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APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES BY SOME OPERATORS OF THE SZASZ-MIRAKJAN TYPE

In this paper we introduce some linear positive operators $L_{\tilde{n}}$ and $U_{\tilde{n}}$ of the Szasz-Mirakjan type in the space of continuous functions of several variables having the polynomial growth. In Sec. 2 we give some properties of these operators. In Sec. 3 some direct approximation theorems are proved.

The operators $L_{\tilde{n}}$ and $U_{\tilde{n}}$ for functions of one variable were introduced in [2].

Key words: function of several variables, linear positive operator, degree of approximation.

1. Preliminaries

1.1. Let $N^1 \equiv N := \{1,2,...\}$, $N_0^1 \equiv N_0 := N \cup \{0\}$, $R^1 \equiv R := (-\infty, +\infty)$ $R_0^1 \equiv R_0 := [0, +\infty)$ and for every fixed $m \in N$ let $N^m := \{\tilde{n} = (n_1, ..., n_m) : n_k \in N$ for $1 \le k \le m\}$. Analogously we define N_0^m, R^m and R_0^m. For $\tilde{x} = (x_1, ..., x_m)$, $\tilde{y} = (y_1, ..., y_m) \in R^m$ and $\lambda \in R$ we define: $\tilde{x} + \tilde{y} = (x_1 + y_1, ..., x_m + y_m)$, $\lambda \tilde{x} = (\lambda x_1, ..., \lambda x_m)$, $\tilde{x} - \tilde{y} = \tilde{x} + (-1)\tilde{y}$, $\tilde{x} < \tilde{y}$ if and only if $x_k < y_k$ for $1 \le k \le m$ (analogously we write $\tilde{x} \le \tilde{y}$ if and only if $x_k \le y_k$ for $1 \le k \le m$), $\tilde{\tilde{\lambda}} = (\lambda, \lambda, ..., \lambda) \in R^m$. For $\tilde{k} = (k_1, ..., k_m) \in N_0^m$ and $\tilde{n} = (n_1, ..., n_m) \in N^n$ we write $\frac{\tilde{k}}{\tilde{n}} := \left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, ..., \frac{k_m}{n_m}\right)$, $\tilde{n} \to \tilde{\tilde{\infty}}$ if and only if $n_k \to +\infty$ for $k = 1, 2, ..., m$.

Let $\sum\limits_{\tilde{k} \ge \tilde{\tilde{0}}} := \sum\limits_{k_1=0}^{\infty} \dots \sum\limits_{k_m=0}^{\infty}$ for $\tilde{k} \in N^m$ and let

$$\int\limits_{\tilde{x}}^{\tilde{y}} f(\tilde{t})\, d\tilde{t} := \int\limits_{x_1}^{y_1} \dots \int\limits_{x_m}^{y_m} f(t_1, ..., t_m)\, dt_1 \dots dt_m$$

for a fixed $\tilde{x}, \tilde{y} \in R^m$.

1.2. Let

(1) $$w_0(x):=1, \quad w_q(x):=(1+x^q)^{-1} \quad \text{for} \quad x\in R_0,\ q\in N,$$

and let for a fixed $\tilde{q}=(q_1,\ldots,q_m)\in N_0^m$

(2) $$w_{\tilde{q}}(\tilde{x}):=\prod_{k=1}^{m} w_{q_k}(x_k), \qquad \tilde{x}=(x_1,\ldots,x_m)\in R_0^m,$$

where $w_{q_k}(x_k)$, $1\le k\le m$, is defined by (1). Next, for fixed $m\in N$ and $\tilde{q}\in N_0^m$, we define the space $C_{\tilde{q},m}$ of all functions f defined on R_0^m for wchich $w_{\tilde{q}}(\cdot)f(\cdot)$ is unuformly continuous and bounded on R_0^m and the norm

(3) $$\| f \|_{\tilde{q},m} := \sup_{\tilde{x}\in R_0^m} w_{\tilde{q}}(\tilde{x})\,|f(\tilde{x})|.$$

The modulus of continuity of $f\in C_{\tilde{q},m}$ we define as usual by the formula

$$\omega(f, C_{\tilde{q},m};\tilde{t}):=\sup_{\tilde{0}\le\tilde{h}\le\tilde{t}} \|\Delta_{\tilde{h}} f\|_{\tilde{q},m}, \quad \tilde{h},\tilde{t}\in R_0^m,$$

$$\Delta_{\tilde{h}} f(\tilde{x}):=f(\tilde{x}+\tilde{h})-f(\tilde{x}), \qquad \tilde{h},\tilde{x}\in R_0^m.$$

For a fixed $\tilde{\alpha}=(\alpha_1,\ldots,\alpha_m)$, $0<\alpha_k\le 1$ for $k=1,2,\ldots,m$, we denote by $\mathrm{Lip}(C_{\tilde{q},m};\tilde{\alpha})$ the class of all functions $f\in C_{\tilde{q},m}$ for which $\omega(f, C_{\tilde{q},m};\tilde{t})=O\left(t_1^{\alpha_1}+\ldots+t_m^{\alpha_m}\right)$ as $t_k\to 0_+$ for $k=1,\ldots,m$.

