FASCICULI MATHEMATICI

Nr 27 1997

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A NOTE ON m-PARABOLIC PROBLEM FOR SPHERICAL SHALL

The subject of the paper is the problem of the existence and of the uniqueness to the polyparabolic problem for spherical shall with the limit conditions of Riquier type. In [4] this problem by another method is solved. By the suitable reduction of the initial conditions the construction of the solution is reduced to the construction given in [7]. By the formulas of [7] the explicite form to the solution is obtained.

The uniqueness theorem is also given, which is not given in [7].

Key words: polyparabolic problem, initial-boundary conditions, uniqueness and existence of the solution, Green function, Green potentials.

1. Introducion

The subject of the paper is the existence and uniqueness of the radial solution

$$U(r,t) = u(x,t)\ , \qquad r = |x|, \qquad \text{to the equation}$$

(1) $$P^m u(x,t) = f(x,t), \quad x = (x_1, x_2, x_3)\ ,$$

$$P = \Delta - D_t, \qquad \Delta = \sum_{i=1}^{3} D_{x_i}^2\ , \qquad P^m = P(P^{m-1})$$

$$m \in N - \{1\},\ P^1 = P,$$

in spherical shall

$$D = \left\{ (x,t): a < |x| < b,\ 0 < a < b,\ t \in (0,T),\ |x| = \left(\sum_{i=1}^{3} x_1^2 \right)^{1/2} \right\}.$$

The radial solution $u(x,t) = U(r,t)$, $r = |x|$ satisfies the initial conditions

(2) $$P^{j-1}u(x,0) = f_j(x) \quad \text{for} \quad (x,t) \in D_1 = \{(x,0): a < |x| < b\},\quad j = 1,2,\ldots,m,$$

and the boundary conditions

(3) $$P^{j-1}u(x,t) = h_j(t) \quad \text{for} \quad (x,t) \in S_1 = \{(x,t): |x| = a, t \in (0,T]\}\ ,\quad j = 1,2,\ldots,m\ ,$$

(4) $P^{j-1}u(x,t)=k_j(t)$ for $(x,t)\in S_2=\{(x,t):|x|=b,\ t\in(0,T]\}$, $j=1,2,\dots,m$.

In [4] the radial solution to the problem (1) – (4) by another method is given.

In [7] the similar limit problem with another initial conditions is examined but without uniqueness.

In the papers [1] and [2] the polyparabolic problem for the strip and for the quart-time-plane with boundary conditions of Lauricella type are considered respectively.

In [3] the polyparabolic problem for the spatial three-dimensional cylinder with boundary conditions of Riquer type is solved.

In the paper [6] the m-parabolic problem for the strip with boundary conditions of even type is examined.

2. The radial problem

Let

(5) $V(r,t)=rU(r,t)$ and $Q=D_r^2-D_t$.

By (5) and by [7], p.118, we obtain the following

Lemma 1. If $u\in C^{2m,m}(D)$, $u(x,t)=U(r,t)$, $r=|x|$ *then*

1° $U\in C^{2m,m}((a,b)\times(0,T))$,

2° $P^m u(x,t)=r^{-1}Q^m(r,t))=r^{-1}Q^m V(r,t)$.

By Lemma 1 and by (1) – (4) problem we obtain

Lemma 2. If the function $U\in C^{2m,m}((a,b)\times(0,T])$ *is the solution of the (1) – (4) problem, then the function* $V(r,t)\in C^{2m,m}(D_2)$, $D_2=\{(r,t):a<r<b,\ t\in(0,T)\}$ *and satisfies the conditions*

(1a) $Q^mV(r,t)=F(r,t),\ F(r,t)=rf(r,t),\ (r,t)\in D_2$,

(2a) $Q^{j-1}V(r,0)=F_j(r),\ r\in(a,b),\ F_j(r)=rf_j(r),\ j=1,2,\dots,m$,

(3a) $Q^{j-1}V(a,t)=H_j(t),\ t\in(0,T],\ H_j(t)=ah_j(t),\ j=1,2,\dots,m$,

(4a) $Q^{j-1}V(b,t)=K_j(t),\ t\in(0,T],\ K_j(t)=bk_j(t),\ j=1,2,\dots,m$.

