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# ON SIMULTANEOUS APPROXIMATION BY COMBINATIONS OF BASKAKOV-SZÁSZ TYPE OPERATORS

In this paper, we study some direct results in simultaneous approximation for the linear combination of Baskakov-Szász type operators.

Key words: Baskakov operators, order of approximation, linear combinations, modulus of continuity.

## 1. Introduction

The integral modification of well known Baskakov operators is defined as

$$(1.1)\qquad S_n(f,x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu}(t)\,f(t)\,dt\,, \qquad x\in[0,\infty)$$

where $p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-n-\nu}$ and $q_{n,\nu}(t) = \dfrac{e^{-nt}(nt)^{\nu}}{\nu!}$.

The above integral modification of Baskakov operator was recently defined by the authors (see e.g.[3]) to approximate Lebesgue integrable functions on $[0,\infty)$. From [3], we may remark that the order of approximation by the operators (1.1) is at best $O(n^{-1})$, howsoever smooth the function may be. Thus if we want to have a better order of approximation for smoothner functions, we have to slacken the positivity conditions. For this we define the linear combinations.

For a fixed natural $k$ and arbitrary but fixed distinct positive integers $d_j$, $j=0,1,2,\dots,k$, the linear combinations $S_n(f,k,x)$ of $S_{d_jn}(f,x)$ are defined as

$$S_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\,S_{d_jn}(f,x)$$

where

$$(1.2)\qquad C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j-d_i}, \qquad k\neq 0, \quad C(0,0)=1$$

Alternately, the linear combination $S_n(f,k,x)$ of $S_{d_j n}(f,x)$, $j = 0,1,\ldots,k$ may be written as

$$(1.3) \qquad S_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} S_{d_0 n}(f,x) & d_0^{-1} & \ldots & d_0^{-k} \\ S_{d_1 n}(f,x) & d_1^{-1} & \ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots \\ S_{d_k n}(f,x) & d_k^{-1} & \ldots & d_k^{-k} \end{vmatrix}$$

where $\Delta$ is the Vondermonde determinant obtained by replacing the operator column of the above determinant by the entries 1.

We denote the class of all measurable functions defined on $[0,\infty)$ such that

$$L = \left\{ f : \int_0^\infty e^{-nt} f(t)\, dt < \infty \text{ for some positive integer n} \right\}.$$

Obviously, $L_1[0,\infty) \subset L$ and hence $S_n$ may be utilised for studying a larger class of functions. The object of the present paper is to study an asymptatic formula and an error estimate for the linear combinations of the operators (1.1).

## 2. Basic Results

To prove the main results, we require the following lemmas and corollary.

*Lemma 2.1 [3]. For* $m \in \mathrm{N}^0$ *(the set of non-negative integers), if*

$$U_{n,m}(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}(x)\left(\frac{\nu}{n} - x\right)^m$$

*then there holds the recurrence relation*

$$n\, U_{n,m+1}(x) = x(1+x)\left[U_{n,m}^{(1)}(x) + m\, U_{n,m-1}(x)\right], \quad \text{for } m \geq 2.$$

Consequently

(i) $U_{n,m}(x)$ is a polynomial in $x$ of degree at most $m$.

(ii) For every fixed $x \geq 0$, $U_{n,m}(x) = O_x(n^{-[(m+1)/2]})$ where $[\alpha]$ denotes the integral part of $\alpha$.

*Lemma 2.2. Let r, $m \in N^0$ (the set of non-negative integers), $n \in N$ and $x \in [0,\infty)$. If we define*

$$(2.1)\qquad T_{r,n,m}(x) = n\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}(t)\,(t-x)^m dt$$

*then there holds*

$$n\,T_{r,n,m+1}(x) = x(1+x)\,T^{(1)}{}_{r,n,m+1}(x) +$$
$$+[m+1+r(1+x)]\,T_{r,n,m}(x) + mxT_{r,n,m-1}(x),\quad m \geq 2.$$

Consequently

(i) We have $T_{r,n,0}(x)=1$, $\qquad T_{r,n,1}(x) = \dfrac{1+r(1+x)}{n}$

and $\qquad T_{r,n,2}(x) = \dfrac{rx(1+x)+1+[1+r(1+x)]^2+nx(2+x)}{n^2}$

(ii) For every fixed $x \geq 0$, $\qquad T_{r,n,m}(x) = O_x(n^{-[(m+1)/2]})$.