1.3. In the papers [1,3] were investigated the Szasz-Mirakjan operators

$$S_n(f;x):=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), x\in R_0, \qquad n\in N,$$

for functions f of one variable, belonging to the space $C_{q,1}$, $q\in N_0$. In [2] we introduced the operators L_n and U_n of the Szasz-Mirakjan type for $f\in C_{0,1}$

(4) $$L_n(f;x):=\sum_{k=0}^{\infty} p_{n,k}(x)\, f\left(\frac{2k}{n}\right),$$

(5) $$U_n(f;x):=\sum_{k=0}^{\infty} p_{n,k}(x)\,\frac{n}{2}\int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t)\,dt,$$

$x\in R_0$ and $n\in N$, where

(6) $$p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in N_0,$$

and $\cosh x$, $\sinh x$, $\tanh x$ are the elementary hyperbolic functions.

In this note we define the operators $L_{\tilde{n}}$ and $U_{\tilde{n}}$ in the space $C_{\tilde{q},m}$ with some fixed $m \in N$ and $\tilde{q} \in N_0^m$. For $\tilde{x} = (x_1, \ldots, x_m) \in R_0^m$, $\tilde{n} = (n_1, \ldots, n_m) \in N^m$, $\tilde{k} = (k_1, \ldots, k_m) \in N_0^m$ we set

(7) $$A_{\tilde{n},\tilde{k}}(\tilde{x}) := \prod_{j=1}^{m} p_{n_j,k_j}(x_j),$$

(8) $$B_{\tilde{n},\tilde{k}}(\tilde{x}) := \prod_{j=1}^{m} p_{n_j,k_j}(x_j) \frac{n_j}{2}.$$

In a given space $C_{\tilde{q},m}$ we introduce the operators

(9) $$L_{\tilde{n}}(f;\tilde{x}) := \sum_{\tilde{k} \geq \tilde{\tilde{0}}} A_{\tilde{n},\tilde{k}}(\tilde{x}) f\left(\frac{2\tilde{k}}{\tilde{n}}\right),$$

(10) $$U_{\tilde{n}}(f;\tilde{x}) := \sum_{\tilde{k} \geq \tilde{\tilde{0}}} B_{\tilde{n},\tilde{k}}(\tilde{x}) \int_{\frac{2\tilde{k}}{n}}^{\frac{2\tilde{k}+\tilde{\tilde{2}}}{n}} f(\tilde{t})\, d\tilde{t}, \quad \tilde{x} \in R_0^m, \quad \tilde{n} \in N^m.$$

$L_{\tilde{n}}$ and $U_{\tilde{n}}$, $\tilde{n} \in N^m$, are a positive linear operators well defined in $C_{\tilde{q},m}$. We shall prove that $L_{\tilde{n}}$ and $U_{\tilde{n}}$ are operators from $C_{\tilde{q},m}$ into $C_{\tilde{q},m}$.

From (6) follows $\sum_{k=0}^{\infty} p_{n,k}(x) = 1$ for $x \in R_0$, $n \in N$, which by (7) – – (10) gives $\sum_{\tilde{k} \geq \tilde{\tilde{0}}} A_{\tilde{n},\tilde{k}}(\tilde{x}) = 1$ and $\sum_{\tilde{k} \geq \tilde{\tilde{0}}} B_{\tilde{n},\tilde{k}}(\tilde{x}) = 1$ for all $\tilde{x} \in R_0^m$ and $\tilde{n} \in N^m$.

