FASCICULI MATHEMATICI
Nr 26 1996

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ON GRONWALL TYPE INEQUALITIES OCCURING IN THE THEORY OF DIFFERENTIAL EQUATIONS

In this paper we present some new two independent variable generalizations of certain Gronwall type integral inequalities occuring in the theory of differential equations. The inequalities obtained here can be used as handy tools in the analysis of certain new classes of partial differential and integral equations involving two independent variables.

Key words: Integral inequalities, theory of differential equations, integral equations, bounds on the solutions.

1. Introduction

The fundamental role played by the following Gronwall type integral inequality in the development of the theory of differential equations is wll know (see [1,3-7]).

Theorem A. Let u and f be real-valued nonnegative continuous functions defined for $t \geq 0$. If

$$u^2(t) \leq c^2 + 2\int_0^t f(s)u(s)ds,$$

for all $t \geq 0$, where $c \geq 0$ is a constant, then

$$u(t) \leq c + \int_0^t f(s)ds,$$

for all $t \geq 0$.

As far as we know, the above inequality was first given by L. Ou-Iang in [7]. In 1979 C.M.Dafermos [3] used the following variant of the above inequality to establish a different connection between stability and second law of thermodynamics.

Theorem B. Assume that the nonnegative function $u(t) \in L^{\infty}[0,s]$ *and* $g(t) \in L^{1}[0,s]$ *satisfy the condition*

$$u^2(t) \leq M^2 u^2(0) + \int_o^t [2\alpha u^2(\sigma) + 2Ng(\sigma)u(\sigma)]d\sigma, \ t \in [0,s], \quad (*)$$

where α, M, N *are nonnegative constants, Then*

$$u(s) \leq Me^{\alpha s} u(0) + Ne^{\alpha s} \int_o^s g(\delta)d\delta .$$

An interesting feature of the inequalities given in Theorems A and B lies in their fruitful utilizations to the situations for which the other available inequalities do not apply directly. For some recent results on such inequalities and their applications, we refer the interested readers to [10, 11]. The upper bound on the inequality $(*)$ written by putting $u^2(t) = z(t)$ in the form:

$$z(t) \leq M^2 z(0) + \int_o^t [2\alpha z(\sigma) + 2Ng(\sigma) z^{\frac{1}{2}}(\sigma)]d\sigma ,$$

was obtained earlier by Willet and Wong [12], see also [2].

In view of the important role played by the above inequalities in Theorems A and B in the theory of differential and integral equations (see[1, 3-7, 10, 11], it is natural to look for some new generalizations of the inequalities in Theorems A and B which would be equally important in certain new applications. The main object of this paper is to establish some new two independent variable generalizations of the inequalities in Theorems A and B which can be used as handy tools in the analysis of certain new classes of partial differential and integral equations in two independent variables for which the inequalities available in the literature do not apply. We also present an immediate application to convey the importance of our results to the literature.

2. Statement of results

We first give some notations and definitions which will be used in our subsequent discussion. Let $\Re$ denote the set of real numbers and $\Re_+ = [0,\infty)$ be a subset of $\Re$. Let $m,n \geq 1$ be integers. We define the differential operators D_1^n and D_2^m by

$$D_1^n z(x,y) = \frac{\partial^n z(x,y)}{\partial x^n}, \quad D_2^m z(x,y) = \frac{\partial^m z(x,y)}{\partial y^m},$$

where *z(x,y)* is some function defined for $x,y \in \Re_+$. For $x,y \in \Re_+$ and some function *q(x,y)* defined for $x,y \in \Re_+$, we set

$$B[x,y,q(s,t]=\int_o^x\int_o^{s_{n-1}}\dots\int_o^{s_1}\int_o^y\int_o^{t_{m-1}}\dots\int_o^{t_1} q(s,t)dtdt_1\dots dt_{m-1}\,dsds_1\dots ds_{n-1},$$

with $s_o = x$ and $t_o = y$.

Our main results are given in the following theorems.

