FASCICULI MATHEMATICI
Nr 26 1996

LUCYNA REMPULSKA, MARIOLA SKORUPKA

ON APPROXIMATION OF FUNCTIONS BY SOME OPERATORS OF THE SZASZ – MIRAKJAN TYPE

Abstract. In this note we define some linear positive operators A_n and B_n of the Szasz - Mirakjan type in the space of functions continuous on $[0, +\infty)$ and having the polynomial growth at infinity. In Sec. 2 we give some fundamental properties of these operators. In Sec. 3 we prove some direct approximation theorems for A_n and B_n.

This note was inspired by the results for the Szasz - Mirakjan operators given in [1].

Key words: linear positive operators, degree of approximation.
A.M.S. Subject classification: 41A36

1. Preliminaries

Let $C = C(R_0)$ be the set of all real-valued functions continuous on $R_0 := [0, +\infty)$. Let $N := \{1,2,...\}$, $N_0 := N \cup \{0\}$ and let

(1) $$w_0(x) := 1, \quad w_q(x) := (1+x^q)^{-1} \quad \text{for } x \in R_0, \quad q \in N.$$

Similarly as in [1] denote by C_q the space defined via the weight w_q

$$C_q := \{f \in C : w_q f \text{ is uniformly continuous and bounded on } R_0\}$$

with the norm

(2) $$\|f\|_{C_q} := \sup_{x \in R_0} w_q(x) |f(x)|$$

For $h, \delta \in R_0$, $0 < \alpha \le 1$ and $f \in C_q$ let as usual $\Delta_h f(x) := f(x+h) - f(x)$, $\omega(f, C_q; \delta) := \sup_{0 < h \le \delta} \|\Delta_h f\|_{C_q}$,

$$\mathrm{Lip}(C_q, \alpha) := \{f \in C_q : \omega(f, C_q; \delta) = O(\delta^\alpha) \text{ as } \delta \to 0+\}.$$

In the papers [1–3] were investigated the Szasz – Mirakjan operators

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$$K_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} n \int_{\frac{k}{n}}^{\frac{(k+1)}{n}} f(t)dt,$$

$n \in N$, $x \geq 0$ for functions $f \in C_q$.

In the present note we define in the space C_q, $q \in N_0$, the following linear positive operators A_n and B_n of the Szasz – Mirakjan type

$$(3) \quad A_n(f;x) := (1+\sinh nx)^{-1} \left\{ f(0) + \sum_{k=0}^{\infty} \frac{(nx)^{2k+1}}{(2k+1)!} f\left(\frac{2k+1}{n}\right) \right\}$$

$$(4) \quad B_n(f;x) := (1+\sinh nx)^{-1} \left\{ f(0) + \sum_{k=0}^{\infty} \frac{(nx)^{2k+1}}{(2k+1)!} \frac{n}{2} \int_{I_{n,k}} f(t)dt \right\},$$

$n \in N$, $x \in R_0$, where $I_{n,k} := \left[\frac{2k+1}{n}, \frac{2k+3}{n}\right]$ and $\sinh x$, $\cosh x$ are the elementary hyperbolic functions, i.e. $\sinh x = \frac{e^x - e^{-x}}{2}$.

The operators A_n and B_n, $n \in N$, are well – defined for all $f \in C_q$ with every $q \in N_0$. In Sec. 2 we shall prove that A_n and B_n are an linear positive operators from C_q into C_q with every fixed $q \in N_0$, and we shall give some fundamental properties of these operators. In Sec. 3 we shall prove some approximation theorems for A_n and B_n using the modulus of continuity $\omega(f;C_q;\cdot)$ of $f \in C_q$.

Below by M_q we shall denote some suitable positive constants depending only on the parameter q.

2. Auxiliary results

Denote by

$$(5) \quad S(nx) := \frac{\sinh nx}{1+\sinh nx}, \qquad T(nx) := \frac{\cosh nx}{1+\sinh nx},$$

for $n \in N$, $x \in R_0$. By elementary calculations from (3) and (4) we obtain the following