Hence

(11) $$L_{\tilde{n}}(1;\tilde{x}) = 1 = U_{\tilde{n}}(1;\tilde{x}) \qquad \text{for all } \tilde{x} \in R_0^m \text{ and } \tilde{n} \in N^m.$$

We observe that if $f(\tilde{x}) = f_1(x_1) \cdot \ldots \cdot f_m(x_m)$ for $\tilde{x} \in R_0^m$ and if $f_k \in C_{q_k,1}$, $q_k \in N_0$ for $k = 1, \ldots, m$, then $f \in C_{\tilde{q},m}$ with $\tilde{q} = (q_1, \ldots, q_m)$ and moreover, for all $\tilde{x} \in R_0^m$ and $\tilde{n} \in N^m$, holds

(12) $$L_{\tilde{n}}(f;\tilde{x}) = \prod_{k=1}^{m} L_{n_k}(f_k;x_k),$$

(13) $$U_{\tilde{n}}(f;\tilde{x})=\prod_{k=1}^{m}U_{n_k}(f_k;x_k)\ .$$

In this paper we shall denote by $M_{a,b}$ the suitable positive constants depending only on indicated parameters.

2. Auxiliary results

2.1. First we shall give some properties of the operators L_n and U_n defined for functions of one variable by (4) and (5). Denote by

(14) $$T(nx):=\tanh nx-1 \qquad \text{for} \quad x\in R_0, \qquad n\in N.$$

By elementary calculations from (4) – (6) we obtain the following

Lemma 1. For all $x\in R_0$ and $n\in N$

$$L_n(1;x)=1=U_n(1;x)\ , \qquad L_n(t;x)=x+x\,T(nx),$$

$$L_n(t^2;x)=x^2+\frac{x}{n}+\frac{x}{n}\ T(nx)\ ,$$

$$U_n(t;x)=x+\frac{1}{n}+x\ T(nx)\ ,$$

$$U_n(t^2;x)=x^2+\frac{3x}{n}+\frac{4}{3n^2}+\frac{3x}{n}\ T(nx)\ ,$$

$$L_n\left((t-x)^2;x\right)=\frac{x}{n}+\left(\frac{x}{n}-2x^2\right)T(nx)\ ,$$

$$U_n\left((t-x)^2;x\right)=\frac{x}{n}+\frac{4}{3n^2}+\left(\frac{3x}{n}-2x^2\right)T(nx)\ . \qquad \blacksquare$$

Using the mathematical induction, we can prove

Lemma 2. For every $p\in N$ there exist the positive numbers $a_{p,k}$, $a^*_{p,k}$, $0\le k\le p$, and $b_{p,k}$, $b^*_{p,k}$, $1\le k\le\left[\frac{p+1}{2}\right]$, depending only on p,k and $a_{p,p}=a^*_{p,p}=1$ and such that for all $x\in R_0$ and $n\in N$

$$L_n(t^p;x)=\sum_{k=1}^{p}a_{p,k}\ \frac{x^k}{n^{p-k}}+T(nx)\sum_{k=1}^{\left[\frac{p+1}{2}\right]}b_{p,k}\ \frac{x^{2k-1}}{n^{p-(2k-1)}}\ ,$$

$$U_n(t^p;x) = \sum_{k=0}^{p} a^*_{p,k} \frac{x^k}{n^{p-k}} + T(nx) \sum_{k=1}^{\left[\frac{p+1}{2}\right]} b^*_{p,k} \frac{x^{2k-1}}{n^{p-(2k-1)}},$$

([y] *denotes the integral part of* y). ■

Applying Lemma 1, (14) and the inequality

(15) $$0 \le x^r \left| T(nx) \right| = \frac{2x^r}{e^{2nx}+1} \le 2^{1-r} r! n^{-r} \quad \text{for } x \ge 0,\, n \in N,\, r \in N,$$

we easily obtain the following

Lemma 3. For all $x \ge 0$ *and* $n \in N$ *we have*

$$L_n\left((t-x)^2;x\right) \le \frac{3(x+1)}{n},$$

$$U_n\left((t-x)^2;x\right) \le \frac{19}{4}\frac{x+1}{n}. \quad ■$$

Applying Lemma 2, (1) – (6) and (15), we shall prove

Lemma 4. For every fixed $q \in N_0$ *there exists a positive constant* M_q *such that for all* $n \in N$

(16) $$\left\| L_n\left(\frac{1}{w_q(t)};\cdot\right) \right\|_{q,1} \le M_q,$$

(17) $$\left\| U_n\left(\frac{1}{w_q(t)};\cdot\right) \right\|_{q,1} \le M_q.$$

Proof. We shall prove only (16), because the proof of (17) is analogous. The inequality (16) is obvious for $q=0$. By Lemma 2, (1) and (15) we get for every fixed $q \ge 1$

$$w_q(x)\, L_n\left(\frac{1}{w_q(t)};x\right) = w_q(x)\left(1+L_n(t^q;x)\right) \le$$

$$\le \frac{1}{1+x^q}\left\{1+x^q+\sum_{k=1}^{q-1} a_{q,k}\frac{x^k}{n^{q-k}} + \left|T(nx)\right| \sum_{k=1}^{\left[\frac{q+1}{2}\right]} b_{q,k} \frac{x^{2k-1}}{n^{q-(2k-1)}}\right\} \le$$