Theorem 1. Let f(x,y) ≥ 0, g(x,y) ≥ 0 be real-valued continuous functions defined for $x,y \in \Re_+$ *and c be a nonnegative real constant. Let u(x,y) ≥ 0 be* a *real-valued continuous function defined for* $x,y \in \Re_+$. *If*

(2.1) $$u^2(x,y) \leq c^2 + 2B[x,y,f(s,t)u^2(s,t)+g(s,t)u(s,t)]$$

for $x,y \in \Re_+$, *then*

(2.2) $$u(x,y) \leq p(x,y)\exp(B[x,y,f(s,t)]),$$

for $x,y \in \Re_+$, *where*

(2.3) $$p(x,y) = c + B[x,y,g(s,t)],$$

for $x,y \in \Re_+$.

Theorem 2. Let f(x,y), g(x,y) and c be as defined in Theorem 1. Let $u(x,y) \geq u_o \geq 0$ *be a real-valued continuous function defined for* $x, y \in \Re_+ ; u_o$ *is a real constant. Let W(u) be a continuous nondecreasing real-valued function defined on an interval* $I = [u_o, \infty)$ *and* $W(u) > 0$ *on* (u_o, ∞), $W(u_o) = 0$. *If*

(2.4) $u^2(x,y) \leq c^2 + 2B[x,y,f(s,t)u(s,t)W(u(s,t)) + g(s,t)u(s,t)]$

for $x, y \in \Re_+$, then for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$, $x, y, x_1, y_1 \in R_1$,

(2.5) $$u(x,y) \leq \Omega^{-1}[\Omega(p(x,y)) + B[x,y,f(s,t)]],$$

where $p(x,y)$ is as defined in (2.3) and

(2.6) $$\Omega(r) = \int_{r_o}^{r} \frac{ds}{W(s)} \quad , \quad r \geq r_o \text{ with } r_o > u_o \quad ,$$

Ω^{-1} is the inverse of Ω and x_1, $y_1 \in \Re_+$ be chosen so that

$$\Omega(p(x,y)) + B[x,y,f(s,t)] \in \text{Dom}(\Omega^{-1}),$$

for all $x, y \in \Re_+$ such that $0 \leq x \leq x_1$, $0 \leq y \leq y_1$.

Theorem 3. Let f(x,y), g(x,y) and c be as defined in Theorem 1. Let $u(x,y) \geq 0$ *be a real-valued continuous function defined for* $x, y \in \Re_+$ *and* $L : \Re_+^3 \to \Re_+$ *be a continuous function satysfying the condition*

(2.7) $$0 \leq L(x,y,v) - L(x,y,w) \leq k(x,y,w)(v-w),$$

for $x, y \in \Re_+$ *and* $v \geq w \geq 0$, *where* $k : \Re_+^3 \to \Re_+$ *is a continuous function. If*

(2.8) $$u^2(x,y) \leq \\ \leq c^2 + 2B[x,y,f(s,t)u(s,t)L(s,t,u(s,t) + g(s,t)u(s,t)],$$

for $x, y \in \Re_+$, *then*

(2.9) $u(x,y) \leq p(x,y) + q(x,y)\exp(B[x,y,f(s,t)k(s,t,p(s,t)]),$

for $x, y \in \Re_+$, *where p(x,y) is defined by* (2.3) *and*

(2.10) $$q(x,y)=B[x,y,f(s,t)L(s,t,p(s,t))],$$

for $x,y\in\Re_+$.

3. Proofs of theorems 1-3

Assume that $c>0$ and define a function *z(x,y)* by

(3.1) $$z(x,y)=c^2+2B[x,y,f(s,t)u^2(s,t)+g(s,t)u(s,t)].$$

From (3.1) and usig the fact that $u(x,y)\le\sqrt{z(x,y)}$, it is easy to observe that

(3.2) $$D_2^m D_1^n z(x,y)\le 2\sqrt{z(x,y)}[f(x,y)\sqrt{z(x,y)}+g(x,y)].$$

From (3.2) and using the facts that $z(x,y)>0, D_2z(x,y)\ge 0$, $D_2^{m-1}D_1^n\ z(x,y)\ge 0$, for $x,y\in R_+$, we observe that

$$\frac{D_2^m D_1^n z(x,y)}{\sqrt{z(x,y)}}\le 2[f(x,y)\sqrt{z(x,y)}+g(x,y)]+$$

$$+\frac{1}{2}\frac{(D_2z(x,y))(D_2^{m-1}D_1^n z(x,y))}{[z(x,y)]^{\frac{3}{2}}}$$

i.e.

