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SINGULAR CAUCHY-NICOLETTI PROBLEM FOR SYSTEM OF DIFFERENTIAL EQUATIONS

In the paper a singular Cauchy-Nicoletti problem for system of ordinary differential equations is considered. The existence of solutions which graph remains in a properly choosen domain is proved. Moreover, the theorem about unicity of solution in this domain is given. The applicability of results in on an illustrative example showed.

Key words: singular Cauchy-Nicoletti problem, unicity of solutions.

1. Introduction

Consider the following singular Cauchy-Nicoletti problem for the system of ordinary differential equations

(1) $$y'(x) = \omega(x,y),$$

(2) $$y_k(a+) = A_k, \quad (k = 1,2,\dots,h); \quad 1 \le h < n;$$
$$y_l(b-) = B_l, \quad (l = h+1,\dots,n),$$

where $x \in I = (a,b)$, $y = (y_1, y_2, \dots, y_n) \in R^n$, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, a, b, A_k, B_l, are real constants and $a < b$. Concerning the function $\omega(x,y)$ we assume that is continuos and satisfy a condition of the Lipschitz type in the second variable in a region D indicated below. For this region the intersection $D \cap \{(x^*, y): y \in R^n\} \neq \emptyset$ for each $x^* \in I$. Under these conditions the solutions of the system (1) are in $\operatorname{int} D$ uniquely determined by their initial data but for $x = a$ or $x = b$ this need not be the case. The notion of solution of the problem (1), (2) we define in the following sense:

Definition. A vector-function $y \in C^1(I)$ is a solution of the problem (1), (2) *if $(x, y(x)) \in D$ on I and, moreover,* $y_k(a+) = A_k$, $(k = 1, 2,\dots,h)$, $y_l(b-) = B_l$, $(l = h+1,\dots,n)$. ■

The Cauchy-Nicoletti problems, the generalized Cauchy problems or the boundary value problems for systems of ordinary differential equations have been considered by many authors. Singular problems of such types have been studied e.g. in works [1-14]. In this paper we give sufficient conditions for solvability and uniquencess of the problem (1),(2). Some estimations of the components of solutions are given too.

2. Main results

We will consider the real functions $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2,\dots,n$, which satisfy the following conditions (H1) – (H3):

(H1):

$$\alpha_i, \beta_i \in C[a,b], \ \alpha_i(x) \le \beta_i(x) \ \text{ on } \ [a,b], \quad i = 1,2,\dots,n,$$
$$\alpha_k(a) = \beta_k(a) = A_k, \quad (k = 1,2,\dots,h),$$
$$\alpha_l(b) = \beta_l(b) = B_l, \quad (l = h+1,\dots,n);$$

(H2):

$$\gamma_k, \delta_k \in C(a,b], \ \gamma_k(x) \le \delta_k(x) \quad \text{on } \ (a,b], \ (k = 1,2,\dots,h),$$
$$\gamma_l, \delta_l \in C[a,b), \ \gamma_l(x) \le \delta_l(x) \quad \text{on } \ [a,b), \ (l = h+1,\dots,n)$$

and, moreover, there are finite (possible improper) integrals

$$\int_a^b \gamma_i(t)dt, \qquad \int_a^b \delta_i(t)dt, \qquad (i = 1,2,\dots,n).$$

in the Riemann's sense;

(H3): on $[a,b]$ it holds

$$\alpha_k(x) \le A_k + \int_a^x \gamma_k(t)dt \le A_k + \int_a^x \delta_k(t)dt \le \beta_k(x), \ (k = 1,2,\dots,h),$$

$$\alpha_l(x) \le B_l - \int_x^b \delta_l(t)dt \le B_l - \int_x^b \gamma_l(t)dt \le \beta_l(x), \ (l = h+1,\dots,n).$$

Now define the domain D:

$$D = \{(x,y) : x \in I, \ \alpha_i(x) \le y_i \le \beta_i(x), \ i = 1,2,\dots,n\} .$$