*Proof.* We first prove (2.1), by using

$$t\,q_{n,\nu}^{(1)}(t) = (\nu - nt)\,q_{n,\nu}(t); \qquad x(1+x)\,p_{n,\nu}^{(1)}(x) = (\nu - nx)\,p_{n,\nu}(x)$$

and from the definition of $T_{r,n,m}(x)$, we have

$$x(1+x)T_{r,n,m}^{(1)} = n\sum_{\nu=0}^{\infty} x(1+x)\,p_{n+r,\nu}^{(1)}(x)\int_0^{\infty} q_{n,\nu+r}(t)(t-x)^m dt -$$

$$-mn\sum_{\nu=0}^{\infty} x(1+x)p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}(t)(t-x)^{m-1}dt =$$

$$= n\sum_{\nu=0}^{\infty}[\nu-(n+r)x]p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}(t)(t-x)^m dt -$$

$$-mx\,(1+m)\,T_{r,n,m-1}(x).$$

Therefore,

$$x(1+x)\left[T_{r,n,m}^{(1)}(x) + m\,T_{r,n,m-1}(x)\right] =$$

$$= n\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty}\,[(\nu+r-nt)+n(t-x)-$$

$$-r(1+x)]q_{n,\nu+r}(t)(t-x)^m dt =$$

$$= n\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} t\, q_{n,\nu+r}^{(1)}(t)(t-x)^m dt +$$

$$+ n\,T_{r,n,m+1}(x) - r(1+x)T_{r,n,m}(x) =$$

$$= n\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}^{(1)}(t)(t-x)^{m+1} dt +$$

$$+ nx\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}^{(1)}(t)(t-x)^m dt +$$

$$+ n\,T_{r,n,m+1}(x) - r(1+x)\,T_{r,n,m}(x) =$$

$$= -(m+1)\,T_{r,n,m}(x) - mx\,T_{r,n,m-1}(x) + n\,T_{r,n,m+1}(x) - r(1+x)\,T_{r,n,m}(x).$$

This leads to the proof of (2.1).

The other consequences follow from the recurrence relaiton (2.1) and the definition of $T_{r,n,m}(x)$. This completes the proof of lemma 2.1.

*Corollary 2.3 [3] Let* $\alpha$ *and* $\delta$ *be positive numbers. Then for every* $m \in N$ *and* $x \in [0,\infty)$, *there exists a positive constant* $M_{m,x}$ *depending on m and x such that*

$$n\sum_{\nu=0}^{\infty} p_{n,\nu}\, q_{n,\nu}\, e^{\alpha t}\, dt \le M_{m,x}\, n^{-m}.$$

*Lemma 2.4. If f is r times* $(r = 1,2,...)$ *differentiable on* $[0,\infty)$, *then we have*

$$S_n^{(r)}(f,x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}(t)\, f^{(r)}(t)\,dt.$$

*Proof.* We have by (1.1) that

$$S_n^{(r)}(f,x) = n\sum_{\nu=0}^{\infty} p_{n,\nu}^{(r)}(x)\int_0^{\infty} q_{n,\nu}(t)\, f(t)\,dt.$$

By using Leibnitz theorem, we have

$$S_n^{(r)}(f,x)=$$

$$=n\sum_{i=0}^{\infty}\sum_{v=i}^{\infty}\binom{r}{i}\frac{(n+v+r-i-1)!}{(n-1)!(v-i)!}(-1)^{r-i}x^{v-i}(1+x)^{-n-v-r-i}\int_0^{\infty}q_{n,v}(t)f(t)dt=$$

$$=n\sum_{v=0}^{\infty}\frac{(n+v+r-1)!}{(n-1)!v!}\frac{x^v}{(1+x)^{n+v+r}}\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}q_{n,v+i}(t)f(t)dt=$$

$$=n\frac{(n+r-1)!}{(n-1)!}\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}q_{n,v+i}(t)f(t)dt\ .$$