$$\leq 1+\sum_{k=1}^{q-1} a_{q,k}\,\frac{x^k}{1+x^q}+\sum_{k=1}^{\left[\frac{q+1}{2}\right]} b_{q,k}\,2^{-2k}(2k-1)!\,n^{-q} \leq$$

$$\leq 1+\sum_{k=1}^{q-1} a_{q,k}+\sum_{k=1}^{\left[\frac{q+1}{2}\right]} b_{q,k}\,2^{-2k}(2k-1)! = M_q\,,\quad n\in N,\ x\in R_0,$$

where M_q = const > 0 depending only on q. From these and by (3) follows (16). ■

Lemma 5. Let $f\in C_{q,1}$ *with some* $q\in N_0$. *Then there exists a positive constant* M_q *such that for all* $n\in N$

$$\|L_n(f;\cdot)\|_{q,1} \leq M_q\,\|f\|_{q,1}\,, \tag{18}$$

$$\|U_n(f;\cdot)\|_{q,1} \leq M_q\,\|f\|_{q,1}\,.$$

Proof. By (1), (3) and (4) we have

$$w_q(x)\left|L_n(f;x)\right| \leq \|f\|_{q,1}\,w_q(x)\sum_{k=1}^{\infty} p_{n,k}\left(w_q\left(\frac{2k}{n}\right)\right)^{-1} =$$

$$=\|f\|_{q,1}\,w_q(x)\,L_n\left(\frac{1}{w_q(t)};x\right) \leq \|f\|_{q,1}\left\|L_n\left(\frac{1}{w_q(t)};\cdot\right)\right\|_{q,1}$$

for all $x\geq 0$ and $n\in N$, which by Lemma 4 and (3) yields (18). The proof for U_n is analogous. ■

Lemma 6. For every fixed $q\in N_0$ *there exists a positive constant* M_q *such that*

$$w_q(x)\,L_n\left(\frac{(t-x)^2}{w_q(t)};x\right) \leq M_q\,\frac{x+1}{n}\,, \tag{19}$$

$$w_q(x)\,U_n\left(\frac{(t-x)^2}{w_q(t)};x\right) \leq M_q\,\frac{x+1}{n}\,,$$

for all $x\in R_0$ *and* $n\in N$.

Proof. We shall only (19) because the proof for U_n is analogous. The inequality (19) holds for $q=0$ by Lemma 3. For $1 \le q \in N_0$, $n \in N$ and $x \ge 0$ we have by Lemma 2

$$L_n\left(\frac{(t-x)^2}{w_q(t)};x\right)=$$
$$=L_n((t-x)^2;x)+L_n(t^{q+2};x)-2x\,L_n(t^{q+1};x)+x^2\,L_n(t^q;x)=$$
$$=L_n((t-x)^2;x)+\sum_{k=1}^{q+1}a_{q+2,k}\frac{x^k}{n^{q+2-k}}-2x\sum_{k=1}^{q}a_{q+1,k}\frac{x^k}{n^{q+1-k}}+$$
$$+x^2\sum_{k=1}^{q-1}a_{q,k}\frac{x^k}{n^{q-k}}+T(nx)\left\{\sum_{k=1}^{\left[\frac{q+3}{2}\right]}b_{q+2,k}\frac{x^{2k-1}}{n^{q+3-2k}}-\right.$$
$$\left.-2x\sum_{k=1}^{\left[\frac{q+2}{2}\right]}b_{q+1,k}\frac{x^{2k-1}}{n^{q+2-2k}}+x^2\sum_{k=1}^{\left[\frac{q+1}{2}\right]}b_{q,k}\frac{x^{2k-1}}{n^{q+1-2k}}\right\}.$$

(We assume that $\sum_{k=s}^{r}a_k=0$ if $r<s$). Now using Lemma 3 and (15), we get