Conversely. If $V \in C^{2m,m}(D_2)$ and satisfies (1a) – (4a), then the function $u(x,t) = U(r,t)|_{r=|x|} = |x|^{-1} V(r,t)$ is the solution of the (1) – (4) problem.

Definition 1. Denote by (K_1) the class of all functions $F : D_2 \to R$ *such that* $F \in C^{1,0}(D_2) \cap C^{0,0}(\overline{D}_2)$.

Definition 2. Denote by (K_2) the class of all functions $E : [a,b] \to R$, *such that* $E \in C^{4m-2}([a,b])$ *and* $D_r^{i-1} E(a) = D_r^{i-1} E(b) = 0$, $i = 1,2,\dots,m,\ m \in N$.

Definition 3. Denote by (K_3) the class of all functions $H : [0,T] \to R$, *such that* $H \in C^1([0,T)]$ *and* $D_r^{i-1} H(0) = 0$, $i = 1,2,\dots,m$, $m \in N$.

Definition 4. Denote by (K_4) the class of all functions $V : D \to R$, *such that* $V \in C^{2m,m}(D_2) \cap C^{m,m-1}(\overline{D}_2)$.

3. Uniqueness theorem

Let $W(r,t) = w_1(r,t) - w_2(r,t)$ and

$$D(t) = \{(r,s) : a < r < b,\ \ 0 < s < t,\ \ t < T\} .$$

Theorem 1. If $w_1(r,t), w_2(r,t) \in (K_4)$ *are solutuions of the problem (1a) – (4a), then* $w_1(r,t) = w_2(r,t)$ *for* $(r,t) \in \overline{D}_2$.

Proof. For the function W(r,s) we have

(6) $Q^m W(r,s) = 0 ,\qquad (r,s) \in D(t) ,$

(7) $Q^{j-1} W(r,0) = 0,\qquad a < r < b ,\qquad j = 1,2,\dots,m ,$

(8) $Q^{j-1} W(a,s) = 0 ,\quad 0 < s < t,\quad t < T ,\qquad j = 1,2,\dots,m ,$

(9) $Q^{j-1} W(b,s) = 0 ,\quad 0 < s < t,\quad t < T,\qquad j = 1,2,\dots,m .$

Multypling by $Q^{m-1} W(r,s)$ both sides of the equation (6) and integrating over $D(t)$, we obtain

(10) $$\int_o^t \int_a^b Q^{m-1} W(r,s) Q^m W(r,s) dr\, ds = I_1 + I_2 = 0,$$

where

$$I_1 = -\int_o^t \int_a^b Q^{m-1} W(r,s)\, D_s\, Q^{m-1} W(r,s) dr\, ds,$$

$$I_2 = \int_o^t \int_a^b Q^{m-1} W(r,s) D_r^2\, Q^{m-1} W(r,s) dr\, ds.$$

Intagrating I_1 by parts we obtain

$$I_1 = -\int_a^b (Q^{m-1} W(r,s))^2 \Big|_{s=o}^{s=t} dr +$$

$$+\int_0^t \int_a^b Q^{m-1} W(r,s)\, D_s\, Q^{m-1} W(r,s) dr\, ds =$$

$$= -\int_a^b (Q^{m-1} W(r,t))^2 dr + \int_a^b (Q^{m-1} W(r,0))^2 dr \ .$$

Consequently by (7) we have

(11) $$I_1 = -\frac{1}{2}\int_a^b (Q^{m-1} W(r,t))^2 dr \ .$$

Integrating I_2 by parts we obtain

$$I_2 = \int_0^t (Q^{m-1} W(r,s) D_r\, Q^{m-1} W(r,s)) \Big|_{r=a}^{r=b} ds -$$

$$-\int_0^t \int_a^b (D_r\, Q^{m-1} W(r,s))^2 dr\, ds \ .$$

Consequently by (8) and (9) we have

(12) $$I_2 = -\int_0^t \left(\int_a^b D_r\, Q^{m-1} W(r,s) \right)^2 dr\, ds \le 0$$

By (10), (11) and (12) we obtain

$$I_1 + I_2 = 0$$

and

$$Q^{m-1} W(r,t) = 0 \ , \qquad (r,t) \in D_2 \ .$$

Similarly we obtain

$$Q^i W(r,t) = 0, \quad (r,t) \in D_2, \qquad i = 1,2,\dots,m-2.$$

For $i = 1$ we have.