Lemma 1. For each $n \in N$ *and* $x \in R_0$ *we have*

(6) $$A_n(1;x) = 1\,, \qquad B_n(1;x) = 1\,,$$

(7) $$A_n(t;x) = x\,T(nx),$$

$$B_n(t;x) = A_n(t;x) + \frac{1}{n}S(nx),$$

(8) $$A_n(t^2;x) = x^2 S(nx) + \frac{x}{n}T(nx)$$

$$B_n(t^2;x) = A_n(t^2;x) + \frac{2}{n}A_n(t;x) + \frac{4}{3n^2}S(nx) =$$

$$= \left(x^2 + \frac{4}{3n^2}\right)S(nx) + \frac{3x}{n}T(nx).$$

Using the mathematical induction, we shall prove

Lemma 2. For every fixed $p \in N$ *there exist the positive numbers* $a_{p,k}$, $b_{p,k}$, $c_{p,k}$ *and* $d_{p,k}$ $, 0 \le k \le \left[\frac{p+1}{2}\right]$, *depending only on p, k and* $a_{2m,m} = c_{2m,m} = 1$ *for* $m \in N$, $b_{2m+1,m+1} = d_{2m+1,m+1} = 1$ *for* $m \in N_0$, *and such that for all* $n \in N$ *and* $x \in R_0$ *holds*

(9) $$A_n(t^p;x) = S(nx)\sum_{k=1}^{\left[\frac{p}{2}\right]} a_{p,k}\,\frac{x^{2k}}{n^{p-2k}} + T(nx)\sum_{k=1}^{\left[\frac{p+1}{2}\right]} b_{p,k}\,\frac{x^{2k-1}}{n^{p-(2k-1)}}\,,$$

(10) $$B_n(t^p;x) = S(nx)\sum_{k=0}^{\left[\frac{p}{2}\right]} c_{p,k}\,\frac{x^{2k}}{n^{p-2k}} + T(nx)\sum_{k=1}^{\left[\frac{p+1}{2}\right]} d_{p,k}\,\frac{x^{2k-1}}{n^{p-(2k-1)}}\,,$$

where $S(nx)$, $T(nx)$ *are defined by (5), [y] denotes the integral part of y. (As usual we assume that* $\sum_{k=i}^{j} a_k = 0$ *if* $i > j$ *and* $a_k \in R$*).*

Proof. We shall prove only (9), because the proof of (10) is analogous. We use the mathematical induction for $p \in N$.

If $p = 1, 2$, then (9) follows by (7) and (8). Assuming (9) for $f(x) = x^j$, $j \in N$ and $j \le p$, we get from (3)

$$A_n(t^{p+1};x)=\frac{1}{(1+\sinh nx)n^{p+1}}\sum_{k=0}^{\infty}\frac{(nx)^{2k+1}}{(2k)!}(2k+1)^p$$

$$=\frac{x}{(1+\sinh nx)n^p}\left\{\cosh nx+\sum_{j=1}^{p}\binom{p}{j}\sum_{k=1}^{\infty}\frac{(nx)^{2k}}{(2k-1)!}(2k)^{j-1}\right\}=$$

$$=\frac{x}{(1+\sinh nx)n^p}\Big\{\cosh nx+npx\sinh nx+$$

$$+\sum_{j=1}^{p-1}\binom{p}{j+1}\sum_{k=0}^{\infty}\frac{(nx)^{2k+2}}{(2k+1)!}(2k+2)^j\Big\}=$$

$$=\frac{x}{n^p}T(nx)+\frac{px^2}{n^{p-1}}S(nx)+$$

$$+\frac{x^2}{(1+\sinh nx)\,n^{p-1}}\sum_{j=1}^{p-1}\binom{p}{j+1}\sum_{r=0}^{j}\binom{j}{r}\sum_{k=0}^{\infty}\frac{(nx)^{2k+1}}{(2k+1)!}(2k+1)^r=$$

$$=\frac{x}{n^p}T(nx)+\frac{px^2}{n^{p-1}}S(nx)+\frac{x^2}{n^{p-1}}S(nx)\sum_{j=1}^{p-1}\binom{p}{j+1}+$$

$$+\sum_{r=1}^{p-1}\alpha_{p,r}\frac{x^2}{n^{p-1-r}}A_n(t^r;x),$$

where $\alpha_{p,r}$ are the positive numbrs depending only on p, r and $\alpha_{p,p-1}=1$. Using the inductive assumption, we can write

$$A_n(t^{p+1};x)=\frac{x}{n^p}T(nx)+\alpha_{p,o}\frac{x^2}{n^{p-1}}S(nx)+$$

$$+\sum_{r=1}^{p-1}\alpha_{p,r}\left\{S(nx)\sum_{k=1}^{\left[\frac{r}{2}\right]}a_{r,k}\frac{x^{2k+2}}{n^{p-1-2k}}+T(nx)\sum_{k=1}^{\left[\frac{r+1}{2}\right]}b_{r,k}\frac{x^{2k+1}}{n^{p-2k}}\right\}$$