(3.3) $$D_2\left[\frac{D_2^{m-1}D_1^n z(x,y)}{\sqrt{z(x,y)}}\right]\le 2[f(x,y)\sqrt{z(x,y)}+g(x,y)].$$

By keeping x fixed in (3.3), we set $y=t$ and then integrating with respect to t from o to y and using the fact that $D_2^{m-1}D_1^n z(x,0)=0$, we have

(3.4) $$\frac{D_2^{m-1}D_1^n z(x,y)}{\sqrt{z(x,y)}}\le 2\int_o^y [f(x,t)\sqrt{z(x,t)}+g(x,t)]dt.$$

Again as above, from (3.4) and using the facts that $z(x,y)>0$, $D_2z(x,y)\ge 0$, $D_2^{m-2}D_1^n z(x,y)\ge 0$ for $x,y\in R_+$, we observe that

(3.5) $$D_2\left[\frac{D_2^{m-2}D_1^n z(x,y)}{\sqrt{z(x,y)}}\right]\le 2\int_o^y[f(x,t)\sqrt{z(x,t)}+g(x,t)]dt.$$

By keeping x fixed in (3.5), set $y = t_1$ and then integrating with respect to t_1 from o to y and using the fact that $D_2^{m-2} D_1^n z(x,0) = 0$, we have

$$\frac{D_2^{m-2} D_1^n z(x,y)}{\sqrt{z(x,y)}} \le 2\int_o^y \int_o^{t_1} [f(x,t)\sqrt{z(x,t)} + g(x,t)] dt dt_1 .$$

Continuing in this way we obtain

$$\text{(3.6)} \qquad \frac{D_1^n z(x,y)}{\sqrt{z(x,y)}} \le 2\int_o^y \int_o^{t_{m-1}} \dots \int_o^{t_1} [f(x,t)\sqrt{z(x,t)} + g(x,t)] dt dt_1 \dots dt_{m-1} .$$

From (3.6) and using the facts that $z(x,y) > 0$, $D_1 z(x,y) \ge 0$, $D_1^{n-1} z(x,y) \ge 0$, for $x, y \in \Re_+$,we observe that

$$\frac{D_1^n z(x,y)}{\sqrt{z(x,y)}} \le 2\int_o^y \int_o^{t_{m-1}} \dots \int_o^{t_1} [f(x,t)\sqrt{z(x,t)} + g(x,t)] dt dt_1 \dots dt_{m-1} + $$

$$+ \frac{1}{2} \frac{(D_1 z(x,y))(D_1^{n-1} z(x,y))}{[z(x,y)]^{\frac{3}{2}}},$$

i.e.

$$\text{(3.7)} \qquad D_1\left[\frac{D_1^{n-1} z(x,y)}{\sqrt{z(x,y)}}\right] \le$$

$$\le 2\int_o^y \int_o^{t_{m-1}} \dots \int_o^{t_1} [f(x,t)\sqrt{z(x,t)} + g(x,t)] dt dt_1 \dots dt_{n-1}$$

Now keeping y fixed in (3.7), set $x = s$ and then integrating with respect to s from o to x using the fact that $D_1^{n-1} z(0,y) = 0$, we have

$$\frac{D_1^{n-1} z(x,y)}{\sqrt{z(x,y)}} \le$$

$$\le 2\int_o^x \int_o^y \int_o^{t_{m-1}} \dots \int_o^{t_1} [f(s,t)\sqrt{z(s,t)} + g(s,t)] . dt dt_1 \dots dt_{m-1} ds .$$