(13) $QW(r,t)=0, \quad (r,t)\in D_2$, and $QW(r,s)=0, \quad (r,s)\in D(t)$.

Multiplying (13) by $W(r,s)$ we obtain

(14) $\quad W(r,s)QW(r,s)=0, \qquad (r,s)\in D(t)$,

Integrating (14) over $D(t)$, we obtain

$$W(r,t)=0,\ \ (r,t)\in D_2,\ \ \text{i.e.}\ \ w_1(r,t)=w_2(r,t)\ \ \text{for}\ \ (r,t)\in D_2.$$

By the continuity of $W(r,t)$ in $\overline{D_2}$, we obtain

$$w_1(r,t) = w_2(r,t) \qquad \text{for} \quad (r,t)\in \overline{D_2} .$$

4. Reduction of the problem (1) – (4) to the problem with homogeneous initial conditions

Let

$$w(r,t) = V(r,t) - R(r,t) ,$$

where

$$R(r,t) = f_o(r)+\sum_{j=1}^{m-1}(t^j/j!)\sum_{k=o}^{j}(-1)^{k+1}\binom{j}{k}\Delta^{j-k} f_k(r).$$

Lemma 3. If $F\in(K_1)$, $F_j\in(K_2)$, $j=0,1,\dots,m-1$, $H_j,K_j\in(K_3)$, $j=0,1,\dots,m-1$ *and* $V\in(K_4)$ *is solution of the (1a) – (4a) problem, then* $w\in(K_4)$ *and satisfies the conditions:*

(1b) $\quad Q^m w(r,t)=L(r,t),\ \ L(r,t)=F(r,t)-Q^m R(r,t),\ \ (r,t)\in D_2$,

(2b) $\quad Q^{j-1} w(r,0)=0,\ \ j=1,2,\dots,m,\ \ a<r<b$,

(3b) $\quad Q^{j-1}w(a,t)=\overline{H}_j(t),\ \ \overline{H}_j(t)=H_j(t)-Q^{j-1}R(a,t),$

$$j=1,2,\dots,m, \qquad 0<t<T ,$$

(4a) $\quad Q^{j-1} w(b,t)=\overline{K}_j(t),\ \ \overline{K}_j(t)=K_j(t)-Q^{j-1}R(b,t)$,

$$j=1,2,\dots,m, \qquad 0<t<T$$

Conversely. If $L\in(K_1)$, $\overline{H}_j,\overline{K}_j\in(K_3)$, $j=0,1,\dots,m-1$ and the function $w\in(K_4)$ satisfies (1b) – (4b), then the function

$V(r,t) = w(r,t)+R(r,t) \in (K_4)$ and satisfies (1a) – (4a).

We omit the simple proof.

5. Green function

By [5], p.476, it is known that the Green function for the equation $QV = 0$ for the strip and Dirichlet boundary conditions is of the form:

$$G(r,t;p,s) = U_0 + H(r,t;p,s)$$

where

$$U_0(r,t;p,s) = (t-s)^{-1/2}\exp(B(t,s)(r_0^1-p)^2),$$

$$B(t,s) = (-4(t-s))^{-1}, \qquad r_o^i = r, \qquad i=1,2.$$

$$H(r,t;p,s) = -U_1^1(r,t;p,s) + \sum_{n=0}^{\infty}(-U_{2n+3}^1(r,t;p,s)+$$

$$+U_{2n+2}^2(r,t;p,s)+\sum_{n=0}^{\infty}(-U_{2n+1}^2(r,t;p,s)+U_{2n+2}^1(r,t;p,s)),$$