Theorem 1. Let functions α_i, β_i, γ_i, δ_i, $i = 1,2,\dots,n$ *satisfy the conditions* (H1) – (H3) *and, moreover,*

a) $\omega_i \in C(D)$,

b) $\gamma_i(x) \le \omega_i(x,y) \le \delta_i(x)$ *where* $(x,y) \in D$

c) for arbitrary points (x,y), $(x,w) \in D$

$$|\omega_i(x,y) - \omega_i(x,w)| \le \sum_{j=1}^n M_{ij}(x)|y_j - w_j|$$

where $i,j = 1,2,\dots,n$, *and* $M_{ij} \in C(I)$ *are nonnegative functions such that*

$$\int_a^b M_{ij}(x)(\beta_j(x) - \alpha_j(x))dx < +\infty .$$

Then there is a solution $y = y(x)$ *of the Cauchy-Nicoletti problem* (1), (2).

Theorem 2. *Let all assumptions of Theorem* 1 *hold and, moreover,*

$$q \int_a^b M(t)\, dt < 1$$

where $q = \max\{h, n-h\}$ *and* $M(t) = \max_{i,j}\{M_{ij}(t)\}$.

Then the solution $y = y(x)$ *of the Cauchy-Nicoletti problem* (1), (2) *with property* $(x, y(x)) \in D$ *on* I *is unique.*

Proof of Theorem 1. In view of a), b) and (H2) the problem (1), (2) is equivalent in D with the following system of integral equations

(3) $$y_k(x) = A_k + \int_a^x \omega_k(t, y(t))dt, \qquad (k = 1,2,\ldots,h),$$

(4) $$y_l(x) = B_l - \int_x^b \omega_l(t, y(t))dt, \qquad (l = h+1,\ldots,n),$$

where the Riemann's integrals can be improper in the points a or b. Define with the aid of (3), (4) the sequences of functions $\{y_i^m(x)\}$, $i = 1,2,\ldots,n$ on I as follows

(5) $$y_i^0(x) = \frac{1}{2}(\alpha_i(x) + \beta_i(x)), \qquad i = 1,2,\ldots,n,$$

(6) $$y_k^{m+1}(x) = A_k + \int_a^x \omega_k(t, y^m(t))dt,$$

(7) $$y_l^{m+1}(x) = B_l - \int_x^b \omega_l(t, y^m(t))dt,$$

where $m = 0,1,2,\ldots$, $k = 1,2,\ldots,h$, $l = h+1,\ldots,n$.

We devide the remaining part of the proof into three parts.

I. By method of induction and elementary properties of integrals it may be easily proved (with the aid of (H1) – (H3), a) and b)) that all elements of these sequences exist, can be continuously continued on the whole interval $[a,b]$ and, moreover, $(x, y^m(x)) \in \overline{D}$ if $x \in [a,b]$, $(m = 0,1,2,\ldots)$. We will take this into account in the next text.

II. We show by Arczel's theorem that there are subsequences $\{y_i^{m_r}(x)\}$ of the sequences $\{y_i^m(x)\}$ which converge uniformly on $[a,b]$. It is necessary to prove that all members of these sequencens are uniformly bounded and equicontinuous. The uniform boundedness follows from the fact that by the previous part $(x, y^m(x)) \in \overline{D}$, $(m = 0,1,2,\ldots)$ on $[a,b]$ and functions $\alpha_i(x)$, $\beta_i(x)$ are bounded (by (H1)) on $\alpha_i(x)$. Prove the equicontinuity. Let $k \in \{1,2,\ldots,h\}$. Define: $\psi_k(x) = \max(|\gamma_k(x)|, |\delta_k(x)|)$ and $\varphi_k(x) = \max(|\alpha_k(x) - A_k|, |\beta_k(x) - A_k|)$. From b), (H1) – (H3) we obtain

$$\left| \int_a^x \omega_k(t, y^m(t))\, dt \right| \le \varphi_k(x),$$

where $x \in [a,b]$, $(m = 0,1,\ldots)$. Choose arbitrary positive number ε_k. Then, because $\varphi_k(a) = 0$, there is a $\eta_k = \eta_k(\varepsilon_k) > 0$, $\eta_k \le (b-a)$ such that $\varphi_k(x) < (1/2)\varepsilon_k$ if $x \in [a, a+\eta_k]$. Let $I_k = (a + (1/2)\eta_k, b]$, $M_k = \sup_{x \in I_k} \psi_k(x)$, $\lambda_k^* \in (0, \varepsilon_k M_k^{-1})$, and $\lambda_k = \min\{\lambda_k^*, (1/2)\eta_k\}$. We obtain for $x', x'' \in [a,b]$, $|x' - x''| < \lambda_k$:

α) if $x', x'' \in I_k$

$$\left| y_k^m(x') - y_k^m(x'') \right| = \left| \int_{x'}^{x''} \omega_k(t, y^{m-1}(t))\, dt \right| \le \left| \int_{x'}^{x''} \left| \omega_k(t, y^{m-1}(t)) \right| dt \right| \le$$

$$\le \left| \int_{x'}^{x''} \psi_k(t) dt \right| \le M_k |x' - x''| \le M_k \lambda_k \le M_k \lambda_k^* < \varepsilon_k\,;$$

β) if $x', x'' \notin I_k$ or $x' \in I_k$, $x'' \notin I_k$, then $x', x'' \in [a, a+\eta_k]$ and

$$\left| y_k^m(x') - y_k^m(x'') \right| \le \left| \int_a^{x'} \omega_k(t, y^{m-1}(t))\, dt \right| + \left| \int_a^{x''} \omega_k(t, y^{m-1}(t))\, dt \right| \le$$

$$\le \varphi_k(x') + \varphi_k(x'') < (1/2)\varepsilon_k + (1/2)\varepsilon_k = \varepsilon_k.$$

The equicontinuity for indices $k \in \{1,2,\ldots,h\}$ is proved. By analogy we can prove the equicontinuity for indices $l \in \{h+1,\ldots,n\}$. Therefore the equicontinuity is proved and by Arczel's theorem above-mentioned subsequences $\{y_i^{m_r}(x)\}$ exist. We denote the limits of these subsequences as $y_i(x)$, $(i = 1,2,\ldots,n)$. In the next reasonings we will use, without loos of generality, the previous sequences instead of these subsequences.

Because $(x, y^m(x)) \in \overline{D}$ for each $m = 0,1,\ldots$ and $x \in [a,b]$ then $(x, y(x)) \in \overline{D}$ on $[a,b]$ too.

III. Prove that the limit function $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))$ satisfy on I the system (3), (4). For each positive $\tilde{\varepsilon}$ there is (in view of uniformly convergence) an index $m_{\tilde{\varepsilon}}$ such that for $m > m_{\tilde{\varepsilon}}$: $|y_i(x) - y_i^m(x)| < \tilde{\varepsilon}$, $(i = 1,2,\ldots,n)$ on $[a,b]$. Because $(x, y(x)) \in \overline{D}$ and $(x, y^m(x)) \in \overline{D}$ on $[a,b]$ then

(8) $$|y_i(x) - y_i^m(x)| < \min\{\tilde{\varepsilon}, |\beta_i(x) - \alpha_i(x)|\}, \qquad (i = 1,2,\ldots,n).$$

From c) and (8) we conclude that for each positive ε there are: an index n_ε (sufficiently large) a value $x_\varepsilon^1 \in [a,b]$ (perhaps sufficiently near to the point a) and value $x_\varepsilon^2 \in [a,b]$, $x_\varepsilon^1 < x_\varepsilon^2$ (perhaps sufficiently near to the point b) such that for $n > n_\varepsilon$, $(i, j = 1,2,\ldots,n)$

(9) $$\int_a^{x_\varepsilon^1} M_{ij}(t)\,|y_i(t) - y_i^n(t)|\,dt < \frac{\varepsilon}{3n}, \qquad \int_{x_\varepsilon^2}^{b} M_{ij}(t)\,|y_i(t) - y_i^n(t)|\,dt < \frac{\varepsilon}{3n}$$

and

(10) $$\int_{x_\varepsilon^1}^{x_\varepsilon^2} M_{ij}(t)\,|y_i(t) - y_i^n(t)|\,dt < \frac{\varepsilon}{3n}.$$