$$w_q(x)\,L_n\left(\frac{(t-x)^2}{w_q(t)};x\right)\le$$
$$\le\frac{3(x+1)}{n}+\frac{x}{n}\left\{\sum_{k=1}^{q+1}a_{q+2,k}\frac{x^{k-1}}{1+x^q}\frac{1}{n^{q+1-k}}+\sum_{k=1}^{q}2a_{q+1,k}\frac{x^k}{1+x^q}\frac{1}{n^{q-k}}+\right.$$
$$\left.+\sum_{k=1}^{q-1}a_{q,k}\frac{x^{k-1}}{1+x^q}\frac{1}{n^{q-1-k}}\right\}+\sum_{k=1}^{\left[\frac{q+3}{2}\right]}b_{q+2,k}\,2^{2-2k}(2k-1)!\,n^{-q-2}+$$
$$+\sum_{k=1}^{\left[\frac{q+2}{2}\right]}2b_{q+1,k}\,2^{1-2k}(2k)!\,n^{-q-2}+\sum_{k=1}^{\left[\frac{q+1}{2}\right]}b_{q,2}\,2^{-2k}(2k+1)!\,n^{-q-2}\le$$
$$\le\frac{3(x+1)}{n}+\frac{x}{n}\left\{\sum_{k=1}^{q+1}a_{q+2,k}+\sum_{k=1}^{q}2a_{q+1,k}+\sum_{k=1}^{q-1}a_{q,k}\right\}+$$

$$+M_q \frac{1}{n^{q+2}} \leq M_q \frac{x+1}{n} \qquad \text{for all } x \geq 0 \text{ and } n \in N.$$

Thus the proof of (19) is completed. ■

2.2. In this part we shall give some properties of the operators $L_{\tilde{n}}$ and $U_{\tilde{n}}$, $\tilde{n} \in N^m$. From (2), (12) and (13) we get for every fixed $\tilde{q} = (q_1, \ldots, q_m) \in N_0^m$, ($m \in N$), and for all $\tilde{x} = (x_1, \ldots, x_m) \in R_0^m$, $\tilde{n} = (n_1, \ldots, n_m) \in N^m$

$$w_{\tilde{q}}(\tilde{x}) L_{\tilde{n}}\left(\frac{1}{w_{\tilde{q}}(\tilde{t})}; \tilde{x}\right) = \prod_{j=1}^{m} w_{q_j}(x_j) L_{n_j}\left(\frac{1}{w_{q_j}(t_j)}; x_j\right),$$

$$w_{\tilde{q}}(\tilde{x}) U_{\tilde{n}}\left(\frac{1}{w_{\tilde{q}}(\tilde{t})}; \tilde{x}\right) = \prod_{j=1}^{m} w_{q_j}(x_j) U_{n_j}\left(\frac{1}{w_{q_j}(t_j)}; x_j\right).$$

From this and by Lemma 4 we immediately obtain the following

Lemma 7. For every fixed $m \in N$ and $\tilde{q} \in N_0^m$ there exists a positive constant $M_{\tilde{q}}$ such that for each $n \in N$

$$\left\| L_{\tilde{n}}\left(\frac{1}{w_{\tilde{q}}(\tilde{t})}; \cdot\right) \right\|_{\tilde{q},m} \leq M_{\tilde{q}},$$

$$\left\| U_{\tilde{n}}\left(\frac{1}{w_{\tilde{q}}(\tilde{t})}; \cdot\right) \right\|_{\tilde{q},m} \leq M_{\tilde{q}}. \quad ■$$

Using Lemma 7 and arguing as in the proof of Lemma 5, we easily derive

Lemma 8. Let $f \in C_{\tilde{q},m}$ with some fixed $\tilde{q} \in N_0^m$ and $m \in N$. Then there exists a positive constant $M_{\tilde{q}}$ such that for all $\tilde{n} \in N^m$

$$\| L_{\tilde{n}}(f; \cdot) \|_{\tilde{q},m} \leq M_{\tilde{q}} \| f \|_{\tilde{q},m},$$

$$\| U_{\tilde{n}}(f; \cdot) \|_{\tilde{q},m} \leq M_{\tilde{q}} \| f \|_{\tilde{q},m}.$$

These inequalities and (9), (10) show that $L_{\tilde{n}}$ *and* $U_{\tilde{n}}$, $\tilde{n} \in N^m$, *are linear positive operators from* $C_{\tilde{q},m}$ *into* $C_{\tilde{q},m}$, $\tilde{q} \in N_0^m$. ■

3. Approximation theorems

In this part we shall prove two theorems on the degree of approximation of functions belonging to a class $C_{\tilde{q},m}$ by $L_{\tilde{n}}$ and $U_{\tilde{n}}$. For fixed $m \in N$ and $\tilde{q} \in N_0^m$ we denote

$$C^1_{\tilde{q},m} := \left\{ f \in C_{\tilde{q},m} : \frac{\partial f}{\partial x_k} \in C_{\tilde{q},m},\ 1 \le k \le m \right\}.$$