Next using again Leibnitz theorem, we get

$$q_{n,v+r}^{(r)}(t)=\sum_{i=0}^{r}\binom{r}{i}(-1)^i n^r\frac{e^{-nt}(nt)^{v+i}}{(v+i)!}=n^r\sum_{i=0}^{r}(-1)^i\binom{r}{i}q_{n,v+i}(t)\ .$$

Therefore,

$$S_n^{(r)}(f,x)=\frac{(n+r-1)!}{(n-1)!n^{r-1}}\sum_{v=0}^{\infty}p_{n+r,v}(x)\int_0^{\infty}q_{n,v+r}^{(r)}(t)(-1)^r f(t)dt$$

integrating by parts $r$ times, we get the desired result.

*Lemma 2.5 [4]. There exist the polynomials* $\phi_{i,j,r}(x)$ *independed of n and v such that*

$$\frac{d^r}{dx^r}\left[x^v(1+x)^{-n-v}\right]=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}n^i(v-nx)^j\phi_{i,j,r}(x)x^{v-r}(1+x)^{-n-v-r}\ .$$

*Lemma 2.6 [5]. If C(j,k), j = 0,1,2,..., are defined as in (1.2), then*

$$\sum_{j=0}^{k}C(j,k)\,d_j^{-m}=\begin{cases}1 & m=0,\\ 0 & m=1,2,\dots,k\ .\end{cases}$$

## 3. Main Results

In this section, we shall prove the main results.

*Theorem 3.1. Let* $f \in L$ *be bounded on every finite subinterval of* $[0,\infty)$ *and* $f^{(2k+r+2)}(x)$ *exists at a fixed point* $x \in (0,\infty)$. *Let* $f(t) = O(e^{\alpha t})$ *as* $t \to \infty$ *for some* $\alpha > 0$, *then we have*

$$\text{(3.1)} \qquad \lim_{n\to\infty} n^{k+1}\left[\alpha(n,r,k)\, S_n^{(r)}(f,k,x) - f^{(r)}(x)\right] = \sum_{i=r+1}^{2k+r+2} Q(i,k,r,x) f^{(i)}(x),$$

$$\text{(3.2)} \qquad \lim_{n\to\infty} n^{k+1}\left[\alpha(n,r,k)\, S_n^{(r)}(f,k,x) - f^{(r)}(x)\right] = 0,$$

*where* $$\alpha(n,r,k) = \left[\sum_{j=0}^{k} C(j,k)\, \frac{(d_j n + r - 1)!}{d_j n^r (d_j n - 1)!}\right]^{-1}$$

*and Q(i,k,r,x) are certain polynomials in x of degree at most i. Further if* $f^{(2k+r+2)}$ *exists and is continuous on* $(a-\eta, b+\eta)$, $\eta > 0$ *then* (3.1) *and* (3.2) *hold uniformly in [a,b].*

*Proof.* First, by the Taylor's expansion of $f$ we have

$$f(t) = \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x)(t-x)^{2k+r+2}$ is of exponential order as $t \to \infty$. Such a formulation of Taylor's theorem has been used by many authors see e.g. May [5, Prop. 3.6], Derriennic [1, Th. II. 5] and Gupta [2, Th 1] etc. Now, using Lemma 2.4 and the above Taylor's expansion, we have

$$n^{k+1}\left[\alpha(n,r,k)\, S_n^{(r)}(f,k,x) - f^{(r)}(x)\right] =$$

$$= n^{k+1}\left[\alpha(n,r,k)\sum_{j=0}^{k} C(j,k)\, S_{d_j n}^{(r)}(f,x) - f^{(r)}(x)\right] =$$

$$= n^{k+1}\left[\alpha(n,r,k) \sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} C(j,k) \frac{(d_j n + r - 1)!}{d_j n^{r-1}(d_j n - 1)!} \times \right.$$