Continuing in this way we obtain

$$(3.8)\qquad \frac{D_1 z(x,y)}{\sqrt{z(x,y)}} \le 2\int_o^x \int_o^{s_{n-2}} \dots \int_o^{s_1} \int_o^y \int_o^{t_{m-1}} \dots \int_o^{t_1} [f(s,t)\sqrt{z(s,t)} +$$

$$+g(s,t)]dtdt_1 \dots dt_{m-1} ds ds_1 \dots ds_{n-2}\ .$$

Now keeping y fixed in (3.8), set $x = s_{n-1}$ and then integrating with respect to s_{n-1} from o to x and using the fact that $\sqrt{z(0,y)} = c$ we have

$$(3.9)\qquad \sqrt{z(x,y)} \le p(x,y) + B[x,y,f(s,t)\sqrt{z(s,t)}].$$

Since $p(x,y)$ is positive and monotone nondecreasing in x and y, from (3.9) we observe that

$$(3.10)\qquad \frac{\sqrt{z(x,y)}}{p(x,y)} \le 1 + B\left[x,y,f(s,t)\frac{\sqrt{z(s,t)}}{p(s,t)}\right].$$

Define a function $v(x,y)$ by

$$(3.11)\qquad v(x,y) = 1 + B\left[x,y,f(s,t)\frac{\sqrt{z(s,t)}}{p(s,t)}\right].$$

From (3.11) and using the fact that $\dfrac{\sqrt{z(x,y)}}{p(x,y)} \le v(x,y)$, we observe that

$$(3.12)\qquad D_2^m D_1^n v(x,y) \le f(x,y)v(x,y).$$

From (3.12) and using the facts that $\mathrm{v(x,y)} > 0, \mathrm{D_2 v(x,y)} \ge 0$, $D_2^{m-1} D_1^n v(x,y) \ge 0$, for $x,y \in \Re_+$, we observe that

$$\frac{D_2^m D_1^n v(x,y)}{v(x,y)} \le f(x,y) + \frac{(D_2 v(x,y))(D_2^{m-1} D_1^n v(x,y))}{v^2(x,y)},$$

i.e.

$$(3.13)\qquad D_2\left[\frac{D_2^{m-1} D_1^n v(x,y)}{v(x,y)}\right] \le f(x,y).$$

Now by following the same steps, below (3.3) upto (3.8), we have

$$\frac{D_1 v(x,y)}{v(x,y)} \le \tag{3.14}$$
$$\le \int_0^x \int_0^{s_{n-2}} \dots \int_0^{s_1} \int_0^y \int_0^{t_{m-1}} \dots \int_0^{t_1} f(s,t) dt dt_1 \dots dt_{m-1} ds ds_1 \dots ds_{n-2} .$$

Now keeping y fixed in (3.14), set $x = s_{n-1}$ and then integrating with respect to s_{n-1} from o to x and using the fact that $v(0,y) = 1$, we have

$$\operatorname{Log} v(x,y) \le B[x,y,f(s,t)],$$

which implies

$$v(x,y) \le \exp(B[x,y,f(s,t)]) . \tag{3.15}$$

Using (3.15) in (3.10) and the fact that $u(x,y) \le \sqrt{z(x,y)}$, we get the required inequality in (2.2).

If c is nonnegative, we carry out the above procedure with $c+\varepsilon$ instead of c, where $\varepsilon > 0$ is an arbitrary small constant, and subsequantly pass to the limit as $\varepsilon \to 0$ to obtain (2.2). The proof of Theorem 1 is complete.

Assume that $c > 0$ and define a function $z(x,y)$ by

$$z(x,y) = c^2 + 2B[x,y,f(s,t)u(s,t)W(u(s,t)) + \tag{3.16}$$
$$+ g(s,t)u(s,t)]$$

From (3.16) and using the fact that $u(x,y) \le \sqrt{z(x,y)}$, it is easy to observe that

$$D_2^m D_1^n z(x,y) \le 2\sqrt{z(x,y)}[f(x,y)W(\sqrt{z(x,y)}) + g(x,y)]. \tag{3.17}$$

Now by following the same steps an in the proof of Theorem 1, below (3.2) upto (3.9) we have

$$\sqrt{z(x,y)} \le p(x,y) + B[x,y,f(s,t)W(\sqrt{z(s,t)})]. \tag{3.18}$$

For an arbitrary $(X,Y) \in \Re_+ \times \Re_+$, it follows from (3.18) that

(3.19) $$\sqrt{z(x,y)} \le p(X,Y) + B[x,y,f(s,t)W(\sqrt{z(s,t)})],$$
$$0 \le x \le X, \qquad 0 \le y \le Y$$