where $\alpha_{p,p-1}=1$, $a_{2m,m}=1$, $b_{2m+1,m+1}=1$. Hence we have

$$A_n(t^{p+1};x) = S(nx) \sum_{k=1}^{\left[\frac{p+1}{2}\right]} a_{p+1,k} \frac{x^{2k}}{n^{p-2k}} + T(nx) \sum_{k=1}^{\left[\frac{p+2}{2}\right]} b_{p+1,k} \frac{x^{2k-1}}{n^{p+1-(2k-1)}}$$

and $a_{2m,m} = 1$ for $p+1 = 2m$, $m \in N$, and $b_{2m+1,m+1} = 1$ for $p+1 = 2m+1$, which shows that (9) holds for $p+1$. Thus, by the mathematical induction, the proof of (9) is completed.

Lemma 3. For all $x \geq 0$ *and* $n \in N$ *holds*

$$A_n((t-x)^2;x) \leq \frac{4(x+1)}{n}, \tag{11}$$

$$B_n((t-x)^2;x) \leq \frac{31}{3}\frac{x+1}{n}. \tag{12}$$

Proof. By (5) – (8) we have

$$A_n((t-x)^2;x) = x^2(1 - S(nx) - 2T(nx)) + \frac{x}{n}T(nx).$$

Since $1 - e^{-nx} \geq 0$ and $|1 - 2e^{-nx}| \leq 1$ for all $n \in N$ and $x \geq 0$, we have

$$0 < T(nx) = \frac{e^{nx} + e^{-nx}}{2 + e^{nx} - e^{-nx}} \leq \frac{e^{nx}+1}{e^{nx}+1} = 1, \tag{13}$$

$$x^2|1 + S(nx) - 2T(nx)| = \frac{2x^2|1 - 2e^{-nx}|}{2 + e^{nx} - e^{-nx}} \leq \frac{2x^2}{e^{nx}+1} \leq \frac{4}{n^2},$$

$$0 \leq S(nx) \leq 1. \tag{14}$$

From these we obtain (11).

The proof of (12) is analogous.

Lemma 4. For every fixed $q \in N_o$ *there exists a positive constant* M_q *depending only on q such that*

$$\left\| A_n\left(\frac{1}{w_q(t)};\cdot\right)\right\|_{C_q} \leq M_q, \tag{15}$$

(16) $$\left\| B_n\left(\frac{1}{w_q(t)};\cdot\right)\right\|_{C_q} \le M_q,$$

for all $n \in N$.

Proof. Because the proofs of (15) and (16) are analogous, we shall prove only (15). From (1), (2), (5) – (7) and (13) we immediately obtain (15) for $q=0$ and $q=1$. Let $2 \le q \in N$ be a fixed integer. Then by Lemma 2 for all $x \in R_0$ and $n \in N$ we have

$$w_q(x)\, A_n\left(\frac{1}{w_q(t)};x\right) = w_q(x)\,\{A_n(1;x) + A_n(t^q;x)\} =$$

$$= \frac{1}{1+x^q} + S(nx)\sum_{k=1}^{\left[\frac{q}{2}\right]} a_{q,k}\,\frac{1}{n^{q-2k}}\,\frac{x^{2k}}{1+x^q} +$$

$$+ T(nx)\sum_{k=1}^{\left[\frac{q+1}{2}\right]} b_{q,k}\,\frac{1}{n^{q-(2k-1)}}\,\frac{x^{2k-1}}{1+x^q}.$$

Using (13) and (14), we get

$$0 \le w_q(x)\, A_n\left(\frac{1}{w_q(t)};x\right) \le 1 + \sum_{k=1}^{\left[\frac{q}{2}\right]} a_{q,k} + \sum_{k=1}^{\left[\frac{q+1}{2}\right]} b_{q,k} = M_q$$

for $x \ge 0$, $n \in N$, where M_q is a positive constant depending only on q. From these and by (2) follows (15).

Lemma 5. Let $q \in N_0$ *be a fixed number. Then there exists a positive constant* M_q *such that for every* $f \in C_q$ *and* $n \in N$ *holds*

(17) $$\|A_n(f;\cdot)\|_{C_q} \le M_q\, \|f\|_{C_q},$$

(18) $$\|B_n(f;\cdot)\|_{C_q} \le M_q\, \|f\|_{C_q},$$

which proves that A_n *and* B_n *are operators from* C_q *into* C_q.