or

$$H(r,t;p,s) = -U_1^2(r,t;p,s)+\sum_{n=0}^{\infty}(U_{2n+3}^2(r,t;p,s)+U_{2n+2}^1(r,t;p,s))+$$

$$+\sum_{n=0}^{\infty}(U_{2n+2}^2(r,t;p,s)-U_{2n+1}^1(r,t;p,s)).$$

$$U_n^j(r,t;p,s) = (t-s)^{-1/2}\exp(B(t,s)(r_n^j-p)^2), \quad j=1,2, \quad n=0,1,2,\ldots$$

and $r_0^1 = r_0^2 = r$

$$r_{2n}^1 = r+2n(b-a)$$

$$r_{2n+1}^1 = -r-2n(b-a)+2a$$

$$r_{2n}^2 = r-2n(b-a)$$

$$r_{2n+1}^2 = -r+2n(b-a)+2b$$

Let us consider the function G_1 given by formulas:

$$G(r,t;p,s) = (i!)(t-s)^i\,G(r,t;p,s), \qquad i=1,2,\ldots,m-1.$$

By [7], p.120, we have.

Lemma 4. When $0<s<t$, *then the functions* G_i *satisfy the folowing conditions:*

1° $Q^i G_{i-1}(r,t;p,s) = 0, \quad i=2,3,\ldots,m, \quad (r,t),(p,s)\in D_1,$

2° $Q^i G_{i+1}(a,t;p,s) = Q^i G_{i+1}(b,t;p,s) = 0, \quad i=0,1,2,\ldots,m-1,$ $(r,t),(p,s)\in D_1.$

By [7], p.121, we have.

Lemma 5. If $0<s<t<T$, *then*

$$D_p\, G(r,t;p,s)\big|_{p=a} = S_1(r,t,s) + S_1^1(r,t,s),$$

where

$$S_1(r,t,s) = D_p(U_0(r,t;p,s)-U_1^1(r,t;p,s))\big|_{p=a}=$$
$$=-2(t-s)^{-1/2}B(t,s)(r-a)\exp(B(t,s)(r-a)^2),$$

$$S_1^1(r,t,s)=-4(t-s)^{-1/2}B(t,s)\sum_{n=0}^{\infty}A_n\exp(B(t,s)(A_n)^2)=$$
$$=-4(t-s)^{-1/2}B(t,s)\sum_{n=0}^{\infty}B_n\exp(B(t,s)(B_n)^2)$$

and $A_n=-r-2(n+1)(b-a)+a, \qquad B_n=r-2(n+1)(b-a)-a$,

$$D_p\, G(r,t;p,s)\big|_{p=b}=S_2(r,t,s)+S_2^1(r,t,s);$$

where

$$S_2(r,t,s) = D_p\,(U_0(r,t;p,s)-U_1^2(r,t;p,s))\big|_{p=b}=$$
$$=-2A(t-s)B(t,s)(r-b)\exp(B(t,s)(r-b)^2),$$

$$S_2^1(r,t,s)=$$
$$=-4(t-s)^{-1/2}B(t,s)\sum_{n=0}^{\infty}\left[D_n\exp\left(B(t,s)(D_n)^2C_n\exp\left(B(t,s)C_n^2\right)\right]\right.,$$

$C_n=r+2(n+1)(b-a)-2a+b, \qquad D_n=r-2(n+1)(b-a)-b$.

6. Green iterated potentials

Similarly to the [7], p.123, let us consider the following potentials

$$w_0^1(r,t)=A_1\int_0^t D_p\,G(r,t;p,s)\big|_{p=a}\overline{H}_0(s)\,ds = T_0^{11}(r,t)+T^{12}(r,t) ,$$

$$w_0^2(r,t)=A_1\int_0^t D_p\,G(r,t;p,s)\big|_{p=b}\overline{K}_0(s)\,ds = T_0^{21}(r,t)+T_0^{22}(r,t) ,$$