Define

(3.20) $$v(x,y) = p(X,Y) + B[x,y,f(s,t)W(\sqrt{z(s,t)})]$$
$$0 \le x \le X, \qquad 0 \le y \le Y.$$

From (3.20) and using the fact that $\sqrt{z(x,y)} \le v(x,y)$, we observe that

(3.21) $$D_2^m D_1^n v(x,y) \le f(x,y)W(v(x,y)),$$
$$0 \le x \le X, \qquad 0 \le y \le Y.$$

Now by following the proof of Theorem 2 given in [q] (see, also[8]) with suitable modifications we get

(3.22) $$\Omega(v(X,Y)) \le \Omega(p(X,Y)) + B[X,Y,f(s,t).$$

Since *(X,Y)* is arbitrary, the inequality (3.22) holds for *(x,y)* = *(X,Y)* for all $(x,y) \in \Re_+ \times \Re_+$ and hence from (3.22) we have

(3.23) $$v(x,y) \le \Omega^{-1}[\Omega(p(x,y)) + B[x,y,f(s,t)]].$$

Using (3.23) in (3.19) and the fact that $u(x,y) \le \sqrt{z(x,y)}$, we get the required inequality in (2.5). The subdomain of $\Re_+ \times \Re_+$ for *(x,y)* is obvious.

The proof of the case when c is nonnegative can be completed as mentioned in the proof of Theorem 1. This completes the proof of Theorem 2.

Assume that $c > 0$ and define a function z(x,y) by

(3.24) $$z(x,y) = c^2 + 2B[x,y,f(s,t)u(s,t)L(s,t,u(s,t)) + $$
$$+ g(s,t)u(s,t)].$$

From (3.24) and using the fact that $u(x,y) \le \sqrt{z(x,y)}$, we observe that

$$(3.25)\qquad D_2^m D_1^n z(x,y) \le \le 2\sqrt{z(x,y)}[f(x,y)L(x,y,\sqrt{z(x,y)}) + g(x,y)].$$

Now by following the same steps as in the proof of Theorem 1, below (3.2) upto (3.9) we have

$$(3.26)\qquad \sqrt{z(x,y)} \le p(x,y) + B[x,y,f(s,t)L(s,t,\sqrt{z(s,t)})].$$

Define

$$(3.27)\qquad v(x,y) = B[x,y,f(s,t)L(s,t,\sqrt{z(s,t)})].$$

From (3.27) and using the fact that $\sqrt{z(x,y)} \le p(x,y) + v(x,y)$ and (2.7) we have

$$(3.28)\qquad \begin{aligned} D_2^m D_1^n v(x,y) &= f(x,y)L(x,y,\sqrt{z(x,y)}) \le \\ &\le f(x,y)L(x,y,p(x,y)+v(x,y)) = \\ &= f(x,y)[L(x,y,p(x,y)+v(x,y)) - L(x,y,p(x,y))] \\ &\qquad + f(x,y)L(x,y,p(x,y)) \le \\ &\le f(x,y)k(x,y,p(x,y))v(x,y) + f(x,y)L(x,y,p(x,y)). \end{aligned}$$

From (3.28) we can very easily obtain that

$$(3.29)\qquad v(x,y) \le q_\epsilon(x,y) + B[x,y,f(s,t)k(s,t))v(s,t)],$$

where $q_\varepsilon(x,y) = \varepsilon + q(x,y)$, in which q(x,y) is defined by (2.10) and $\varepsilon > 0$ an arbitrary small constant. Since $q_\varepsilon(x,y)$ is positive and monotone nondecreasing for $x, y \in \Re_+$, from (3.29) we observe that

$$(3.30)\qquad \frac{v(x,y)}{q_\varepsilon} \le 1 + B\left[x,y,f(s,t)k(s,t,p(s,t))\frac{v(s,t)}{q_\varepsilon(s,t)}\right].$$

The inequality (3.30) implies the estimate

$$(3.31)\qquad v(x,y) \le q_\varepsilon(x,y)\exp(B[x,y,f(s,t)k(s,t,p(s,t))]).$$

The desired inequality in (2.9) now follows by using (3.31) in (3.26) and then letting $\varepsilon \to 0$ in the resulting inequality and using the fact that $u(x,y) \le \sqrt{z(x,y)}$.