Proof. By (1) – (3) we see that

$$w_q(x)\,|\,A_n(f(t);x)\,|\le w_q(x)\,A_n(|f(t)|;x)=$$

$$=w_q(x)\,A_n\left(w_q(t)\,|f(t)|\frac{1}{w_q(t)};x\right)\le\|f\|_{C_q}\,w_q(x)\,A_n\left(\frac{1}{w_q(t)};x\right),$$

for $x\ge 0$ and $n\in N$. Now using (15), we obtain (17).

The proof of (18) is analogous.

Lemma 6. For every fixed $q\in N_0$ there exists a positive constant M_q depending only on q such that for all $x\ge 0$ and $n\in N$ holds

$$0\le w_q(x)\,A_n\left(\frac{(t-x)^2}{w_q(t)};x\right)\le M_q\,\frac{x+1}{n}\,, \tag{19}$$

$$0\le w_q(x)\,B_n\left(\frac{(t-x)^2}{w_q(t)};x\right)\le M_q\,\frac{x+1}{n}\,. \tag{20}$$

Proof. The inequalities (19) and (20) for $q=0$ are proved in Lemma 3. If $q\ge 1$, then by (1) and (3) follows

$$A_n\left(\frac{(t-x)^2}{w_q(t)};x\right)=A_n((t-x)^2;x)+A_n(t^q(t-x)^2;x), \tag{21}$$

$$A_n(t^q(t-x)^2;x)=A_n(t^{q+2};x)-2x\,A_n(t^{q+1};x)+x^2A_n(t^q;x).$$

Let $q=2r$, $r\in N$, be a fixed number. Then by Lemma 2 we get

$$A_n(t^{2r}(t-x)^2;x)=S(nx)\left\{x^{2r+2}+\sum_{k=1}^{r}a_{2r+2,k}\frac{x^{2k}}{n^{2r+2-2k}}-\right.$$

$$\left.-2\sum_{k=1}^{r}a_{2r+1,k}\frac{x^{2k+1}}{n^{2r+1-2k}}+x^{2r+2}+\sum_{k=1}^{r-1}a_{2r,k}\frac{x^{2k+2}}{n^{2r-2k}}\right\}+$$

$$+T(nx)\left\{\sum_{k=1}^{r+1}b_{2r+2,k}\frac{x^{2k-1}}{n^{2r+2-(2k-1)}}-2x^{2r+2}-\right.$$

$$-2\sum_{k=1}^{r} b_{2r+1,k}\frac{x^{2k}}{n^{2r+1-(2k-1)}}+\sum_{k=1}^{r} b_{2r,k}\frac{x^{2k+1}}{n^{2r-(2k-1)}}\Bigg\}=$$

$$=2x^{2r+2}\{S(nx)-T(nx)\}+$$

$$+S(nx)\left\{\frac{1}{n^2}\sum_{k=1}^{r} a'_{2r,k}\frac{x^{2k}}{n^{2r-2k}}-\frac{x}{n}\sum_{k=1}^{r} a''_{2r,k}\frac{x^{2k}}{n^{2r-2k}}\right\}+$$

$$+T(nx)\left\{\frac{x}{n}\sum_{k=0}^{r} b'_{2r,k}\frac{x^{2k}}{n^{2r-2k}}-\frac{1}{n^2}\sum_{k=0}^{r} b''_{2r,k}\frac{x^{2k}}{n^{2r-2k}}\right\},$$

for $x \geq 0$ and $n \in N$, where $a'_{2r,k}$, $a''_{2r,k}$, $b'_{2r,k}$ and $b''_{2r,k}$ are a positive numbers depending only on r and k.