Theorem 1. Suppose that $g \in C^1_{\tilde{q},m}$ *with some fixed* $m \in N$ *and* $\tilde{q} \in N_0^m$. *Then there exists a positive constant* $M_{\tilde{q}}$ *depending only on* $\tilde{q}$ *such that for all* $\tilde{x} = (x_1, \dots, x_m) \in R_0^m$ *and* $\tilde{n} = (n_1, \dots, n_m) \in N^m$

$$\text{(20)} \qquad w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(g(\tilde{t}); \tilde{x}) - g(\tilde{x}) \right| \le M_{\tilde{q}} \sum_{k=1}^{m} \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \left(\frac{x_k + 1}{n_k} \right)^{\frac{1}{2}},$$

$$\text{(21)} \qquad w_{\tilde{q}}(\tilde{x}) \left| U_{\tilde{n}}(g(\tilde{t}); \tilde{x}) - g(\tilde{x}) \right| \le M_{\tilde{q}} \sum_{k=1}^{m} \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \left(\frac{x_k + 1}{n_k} \right)^{\frac{1}{2}}.$$

Proof. We shall prove only (20) because the proof of (21) is analogous. Fix $\tilde{x} = (x_1, \dots, x_m) \in R_0^m$. For every $\tilde{t} = (t_1, \dots, t_m) \in R_0^m$ we have

$$g(\tilde{t}) - g(\tilde{x}) = \sum_{k=1}^{m} \int_{x_k}^{t_k} \frac{\partial g}{\partial u_k}(\tilde{y}_k)\, du_k,$$

where $\tilde{y}_k = (x_1, \dots, x_{k-1}, u_k, t_{k+1}, \dots, t_m)$. Further by (9) and (11) we get for every $\tilde{n} = (n_1, \dots, n_m) \in N^m$

$$L_{\tilde{n}}\left(g(\tilde{t}); \tilde{x} \right) - g(\tilde{x}) = \sum_{k=1}^{m} L_{\tilde{n}} \left(\int_{x_k}^{t_k} \frac{\partial g}{\partial u_k}(\tilde{y}_k)\, du_k ; \tilde{x} \right)$$

and

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}\left(g(\tilde{t}); \tilde{x} \right) - g(\tilde{x}) \right| \le \sum_{k=1}^{m} w_{\tilde{q}}(\tilde{x})\, L_{\tilde{n}} \left(\left| \int_{x_k}^{t_k} \frac{\partial g}{\partial u_k}(\tilde{y}_k)\, du_k \right| ; \tilde{x} \right).$$

But by (1) – (3) follows

$$\left| \int_{x_k}^{t_k} \frac{\partial g}{\partial u_k}(\tilde{y}_k)\,du_k \right| \le \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \left| \int_{x_k}^{t_k} \frac{1}{w_{\tilde{q}}(\tilde{y}_k)}\,du_k \right| =$$

$$= \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \frac{1}{w_{q_1}(x_1)\ldots w_{q_{k-1}}(x_{k-1})\,w_{q_{k+1}}(t_{k+1})\ldots w_{q_m}(t_m)} \left| \int_{x_k}^{t_k} \frac{1}{w_{\tilde{q}}(u_k)}\,du_k \right|,$$

$$\left| \int_{x_k}^{t_k} \frac{1}{w_{q_k}(u_k)}\,du_k \right| \le \left(\frac{1}{w_{q_k}(t_k)} + \frac{1}{w_{q_1}(x_k)} \right) | t_k - x_k |,$$

for $k = 1, 2, \ldots, m$. From this and (12) and Lemma 4 we derive

$$w_{\tilde{q}}(\tilde{x})\, L_{\tilde{n}}\left(\left| \int_{x_k}^{t_k} \frac{\partial g}{\partial u_k}(\tilde{y}_k)\,du_k \right|; \tilde{x} \right) \le$$

$$\le \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \left(\prod_{j=k+1}^{m} w_{q_j}(x_j)\, L_{n_j}\left(\frac{1}{w_{q_j}(t_j)}; x_j \right) \right) \times$$

$$\times \left\{ w_{q_k}(x_k)\, L_{n_k}\left(\frac{|t_k - x_k|}{w_{q_k}(t_k)}; x_k \right) + L_{n_k}\left(|t_k - x_k|; x_k \right) \right\} \le$$

$$\le M_{\tilde{q}} \left\| \frac{\partial g}{\partial x_k} \right\|_{\tilde{q},m} \left\{ w_{q_k}(x_k)\, L_{n_k}\left(\frac{|t_k - x_k|}{w_{q_k}(t_k)}; x_k \right) + L_{n_k}\left(|t_k - x_k|; x_k \right) \right\}.$$