$$\left. \times \sum_{v=0}^{\infty} p_{d_j n+r,v}(x) \int_0^\infty q_{d_j n, v+r}(t) \frac{d^r}{dx^r}(t-x)^i\, dt - f^{(r)}(x)\right] +$$

$$+\alpha(n,r,k)n^{k+1}\sum_{j=0}^{k}C(j,k)\sum_{\nu=0}^{\infty}p_{d_j n,\nu}^{(r)}(x)\int_0^{\infty}q_{d_j n,\nu}(t)\varepsilon(t,x)(t-x)^{2k+r+2}dt=$$

$$=\sum_{i=r+1}^{2k+r+2}Q(i,k,r,x)f^{(i)}(x)+E_{n,r,k}(x)+o(1),$$

by Lemma 2.2, where

$$E_{n,r,k}(x)=\alpha(n,r,k)\,n^{k+1}\sum_{j=0}^{k}C(j,k)\sum_{\nu=0}^{\infty}p_{d_j n,\nu}^{(r)}\times$$

$$\times\int_0^{\infty}q_{d_j n,\nu}(t)\,\varepsilon(t,x)(t-x)^{2k+r+2}dt\;.$$

Thus to complete the proof of (3.1), it suffices to show that

$$I=n^{k+1}\sum_{\nu=0}^{\infty}p_{n,\nu}^{(r)}(x)\int_0^{\infty}q_{n,\nu}(t)\varepsilon(t,x)(t-x)^{2k+r+2}dt$$

tends to zero as $n\to\infty$.

Next, by Lemma 2.5 we have

$$|I|\le n^{k+1}\sum_{\nu=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge0}}n^i|\nu-nx|^j\,|\phi_{i,j,r}(x)|p_{n,\nu}(x)\int_0^{\infty}q_{n,\nu}(t)|\varepsilon(t,x)||t-x|^{2k+r+2}dt\le$$

$$\le n^{k+1}K(x)\sum_{\substack{2i+j\le r\\ i,j\ge0}}n^i\sum_{\nu=0}^{\infty}p_{n,\nu}(x)|\nu-nx|^j\int_0^{\infty}q_{n,\nu}(t)|\varepsilon(t,x)||t-x|^{2k+r+2}dt\le$$

$$\le n^{k+1}K(x)\sum_{\substack{2i+j\le r\\ i,j\ge0}}n^i\left[\sum_{\nu=0}^{\infty}p_{n,\nu}(x)(\nu-nx)^{2j}\right]^{1/2}\times$$

$$\times\left[\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\left(\int_0^{\infty}q_{n,\nu}(t)|\varepsilon(t,x)||t-x|^{2k+r+2}dt\right)^2\right]^{1/2}$$

where $K(x)=\sup\limits_{\substack{2i+j\le r\\ i,j\ge0}}|\phi_{i,j,r}(x)|$.

For a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. For $|t-x|\ge\delta$ we have $|\varepsilon(t,x)||t-x|^{2k+r+2}\le$ $\le Ce^{\alpha t}$. Thus

$$\left[\int_0^\infty q_{n,v}(t)|\varepsilon(t,x)||t-x|^{2k+r+2}\,dt\right]^2 \le$$

$$\le \frac{1}{n}\left[\int_{|t-x|<\delta} + \int_{|t-x|\ge\delta}\right] q_{n,v}(t)(\varepsilon(t,x))^2(t-x)^{4k+2r+4}\,dt \le$$

$$\le \frac{1}{n}\left\{\int_{|t-x|<\delta} q_{n,v}(t)\,\varepsilon^2(t-x)^{4k+2r+4}\,dt + \int_{|t-x|\ge\delta} q_{n,v}(t)\,C^2 e^{\alpha t}\,dt\right\}.$$