In view of c), (9) and (10) we have for $x \in I$, $(k = 1,2,\ldots,h)$:

$$\left| \int_a^x \omega_k(t, y(t))\,dt - \int_a^x \omega_k(t, y^n(t))\,dt \right| \le$$

$$\le \left| \int_a^x \sum_{j=1}^n M_{kj}(t)\,|y_j(t) - y_j^n(t)|\,dt \right| \le$$

$$\le \sum_{j=1}^n \left\{ \int_a^{x_\varepsilon^1} + \int_{x_\varepsilon^1}^{x_\varepsilon^2} + \int_{x_\varepsilon^2}^{b} \right\} M_{kj}(t)\,|y_j(t) - y_j^n(t)|\,dt < \varepsilon.$$

Consequently, if $\varepsilon \to 0$ then

$$\int_a^x \omega_k(t, y^n(t))\,dt \to \int_a^x \omega_k(t, y(t))\,dt.$$

By analogy we can prove that for $l \in \{h+1,\dots,n\}$

$$\int_x^b \omega_l(t, y^n(t))\, dt \to \int_x^b \omega_l(t, y(t))\, dt$$

if $x \in I$ and $\varepsilon \to 0$.

Therefore the vector-function $y(x)$ is a solution of (3), (4), and, consequently, a solution of the problem (1), (2) with mentioned properties too. The theorem is proved. ■

Proof of Theorem 2. Let there exist two different solutions $y(x)$ and $u(x)$ of the problem (1), (2) with properties indicated in Theorem 1. Then by condition c) of Theorem 1 and from (3) for $k \in \{1,2,\dots,h\}$ on I follows

$$|y_k(x) - u_k(x)| \le \int_a^x M(t) \sum_{j=1}^n |y_j(t) - u_j(t)|\, dt = \int_a^x M(t)\, \Delta(t)\, dt \tag{11}$$

where $\Delta(t) = \sum_{j=1}^n |y_j(x) - u_j(x)|$. Analogously from (4) for $l \in \{h+1, \dots, n\}$ on I follows

$$|y_l(x) - u_l(x)| \le \int_x^b M(t)\, \Delta(t)\, dt \tag{12}$$

Denote $\Delta = \sup_{x \in I} \Delta(x)$. Then in view of (11), (12) we have

$$\sum_{k=1}^h |y_k(x) - u_k(x)| + \sum_{l=h+1}^n |y_l(x) - u_l(x)| = \Delta(x) \le$$

$$\le \Delta \left(\sum_{k=1}^h \left(\int_a^x M(t)\, dt \right) + \sum_{l=h+1}^n \left(\int_x^b M(t)\, dt \right) \right) \le \Delta q \int_a^b M(t)\, dt\,.$$

We obtain a contradiction, becouse

$$0 < \Delta \le \Delta q \int_a^b M(t)\, dt < \Delta$$

The theorem is proved.

3. Applications

A. Prove, that there is a nontrivial solution of Cauchy-Nicoletti problem

(13)
$$y_1' = \frac{y_1}{x} + \frac{x^2 y_2}{2(1-x)},$$
$$y_2' = \frac{(1-x)^2 y_1}{2x} - \frac{y_2}{(1-x)},$$

(14) $$y_1(0+) = y_2(1-) = 0.$$

All conditions of the Theorem 1 are valid if we put $n=2$, $h=1$, $a=0$, $b=1$, $A_1 = B_2 = 0$, $\alpha_1(x) = x - x^2$; $\beta_1(x) = x + x^2$, $\alpha_2(x) = (1-x) - -(1-x)^2$, $\beta_2(x) = (1-x)+(1-x)^2$, $\gamma_1(x) = 1-x$, $\delta_1(x) = 1+2x$, $\gamma_2(x) = -2+x$ and $\delta_2(x) = 1-2x$. Therefore, there is a solution $y = y(x)$, of the problem (13), (14) and on $(0,1)$:

$$|y_1(x) - x| \le x^2, \qquad |y_2(x) - (1-x)| \le (1-x)^2.$$

B. Consider the Cauchy-Nicoletti problem (15), (16):

(15) $$y_1' = f(x)y_1 + F(x, y_1, y_2, y_3), \qquad y_2' = y_1, \qquad y_3' = y_2,$$

