\ldots,6,$$

where

$$F^1(t) = (A_{11}(t))^{-1}\left[\overline{H^1(t)} - A_{12}(t)F^2(t) - A_{13}(t)F^3(t)\right],$$

$$F^2(t) = (A_{22}(t))^{-1}\left[\overline{H^1(t)} - A_{23}(t)F^3(t)\right],$$

$$F^3(t) = (A_{33}(t))^{-1}\overline{H^3(t)},$$

$$F^4(t) = (A_{44}(t))^{-1}\left[\overline{H^4(t)} - A_{45}(t)F^5(t) - A_{46}(t)F^6(t)\right],$$

$$F^5(t) = (A_{55}(t))^{-1}\left[\overline{H^5(t)} - A_{56}(t)F^6(t)\right],$$

$$F^6(t) = (A_{66}(t))^{-1}\overline{H^6(t)},$$

$$N_{1j}(t,s) = (A_{11}(t))^{-1}[A_{12}(t)N_{2j}(t,s) + A_{13}(t)N_{3j}(t,s) + D_tS_{1j}(t,s)],$$

$$j = 1,2,\ldots,6,$$

$$N_{2j}(t,s) = (A_{22}(t))^{-1}[N_{3j}(t,s) - D_tS_{2j}(t,s)], \qquad j = 1,2,\ldots,6,$$

$$N_{3j}(t,s) = (A_{33}(t))^{-1} D_t S_{3j}(t,s), \qquad j = 1,2,\dots,6,$$

$$N_{4j}(t,s) = (A_{44}(t))^{-1}[A_{45}(t) N_{3j}(t,s) + A_{46}(t) N_{6j}(t,s) + D_t S_{4j}(t,s)], \qquad j = 1,2,\dots,6,$$

$$N_{5j}(t,s) = (A_{55}(t))^{-1}[N_{6j}(t,s) - D_t S_{5j}(t,s)], \qquad j = 1,2,\dots,6,$$

$$N_{6j}(t,s) = (A_{66}(t))^{-1} D_t S_{6j}(t,s), \qquad j = 1,2,\dots,6.$$

Proof. Applying the method of the successive elimination of the unknown functions q_i, $i = 1,2,\dots,6$, we shall determine successively the densities q_i.

By the equation $(I_3 b)$, we obtain

$$q_3(t) = F^3(t) + \sum_{j=1}^{6} \int_0^t q_j(s) N_{3j}(t,s) ds,$$

where $F^3(t)$ and $N_{3j}(t,s)$ are defined above.

By the equation $(I_2 b)$, we obtain

$$q_2(t) = F^2(t) + \sum_{j=1}^{6} \int_0^t q_j(s) N_{2j}(t,s) ds,$$

where $F^2(t)$ and $N_{2j}(t,s)$ are defined above.

By the equation $(I_1 b)$, we obtain

$$q_1(t) = F^1(t) + \sum_{j=1}^{6} \int_0^t q_j(s) N_{1j}(t,s) ds,$$

where $F^1(t)$ and $N_{1j}(t,s)$ are defined above.

Similarly by the equations $(I_6 b)$, $(I_5 b)$ and $(I_4 b)$ we obtain the equations $(I_6 c)$, $(I_5 c)$ and $(I_4 c)$ respectively.

11. Solution of the system $(I_i c)$, $i = 1,2,\dots,6$

Let us consider the system with parameter λ compatible to the system $(I_i c)$ of the form

$$(\mathrm{I}_i\lambda) \qquad q_i(t) = F^i(t) + \lambda \sum_{j=1}^{6} \int_0^t q_i(s) N_{ij}(t,s) ds, \quad i = 1,2,\ldots,6.$$

Let

$$N_{ij}^1(t,s) = \sum_{k=1}^{6} \int_s^t N_{ik}(t,s_1) N_{kj}(s_1,s) ds_1 ,$$

$$N_{ij}^{n+1}(t,s) = \sum_{k=1}^{6} \int_s^t N_{ik}(t,s_1) N_{kj}^n(s_1,s) ds_1 , \qquad n = 1,2,\ldots, \qquad i,j = 1,2,\ldots,6,$$

$$R_{ij}(t,s,\lambda) = N_{ij}(t,s) + \lambda N_{ij}^1(t,s) + \ldots \lambda^n N_{ij}^n(t,s) + \ldots,$$

$$n = 1,2,\ldots, \qquad i,j = 1,2,\ldots,6$$

By [14], Vol. I, p. 4 or by [5], p. 97, we obtain

Lemma 7. If the functions $F^i \in C^{3-j}([0,T])$, $H^i \in C^3([0,T])$ *and* $D_t^{2-j} H^i(0) = 0$, *where* $i = 1,2,\ldots,6$ *and* $j = 0$ *for* $k = 1,4$, $j = 1$ *for* $k = 2,5$, $j = 2$ *for* $k = 3,6$, *then functions*

$$q_i(t,\lambda) = F^i(t) + \lambda \sum_{j=1}^{6} \int_0^t R_{ij}(t,s,\lambda) F^j(s) ds, \quad i = 1,2,\ldots,6,$$

are the solutions of class $C([0,T])$ *for every* $\lambda \in \mathfrak{R}$ *of the system* $(\mathrm{I}_i\lambda)$ *and the functions*

$$q_i(t) = q_i(t,1), \quad i = 1,2,\ldots,6,$$

are the unique solutions of class $C([0,T])$ *of the system* $(\mathrm{I}_i\mathrm{c})$.

12. Fundamental theorem

By Lemmas 1-7, we obtain the following result.

Theorem. If the assumptions of Lemmas 1-7 *are satisfied, then the functions*

$$w(x,t) = \sum_{i=1}^{7} w_i(x,t)$$

is the solution of the problem (1a) - (4a) *and the function*

$$u(x,t) = w(x,t) + r(x,t)$$

is the solutions of the (1) - (4) *problem.*

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