Using the Hölder inequality and Lemmas 6 and 4, we obtain

$$w_{q_k}(x_k)\, L_{n_k}\left(\frac{|t_k - x_k|}{w_{q_k}(t_k)}; x_k \right) \le$$

$$\le \left\{ w_{q_k}(x_k)\, L_{n_k}\left(\frac{(t_k - x_k)^2}{w_{q_k}(t_k)}; x_k \right) \right\}^{\frac{1}{2}} \left\{ w_{q_k}(x_k)\, L_{n_k}\left(\frac{1}{w_{q_k}(t_k)}; x_k \right) \right\}^{\frac{1}{2}} \le$$

$$\le M_{q_k} \left(\frac{x_k + 1}{n_k} \right)^{\frac{1}{2}}$$

and analogously by Lemma 3 and (11),

$$L_{n_k}\left(\left|t_k - x_k\right|; x_k\right) \le \left\{L_{n_k}\left((t_k - x_k)^2 ; x_k\right)\right\}^{\frac{1}{2}} \left\{L_{n_k}\left(1; x_k\right)\right\}^{\frac{1}{2}} \le$$

$$\le \left\{\frac{3(x_k+1)}{n_k}\right\}^{\frac{1}{2}},$$

for all $n_k \in N$ and $x_k \ge 0$, $k = 1, 2, \ldots, m$.

Summarizing, we obtain the inequality (20) for all $\tilde{n} = (n_1, \ldots, n_m) \in N^m$ and $\tilde{x} = (x_1, \ldots, x_m) \in R_0^m$. ■

Theorem 2. Suppose that $f \in C_{\tilde{q},m}$ *with some* $m \in N$ *and* $\tilde{q} = (q_1, \ldots, q_m) \in N_0^m$. *Then there exists a positive constant* $M_{\tilde{q}}$ *depending only on* $\tilde{q}$ *such that for all* $\tilde{x} = (x_1, \ldots, x_m) \in R_0^m$ *and* $\tilde{n} = (n_1, \ldots, n_m) \in N^m$

(22) $$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f; \tilde{x}) - f(\tilde{x}) \right| \le$$

$$\le M_{\tilde{q}}\, \omega\left(f, C_{\tilde{q},m}; \left(\frac{x_1+1}{n_1}\right)^{\frac{1}{2}}, \ldots, \left(\frac{x_m+1}{n_m}\right)^{\frac{1}{2}}\right),$$

(23) $$w_{\tilde{q}}(\tilde{x}) \left| U_{\tilde{n}}(f; \tilde{x}) - f(\tilde{x}) \right| \le$$

$$\le M_{\tilde{q}}\, \omega\left(f, C_{\tilde{q},m}; \left(\frac{x_1+1}{n_1}\right)^{\frac{1}{2}}, \ldots, \left(\frac{x_m+1}{n_m}\right)^{\frac{1}{2}}\right).$$

Proof. Let $f_{\tilde{h}}$ be the Steklov mean of $f \in C_{\tilde{q},m}$, i.e.

$$f_{\tilde{h}}(\tilde{x}) := \left(\prod_{k=1}^{m} h_k\right)^{-1} \int_{\tilde{\tilde{0}}}^{\tilde{h}} f(\tilde{x} + \tilde{u})\, d\tilde{u}$$

for $\tilde{h} \equiv (h_1, \ldots, h_m) > \tilde{\tilde{0}}$ and $\tilde{x} \in R_0^m$. We have

$$f_{\tilde{h}}(\tilde{x}) - f(\tilde{x}) = \left(\prod_{k=1}^{m} h_k\right)^{-1} \int_{\tilde{\tilde{0}}}^{\tilde{h}} \left(f(\tilde{x} + \tilde{u}) - f(\tilde{x})\right) d\tilde{u}$$

and for $k = 1, \ldots, m$

$$\frac{\partial}{\partial x_k} f_{\tilde{h}}(\tilde{x}) =$$

$$= \left(\prod_{k=1}^{m} h_k\right)^{-1} \int_0^{h_1} \dots \int_0^{h_{k-1}} \int_0^{h_{k+1}} \dots \int_0^{h_m} \left(f(\tilde{x}+\tilde{u}') - f(\tilde{x}+\tilde{u}'')\right) du_1 \dots du_{k-1} du_{k+1} \dots du_m$$

where $\tilde{u}' = (u_1, \dots, u_{k-1}, h_k, u_{k+1}, \dots, u_m)$, $\tilde{u}'' = (u_1, \dots, u_{k-1}, 0, u_{k+1}, \dots, u_m)$. From this we get

$$(24) \qquad \left\| f_{\tilde{h}} - f \right\|_{\tilde{q},m} \le \omega(f, C_{\tilde{q},m}; \tilde{h}),$$

$$(25) \qquad \left\| \frac{\partial f_{\tilde{h}}}{\partial x_k} \right\|_{\tilde{q},m} \le 2h_k^{-1}\, \omega(f, C_{\tilde{q},m}; \tilde{h}), \qquad \tilde{h} > \tilde{\tilde{0}},$$

which implies $f_{\tilde{h}} \in C^1_{\tilde{q},m}$. Hence, for every $\tilde{h} > \tilde{\tilde{0}}$, $\tilde{x} \in R_0^m$ and $n \in N^m$, we can write