The proof of the case when c is nonnegative can be completed an mentioned in the proof of Theorem 1. This completes the proof of Theorem 3.

4. An application

In this section we indicate an application of our Theorem 1 to obtain a bound on the solution of certain differential equation involving two independent variables, for which the inequalities available in the literature do not apply directly. For example consider the following partial differential equation of the form

(4.1) $$D_2^m D_1^n u^2(x,y) = u(x,y)F(x,y,u(x,y)) + G(x,y,u(x,y)),$$

with the given initial conditions

(4.2) $$D_2^j u^2(x,0) = \alpha_j(x), \quad 0 \le j \le m-1,$$

(4.3) $$D_1^i u^2(0,y) = \beta_1(y), \quad 0 \le i \le n-1,$$

where $F \in C[\mathfrak{R}_+^2 \times \mathfrak{R}, \mathfrak{R}]$, $G \in C[\mathfrak{R}_+^2 \times \mathfrak{R}, \mathfrak{R}]$, $\alpha_j \in C^{(n)}[\mathfrak{R}_+, \mathfrak{R}]$, $\beta_i \in C^{(m)}[\mathfrak{R}_+, \mathfrak{R}]$ and

(4.4) $$\alpha_j^{(i)}(0) = \beta_i^{(j)}(0); \quad 0 \le j \le m-1, \quad 0 \le i \le n-1.$$

It is easy to observe that the problem (4.1) - (4.4) is equivalent to the integral equation

(4.5) $$u^2(x,y) = q(x,y) + B[x,y,u(s,t)F(s,t,u(s,t)) + G(s,t,u(s,t))],$$

where

$$q(x,y) = \sum_{i=1}^{n} \frac{x^{i-1}}{(i-1)!}\beta_{i-1}(y) + \sum_{j=1}^{m} \frac{y^{j-1}}{(j-1)!}\alpha_{j-1}(x)$$

$$-\sum_{i=1}^{n} \frac{x^{i-1}}{(i-1)!}\sum_{j=1}^{m} \frac{y^{j-1}}{(j-1)!}\alpha_{j-1}^{(i-1)}(0).$$

If *u(x,y)* is a solution of the problem (4.1) - (4.4), then it satosfies the equation (4.5) we assume that

(4.6) $\quad |q(x,y)| \leq c^2,$

(4.7) $\quad |F(x,y,u(x,y))| \leq 2f(x,y)|u(x,y)|,$

(4.8) $\quad |G(x,y,u(x,y))| \leq 2g(x,y)|u(x,y)|,$

where $f(x,y)$ and $g(x,y)$ are real-valued nonnegative continuous functions defined for $x,y \in \Re_+$ and c is a nonnegative real constant.

From (4.5) - (4.8) we observe that

(4.9) $\quad |u(x,y)|^2 \leq c^2 + 2B[x,y,f(s,t)|u(s,t)|^2 + g(s,t)|u(s,t)|].$

Now an application of Theorem 1 yields

(4.10) $\quad |u(x,y)| \leq p(x,y)\exp(B[x,y,f(s,t)]),$

where $p(x,y)$ is as defined in (2.3). The inequality (4.10) obtains the bound on the solution u(x,y) of (4.1) - (4.4) in terms of the known functions.

We note that, there are many possible applications of the inequalities given in Theorems 1 - 3 to certain new classes of partial differential and integral equations similar to that of given in [4]. Various other applications of these inequalities will appear elsewhere.

In concluding this paper, we note that the inequalities established in Theorems 1 - 3 can be extended very easily to r $(r \geq 3)$ independent variables. The precise formulations of these results is very close to that of the results given in Theorems 1 - 3 with suitable modifications. We leave it for the reader to fill in where needed.

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