Applying Lemma 2.2 and Corollary 2.3, we get

$$\sum_{v=0}^{\infty} p_{n,v}(x)\left[\int_0^\infty q_{n,v}(t)|\varepsilon(t,x)||t-x|^{2k+r+2}\,dt\right]^2 \le$$

$$\le \frac{1}{n}\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^\infty q_{n,v}(t)\,\varepsilon^2(t-x)^{4k+2r+4}\,dt +$$

$$+\frac{C^2}{n}\sum_{v=0}^{\infty} p_{n,v}(x)\int_{|t-x|\ge\delta} q_{n,v}(t)e^{\alpha t}\,dt =$$

$$= \varepsilon^2 O(n^{-(2k+r+4)}) + O(n^{-m-2}).$$

Thus, using Lemma 2.1, we have

$$|I| \le n^{k+1}K(x)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j}\cdot O(n^{-j/2})\left\{\varepsilon^2 O(n^{-(2k+r+4)}) + O(n^{-m-2})\right\}^{1/2} =$$

$$= \varepsilon^2 O(1),$$

choosing $m > 2k+r+2$ and $n$ sufficiently large, this implies that $I \to \infty$ as $n \to \infty$. This completes the proof of (3.1).

Next, we prove (3.2), using Lemma 2.2, we have

$$S_{d_j n}((t-x)^i, x) = \frac{P_0(x)}{(d_j n)^{[(i+1)/2]}} + \frac{P_1(x)}{(d_j n)^{[(i+1)/2]+1}} + \cdots + \frac{P_{[i/2]}(x)}{(d_j n)^i}$$

for certain polynomials $P_m$; $m = 0,1,2,\ldots[i/2]$ in $x$ of degree at most $[i/2]$. Clearly

$$S_n((t-x)^i, k+1, x) = \sum_{j=0}^{k+1} C(j,k+1)\,S_{d_j n}((t-x)^i, x) =$$

$$= \frac{1}{\Delta}\begin{vmatrix} \frac{P_0(x)}{(d_0 n)^{[(i+1)/2]}} & +\frac{P_1(x)}{(d_0 n)^{[(i+1)/2]+1}} & +\cdots & +\frac{P_{[i/2]}(x)}{(d_0 n)^i} & d_0^{-1} & \cdots & d_0^{-(k+1)} \\ \frac{P_0(x)}{(d_1 n)^{[(i+1)/2]}} & +\frac{P_1(x)}{(d_1 n)^{[(i+1)/2]+1}} & +\cdots & +\frac{P_{[i/2]}(x)}{(d_1 n)^i} & d_1^{-1} & \cdots & d_1^{-(k+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{P_0(x)}{(d_{k+1} n)^{[(i+1)/2]}} & +\frac{P_1(x)}{(d_{k+1} n)^{[(i+1)/2]+1}} & +\cdots & +\frac{P_{[i/2]}(x)}{(d_{k+1} n)^i} & d_{k+1}^{-1} & \cdots & d_{k+1}^{-(k+1)} \end{vmatrix}$$

$$= n^{-(k+2)}\{Q(i,k,x)+o(1)\}, \qquad i = k+2,\ k+3\ ,\ldots,\ 2k+r+2$$

where $\Delta$ is as defined in (1.3).

Hence the assertation (3.2) can be proved along similar lines as that of (3.1) by using the above fact.

The last assertion follows due to uniform continuity of $f^{(2k+r+2)}$ on $[a,b]$, (enabling $\delta$ to become independent of $x \in [a,b]$).

*Theorem* 3.2. *Let $f \in L$ be bounded on every finite subinterval of $[0,\infty)$. Also let $f(t) = O(e^{\alpha t})$ as $t \to \infty$ for some $\alpha > 0$. If $f^{(p+r)}$, $1 \le p \le 2k+2$ exists and is continuous on $(a-\eta, b+\eta), \eta > 0$, having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $(a-\eta, b+\eta)$. Then for n sufficiently large:*

$$\left\| \alpha\,(n,r,k)\, S_n^{(r)}(f,k,\cdot) - f^{(r)} \right\|_{C[a,b]} \le \mathrm{Max}\left\{C_1 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}), C_2 n^{-(k+1)}\right\}$$

*where* $C_1 = C_1(k,p,r)$, $C_2 = C_2(k,p,r,f)$ *and* $\alpha(n,r,k)$ *as defined in Theorem* 3.1.