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f; \tilde{x}) - f(\tilde{x}) \right| \le w_{\tilde{q}}(\tilde{x}) \Big\{ \left| L_{\tilde{n}}\left(f - f_{\tilde{h}}; \tilde{x}\right) \right| +$$

$$+ \left| L_{\tilde{n}}\left(f_{\tilde{h}}; \tilde{x}\right) - f_{\tilde{h}}(\tilde{x}) \right| + \left| f_{\tilde{h}}(\tilde{x}) - f(\tilde{x}) \right| \Big\}$$

By Lemma 8 and (24) it follows that

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f - f_{\tilde{h}}; \tilde{x}) \right| \le M_{\tilde{q}} \left\| f - f_{\tilde{h}} \right\|_{\tilde{q},m} \le M_{\tilde{q}}\, \omega\left(f, C_{\tilde{q},m}; \tilde{h}\right).$$

Using Theorem 1 and (25), we get

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f; \tilde{x}) - f(\tilde{x}) \right| \le M_{\tilde{q}} \sum_{k=1}^{m} \left\| \frac{\partial f_{\tilde{h}}}{\partial x_k} \right\|_{\tilde{q},m} \left(\frac{x_k + 1}{n_k}\right)^{\frac{1}{2}} \le$$

$$\le 2M_{\tilde{q}}\, \omega\left(f, C_{\tilde{q},m}; \tilde{h}\right) \sum_{k=1}^{m} h_k^{-1} \left(\frac{x_k + 1}{n_k}\right)^{\frac{1}{2}}.$$

Consequently,

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f_{\tilde{h}}; \tilde{x}) - f_{\tilde{h}}(\tilde{x}) \right| \le$$

$$\le M_{\tilde{q}}\, \omega\left(f, C_{\tilde{q},m}; h_1, \dots, h_m\right) \left\{ 1 + \sum_{k=1}^{m} h_k^{-1} \left(\frac{x_k + 1}{n_k}\right)^{\frac{1}{2}} \right\}$$

for all $\tilde{h} \equiv (h_1, \ldots, h_m) > \tilde{\tilde{0}}$. Choosing $\tilde{h}$ with $h_k = \left(\frac{x_k+1}{n_k}\right)^{\frac{1}{2}}$, $k = 1, \ldots, m$, we obtain the desired estimation (22).

The proof of (23) is analogous. ∎

From Theorem 2 we can derive the following two corollaries

Corollary 1. Let $f \in C_{\tilde{q},m}$ with some fixed $m \in N$ and $\tilde{q} \in N_0^m$. Then

$$\lim_{\tilde{n} \to \tilde{\tilde{\infty}}} L_{\tilde{n}}(f;\tilde{x}) = f(\tilde{x}),$$

$$\lim_{\tilde{n} \to \tilde{\tilde{\infty}}} U_{\tilde{n}}(f;\tilde{x}) = f(\tilde{x}),$$

for every $\tilde{x} \in R_0^m$.

Corollary 2. Let $f \in \mathrm{Lip}\left(C_{\tilde{q},m}, \tilde{\alpha}\right)$ with some fixed $m \in N$, $\tilde{q} \in N_0^m$ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $0 < \alpha_k \le 1$ for $k = 1, \ldots, m$. Then there exists a positive constant $M_{\tilde{q}}$ such that for all $\tilde{x} = (x_1, \ldots, x_m) \in R_0^m$ and $\tilde{n} = (n_1, \ldots, n_m) \in N^m$

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}(f;\tilde{x}) - f(\tilde{x}) \right| \le M_{\tilde{q}} \sum_{k=1}^{m} \left(\frac{x_k+1}{n_k}\right)^{\frac{\alpha_k}{2}},$$

$$w_{\tilde{q}}(\tilde{x}) \left| U_{\tilde{n}}(f;\tilde{x}) - f(\tilde{x}) \right| \le M_{\tilde{q}} \sum_{k=1}^{m} \left(\frac{x_k+1}{n_k}\right)^{\frac{\alpha_k}{2}}.$$

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