(16) $$y_1(0+) = 0, \qquad y_2(T-) = y_3(T-) = -\alpha$$

where $0 < T$, $0 \le \alpha$. Let there exist positive function $h \in C(I_1)$, $I_1 = (0,T)$ and negative function $\omega \in C(I_1)$ such that $h(0+) = 0$, $\omega(x) < -\alpha$ on I_1, $\omega(T-) = -\alpha$. Let, moreover, exist integrals $\int_0^T h(s)\,ds$, $\int_0^T \omega(s)ds$ on I_1

$$\int_x^T h(s)ds \le -\alpha - \omega(x), \qquad \alpha(T-x) \le -\int_x^T \omega(s)ds.$$

Introduce domain

$$D = \{(x, y_1, y_2, y_3) : x \in I_1,\ \alpha_i(x) \le y_i \le \beta_i(x),\ i = 1,2,3\}$$

where $\alpha_1(x) \equiv 0$, $\alpha_2(x) = \omega(x)$, $\alpha_3(x) = -\alpha + \alpha(T-x)$, $\beta_1(x) = h(x)$, $\beta_2(x) \equiv -\alpha$, $\beta_3(x) = -\alpha - \int_x^T \omega(s)ds$.

Theorem 3. *Let $f \in C(I_1)$ is nonnegative function, $\int_0^T f(s)h(s)\,ds < <+\infty$, $F \in C(D)$, $0 \le F(x,y) \le p(x)$ on D where $p(x) \in C(I_1)$, $\int_0^T p(s)\,ds < \infty$ and $\int_0^x (f(s)h(s) + p(s))\,ds \le h(x)$ on I_1. Let, moreover, for all $(x,y) \in D$, $(x,w) \in D$*

$$|F(x,y) - F(x,w)| \le M_1(x)|y_1 - w_1| + M_2(x)|y_2 - w_2| + M_3(x)|y_3 - w_3|$$

where $M_i \in C(I_1)$, $(i = 1,2,3)$ *are nonnegative functions such that*

$$\int_0^T M_i(s)(\beta_i(s) - \alpha_i(s))\,ds < +\infty .$$

Then there is a solution $y = y(x)$ *of the problem* (15), (16), *such that* $(x, y(x)) \in D$ *on* I_1.

Proof. It is not difficult to verify all assumptions of Theorem 1 if except above introduced functions is put $a = 0$, $b = T$, $n = 3$, $h = 1$, $A_1 = 0$, $B_2 = B_3 = -\alpha$, $\gamma_1(x) = \gamma_2(x) \equiv 0$, $\gamma_3(x) = \omega(x)$, $\delta_1(x) = f(x)h(x) + p(x)$, $\delta_2(x) = h(x)$ and $\delta_3(x) \equiv -\alpha$. From its conclusion follows the conclusion of Theorem 3.

Consider the concrete problem of the type of (15), (16):

$$y_1' = (1/x)y_1 + x^4 y_3^2 , \quad y_2' = y_1 , \quad y_3' = y_2 ,$$
$$y_1(0-) = 0 , \qquad y_2(1-) = y_3(1-) = -1 .$$

The conditions of Theorem 3 are valid for $\alpha = 1$, $T = 1$, $f(x) = x^{-1}$, $F(x,y) = x^4 y_3^2, \omega(x) = -2 + x$, $h(x) = x^4$, $p(x) = x^4$. Therefore, there is a solution $y = y(x)$ on $(0,1)$ of this problem for which $0 \le y_1(x) \le x^4$, $-2 + x \le y_2(x) \le -1$ and $-x \le y_3(x) \le (1/2)(1 - 4x + x^2)$.

At the end we note, that the singular problem for the equation of third order

$$y''' = f(x)y'' + F(x, y'', y', y),$$
$$y(T-) = y'(T-) = -\alpha, \; y'''(0+) = 0$$

can be transformed to the problem (15), (16) by means of transformations $y_3 = y$, $y_2 = y'$, $y_1 = y''$.

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