*Proof.* For every $t \in [0,\infty)$ and $x \in [a,b]$, we have by Taylor's expansion

$$(3.3) \qquad f(t) = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + $$

$$+ \frac{\left\{f^{(p+r)}(\xi) - f^{(p+r))}(x)\right\}}{(p+r)!}(t-x)^{p+r}\chi + h(t,x)(1-\chi(t)),$$

where $\xi$ lies between $t$ and $x$ and $\chi(t)$ denotes the characteristic function of $(a-\eta, b+\eta)$. For $t \in [0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i .$$

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we have

$$f(t) = \sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{\{f^{(p+r)}(\xi) - f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r} .$$

Now, using (3.3) and Lemma 2.4, we have

$$\alpha(n,r,k)\, S_n^{(r)}(f,k,x) - f^{(r)}(x) =$$

$$= \alpha(n,r,k)\left[\left\{\sum_{i=0}^{p+r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} C(j,k) \frac{(d_j n + r - 1)!}{d_j n^{r-1}(d_j n - 1)!} \sum_{v=0}^{\infty} p_{d_j n+r,v}(x) \times \right.\right.$$

$$\left. \times \int_0^\infty q_{d_j n,v+r}(t) \frac{d^r}{dx^r}(t-x)^i dt - f^{(r)}(x) \right\} +$$

$$+ \sum_{j=0}^{k} C(j,k) d_j n \sum_{v=0}^{\infty} p_{d_j n,v}^{(r)}(x) \int_0^\infty q_{d_j n,v}(t) \frac{\{f^{(p+r)}(\xi) - f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}\chi(t)dt +$$

$$\left. + \sum_{j=0}^{k} C(j,k)\, d_j n \sum_{v=0}^{\infty} P_{d_j n,v}^{(r)}(x) \int_0^\infty q_{d_j n,v}(t) h(t,x)(1-\chi(t))\, dt \right] =$$

$= J_1 + \alpha(n,r,k)(J_2 + J_3)$, say.

Making use of Lemma 2.2 and Lema 2.6, we obtain

$$J_1 \le K_1 n^{-(k+1)}, \text{ uniformly for } x \in [a,b]$$

where $K_1 = K_1(k,p,r,f)$.

Next, to estimate $J_2$, let

$$J_4 = n \sum_{v=0}^{\infty} p_{n,v}^{(r)}(x) \int_0^\infty q_{n,v}(t) \frac{\{f^{(p+r)}(\xi) - f^{(p+r)}(x)\}}{(p+r)!}(t-x)^{p+r}\chi(t)\, dt .$$

Using Lemma 2.5, we get

$$|J_4| \le n \sum_{v=0}^{\infty} \sum_{\substack{2i+j \le r \\ i,j \ge 0}} n^i |v - nx|^j \frac{|\phi_{i,j,r}(x)|}{x^r(1+x)^r} p_{n,v}(x) \times$$

$$\times \int_0^\infty q_{n,v}(t) \frac{\left|f^{(p+r)}(\xi) - f^{(p+r)}(x)\right|}{(p+r)!} |t-x|^{p+r} \chi(t)\, dt \le$$

$$\leq \frac{nC_1}{(p+r)!} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j \times$$

$$\times \int_0^{\infty} q_{n,v}(t)\left[1+\frac{|t-x|}{\delta}\right]\omega_f(p+r)(\delta)|t-x|^{p+r}\,dt =$$

$$= \frac{nC_1\,\omega_f(p+r)^{(\delta)}}{(p+r)!} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j \times$$

$$\times \int_0^{\infty} q_{n,v}(t)\left\{|t-x|^{p+r} + \frac{|t-x|^{p+r+1}}{\delta}\right\}dt \quad \text{dla każdego } \delta > 0$$

where

$$C_1 = \operatorname*{Sup}_{\substack{2i+j\leq r \\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|\phi_{i,j,r}(x)|}{x^r(1+x)^r}.$$