Hence and by (13) and (14) we get

$$w_{2r}(x)A_n(t^{2r}(t-x)^2;x)\leq\frac{x^{2r+2}}{1+x^{2r}}|S(nx)-T(nx)|+$$

$$+\frac{1}{n^2}\sum_{k=1}^{r} a_{2r,k}\frac{x^{2k}}{1+x^{2r}}+\frac{x}{n}\sum_{k=1}^{r} a''_{2r,k}\frac{x^{2k}}{1+x^{2r}}+$$

$$+\frac{x}{n}\sum_{k=0}^{r} b_{2r,k}\frac{x^{2k}}{1+x^{2r}}+\frac{1}{n^2}\sum_{k=1}^{r} b''_{2r,k}\frac{x^{2k}}{1+x^{2r}}.$$

Arguing as in the proof of Lemma 3 we obtain

$$\frac{x^{2r+2}}{1+x^{2r}}|S(nx)-T(nx)|=\frac{x^{2r}}{1+x^{2r}}\frac{2x^2e^{-nx}}{2+e^{nx}-e^{-nx}}\leq\frac{2x^2}{e^{nx}+1}\leq\frac{4}{n^2},$$

$$x\geq 0,\ n\in N.$$

From these follows

$$w_{2r}(x)A_n(t^{2r}(t-x)^2;x)\leq\frac{4}{n^2}+M_q\left\{\frac{1}{n^2}+\frac{x}{n}\right\}\leq$$

$$\leq M_q\frac{x+1}{n}\quad\text{for}\quad x\geq 0,\ n\in N.$$

Using this inequality and (11) to (21) we obtain (19) for $q=2r$, $r\in N$.

Analogously, applying Lemmas 2 and 3, we obtain (19) for every $q=2r+1$, $r\in N_0$.

The proof of (20) is similar.

3. Approximation theorems

In this part we shall prove two direct approximation theorems for A_n and B_n.

Theorem 1. Suppose that $q \in N_0$ *is a fixed number and* $g \in C_q^1 := \{f \in C_q : f' \in C_q\}$. *Then there exists a positive constant* M_q *(depending only on q) such that*

$$w_q(x)\,|A_n(g;x) - g(x)| \le M_q \|g'\|_{C_q} \left(\frac{x+1}{n}\right)^{\frac{1}{2}}, \tag{22}$$

$$w_q(x)\,|B_n(g;x) - g(x)| \le M_q \|g'\|_{C_q} \left(\frac{x+1}{n}\right)^{\frac{1}{2}}, \tag{23}$$

for all $x \ge 0$ *and* $n \in N$.

Proof. We shall prove only (22) because the proof of (23) is analogous.

Let $x \ge 0$ be a fixed point. For $t \ge 0$ we have

$$g(t) - g(x) = \int_x^t g'(u)\,du$$

and by (3) and (6) we get

$$A_n(g(t);x) - g(x) = A_n\left(\int_x^t g'(u)\,du; x\right), \qquad n \in N. \tag{24}$$

Since

$$\left|\int_x^t g'(u)\,du\right| \le \|g'\|_{C_q} \left|\int_x^t (w_q(u))^{-1}\,du\right| \le$$

$$\leq \|g'\|_{C_q}\left(\frac{1}{w_q(x)}+\frac{1}{w_q(t)}\right)|t-x|,$$

we get from (24)

$$w_q(x)\,|A_n(g(t);x)-g(x)|\leq \|g'\|_{C_q}\,\Big\{A_n(|t-x|;x)+$$

$$+w_q(x)\,A_n\left(\frac{|t-x|}{w_q(t)};x\right)\Big\}.$$

But from (3) follows

$$A_n(|t-x|;x)\leq \{A_n((t-x)^2;x)\}^{\frac{1}{2}},$$

$$A_n\left(\frac{|t-x|}{w_q(t)};x\right)\leq \left\{A_n\left(\frac{1}{w_q(t)};x\right)\right\}^{\frac{1}{2}}\left\{A_n\left(\frac{(t-x)^2}{w_q(t)};x\right)\right\}^{\frac{1}{2}},$$

for all $x\geq 0$, $n\in N$ and $q\in N_0$, which by Lemma 3 and Lemma 5 yield

$$w_q(x)A_n(|t-x|;x)\leq \left(\frac{4(x+1)}{n}\right)^{\frac{1}{2}}$$

$$w_q(x)A_n\left(\frac{|t-x|}{w_q(t)};x\right)\leq M_q\left(\frac{x+1}{n}\right)^{\frac{1}{2}}$$

for all $x\geq 0$, $n\in N$ and $q\in N_0$. Combining these, we obtain (22).

Theorem 2. Suppose that $f\in C_q$, with a fixed $q\in N_0$. Then there exists a positive constant M_q (depending only on q) such that

$$\text{(25)}\qquad w_q(x)\,|A_n(f;x)-f(x)|\leq M_q\,\omega\left(f,C_q;\sqrt{\frac{x+1}{n}}\right),$$

$$\text{(26)}\qquad w_q(x)\,|B_n(f;x)-f(x)|\leq M_q\,\omega\left(f,C_q;\sqrt{\frac{x+1}{n}}\right),$$

for all $x\geq 0$ and $n\in N$.