Now, we shall show that for $m = 0,1,2,\ldots$

$$n\sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j \int_0^{\infty} q_{n,v}(t)\,|t-x|^m\,dt = O\left[n^{(j-m)/2}\right].$$

We have

$$n\sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j \int_0^{\infty} q_{n,v}(t)\,|t-x|^m\,dt \leq$$

$$\leq n\sum_{v=0}^{\infty} p_{n,v}(x)\,|v-nx|^j\left[\left(\int_0^{\infty} q_{n,v}(t)dt\right)^{1/2}\left(\int_0^{\infty} q_{n,v}(t)(t-x)^{2m}dt\right)^{1/2}\right] =$$

$$= \frac{n}{\sqrt{n}}\sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j\left[\int_0^{\infty} q_{n,v}(t)(t-x)^{2m}dt\right]^{1/2} \leq$$

$$\leq \left[\sum_{v=0}^{\infty} p_{n,v}(x)(v-nx)^{2j}\right]^{1/2}\left[n\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^{2m}dt\right]^{1/2} =$$

$$= O(n^{j/2})\cdot O(n^{-m/2}) = O(n^{(j-m)/2}),$$

uniformly in $x$, by Lemma 2.1 and Lemma 2.2. Hence

$$|J_4| \le \frac{C_1\, \omega_{f^{(p+r)}}(\delta)}{(p+r)!} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left[ O(n^{(j-p-r)/2}) + \frac{1}{\delta} O(n^{(j-p-r-1)/2}) \right].$$

Choosing $\delta = n^{-1/2}$, we get

$$|J_4| \le C_2\, n^{-p/2}\, \omega_{f^{(p+r)}}(n^{-1/2}).$$

Therefore

$$|J_2| \le K_2\, n^{-p/2}\, \omega_{f^{(p+r)}}(n^{-1/2}), \quad \text{where } K_2 = K_2(k,p,r).$$

Finally, we estimate $J_3$. Since $t \in [0,\infty)\setminus(a-\eta, b+\eta)$, we can choose a $\delta > 0$ in such a way that $|t-x| \ge \delta$ for all $x \in [a,b]$. Let

$$J_5 = n \sum_{v=0}^{\infty} p_{n,v}^{(r)}(x) \int_0^{\infty} q_{n,v}(t) h(t,x)(1-\chi(t))dt .$$

Using Lemma 2.4, we get

$$|J_5| \le n \sum_{v=0}^{\infty} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, |v-nx|^j\, \frac{|\phi_{i,j,r}(x)|}{x^r(1+x)^r} p_{n,v}(x) \int_{|t-x|\ge\delta} q_{n,v}(t) |h(t,x)| dt .$$

If $\beta$ is any integer $\ge \max\{\alpha, 2k+r+2\}$ then we can find a constant $C_3$ such that $|h(t,x)| \le C_3\, e^{\beta t}$ for $|t-x| \ge \delta$.

Applying Cauchy-Schwarz inquality, Lemma 2.7 and Corollory 2.3, we obtain

$$|J_5| \le C_1\, n \sum_{v=0}^{\infty} \sum_{\substack{2i+j\le r\\ i,j\ge o}} n^i\, |v-nx|^j\, p_{n,v}(x) \int_{|t-x|\ge\delta} q_{n,v}(t)\, C_3\, e^{\beta t} dt \le$$

$$\le C_4\, n \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{v=0}^{\infty} |v-nx|^j\, p_{n,v}(x) \int_{|t-x|\ge\delta} q_{n,v}(t)\, C_3\, e^{\beta t} dt =$$

$$= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O(n^{j/2}) O(n^{-m}) = O(n^{(r-2m)/2}) = O(n^{-(k+1)})$$

uniformly on $[a,b]$, where $m$ is a natural bigger than $\beta/2$.

Hence, $$|J_3| \le K_3\, n^{-(k+1)} .$$

Combining the estimates of $J_1$, $J_2$ and $J_3$ we get the required result.

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