Proof. Because the proofs of (25) and (26) are analogous, we shall prove only (25). Let f_h be the Stieklov mean of $f \in C_q$, i.e.

$$f_h(x) = h^{-1}\int_0^h f(x+t)\,dt \quad \text{for} \quad x \geq 0, \;\; h > 0 .$$

We have

$$f_h(x) - f(x) = h^{-1}\int_0^h (f(x+t) - f(x))\,dt ,$$

$$f_h'(x) = h^{-1}\{f(x+h) - f(x)\},$$

for $x \geq 0$, $h > 0$. It is easily observed that if $f \in C_q$, then $f_h \in C_q^1$ for every fixed $h > 0$. Moreover, for $h > 0$ holds

(27) $$\|f_h - f\|_{C_q} \leq \sup_{x \geq 0}\left[h^{-1}\int_0^h w_q(x)\left|f(x) - f(x)\right|dt \right] \leq \omega(f, C_q; h),$$

(28) $$\|f_h'\|_{C_q} \leq h^{-1}\,\omega(f, C_q; h).$$

Since A_n is a linear operator, we have

$$w_q(x)|A_n(f;x) - f(x)| \leq w_q(x)\{|A_n(f - f_h;x)| + \\ +|A_n(f_h;x) - f_h(x)| + |f_h(x) - f(x)|\}$$

for $x \geq 0$, $n \in N$ and $h > 0$.

Using Lemma 5 and (27), we get

$$w_q(x)|A_n(f - f_h;x)| \leq M_q\|f - f_h\|_{C_q} \leq M_q\,\omega(f, C_q; h).$$

From Theorem 1 and (28) follows

$$w_q(x)|A_n(f_h;x) - f_h(x)| \leq M_q\|f_h'\|_{C_q}\left(\frac{x+1}{n}\right)^{\frac{1}{2}} \leq$$

$$\leq M_q\,\omega(f, C_q; h)\,h^{-1}\left(\frac{x+1}{n}\right)^{\frac{1}{2}}.$$

From these and by (27) we obtain

$$(29)\qquad w_q(x)\,|A_n(f;x)-f(x)|\le M_q\,\omega(f,C_q;h)\left\{1+h^{-1}\left(\frac{x+1}{n}\right)^{\frac{1}{2}}\right\}$$

for every $x\ge 0$ and $h>0$. Setting, for every fixed $x\ge 0$, $h=\left(\frac{x+1}{n}\right)^{\frac{1}{2}}$ to (29), we obtain the desired estimation (25) for $x\ge 0$, and $n\in N$.

From Theorem 2 we obtain the following two corollaries.

Corollary 1. If $f\in C_q$ *with some* $q\in N_0$, *then*

$$(30)\qquad \lim_{n\to\infty} A_n(f;x)=f(x),$$

$$(31)\qquad \lim_{n\to\infty} B_n(f;x)=f(x),$$

for all $x\ge 0$. *Moreover, the statements (30) and (31) hold uniformly on every* $[0,a]$, $a>0$.

Corollary 2. If $f\in \mathrm{Lip}\,(C_q,\alpha)$ *with some* $q\in N_0$ *and* $0<\alpha\le 1$, *then there exists a positive constant* $M_{q,\alpha}$ *(depending only an* q *and* α*) such that*

$$w_q(x)|A_n(f;x)-f(x)|\le M_{q,\alpha}\left(\frac{x+1}{n}\right)^{\frac{\alpha}{2}}$$

$$w_q(x)|B_n(f;x)-f(x)|\le M_{q,\alpha}\left(\frac{x+1}{n}\right)^{\frac{\alpha}{2}},$$

for all $x\ge 0$ *and* $n\in N$.

References

[1] M. Becker, Global approximation theorems for Szasz – Mirakjan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.*, 27(1) (1978), 127–142.

[2] V. Totik, Uniform approximation by Szasz–Mirakjan operators, *Acta Math. Hung*, 41(3–4) (1983), 291–307.

[3] V. Totik, Approximation by Szasz – Mirakjan – Kantorovich operators in L^p $(p > 1)$, *Analysis Math.*, 9 (1983), 147–167.

(Poznan University of Technology, Institute of Mathematics, 60 – 965 Poznań, Poland)
Received on 14.12.1994.