where

$$T_0^{11}(r,t) = A_1\int_0^t S_1(r,t,s)\,\overline{H}_0(s)\,ds ,$$

$$T^{12}(r,t) = A_1\int_0^t S_1^1(r,t,s)\,\overline{H}_0(s)\,ds ,$$

$$T_0^{21}(r,t) = A_1 \int_0^t S_2(r,t,s)\, \overline{K}_0(s)\, ds,$$

$$T_0^{22}(r,t) = A_1 \int_0^t S_2^1(r,t,s)\, \overline{K}_0(s)\, ds.$$

Let

$$w_1^1(r,t) = A_2 \int_o^t \int_a^b G(r,t;p,s)\, M_1(p,s)\, dp\, ds,$$

where

$$M_1(p,s) = M_1^1(p,s) + M_1^2(p,s)\ ,$$

and

$$M_1^1(p,s) = \int_o^s S_1(p,s,s_1)\, \overline{H}_1(S_1)\, ds_1\ ,$$

$$M_1^2(p,s) = \int_0^s S_1^1(p,s,s_1)\, \overline{H}_1(s_1)\, ds_1\ ,$$

and
$$w_1^2(r,t) = A_2 \int_0^t \int_a^b G(r,t;p,s)\, N_1(p,s)\, dp\, ds,$$

where

$$N_1(p,s) = N_1^1(p,s) + N_1^2(p,s)$$

and

$$N_1^1(p,s) = \int_0^s S_2(p,s,s_1)\, \overline{K}_1(s_1)\, ds_1\ ,$$

$$N_1^2(p,s) = \int_0^s S_2^1(p,s,s_1)\, \overline{K}_1(s_1)\, ds_1\ .$$

Let

$$w_2^1(r,t) = A_3 \int_0^t \int_a^b G_1(r,t;p,s)\, M_2(p,s)\, dp\, ds\ ,$$

where

$$M_2(p,s) = \int_0^s D_{p_1} G(p,s;p_1,s_1)\Big|_{p_1=b}\, \overline{H}(S_1)\, ds_1\ ,$$

and so one ....

$$w^1_{m-1}(r,t) = A_{m-1}\int_0^t\int_a^b G_{m-2}(r,t;p,s)\, M_{m-1}(p,s)\,dp\,ds,$$

where

$$M_{m-1}(p,s) = \int_0^s D_{p1}\, G(p,s;p_1,s_1)\big|_{p_1=a}\, \overline{H}_{m-1}(s_1)\,ds_1\ ,$$

$$w^2_{m-1}(r,t) = A_{m-1}\int_b^t\int_a^b G_{m-2}(r,t;p,s)\, N_{m-1}(p,s)\,dp\,ds\ ,$$

where

$$N_{m-1}(p,s) = \int_o^s D_{p1}\, G(p,s;p_1,s_1)\big|_{p_1=b}\, \overline{K}_{m-1}(s_1)\,ds_1\ ,$$

and

$$w_m(r,t) = A_{m-1}\int_o^t\int_a^b G_{m-1}(r,t;p,s)\, L(p,s)\,dp\,ds\ ,$$

$$A_i\left[2\sqrt{\pi}\right]^{-1}(i!)^{-1}, \qquad i=1,2,\ldots,m-1.$$

Let

$$w_m(r,t) = \sum_{i=1}^{4} w^1_m(r,t),$$

where

$$w^1_m(r,t) = A_{m-1}\int_0^t\int_a^b (t-s)^{m-1}\, U_0(r,t;p,s)\, L(p,s)\,dp\,ds =$$

$$= A_{m-1}\int_0^t\int_a^b (t,s)^{m-1}\exp(B(t,s)(r-p))^2\, L(p,s)\,dp\,ds,$$

$$w^2_m(r,t) = A_{m-1}\int_0^t\int_a^b (t-s)^{m-1}\, U^1_1(r,t;p,s)\, L(p,s)\,dp\,ds=$$

$$= A_{m-1}\int_o^t\int_a^b (t,s)^{m-1}\exp(B(t,s)(-r+2a-p))^2\, L(p,s)\,dp\,ds,$$

$$w^3_m(r,t) = A_{m-1}\int_0^t\int_a^b (t-s)^{m-1}\, U^2_1(r,t;p,s)\, L(p,s)\,dp\,ds =$$

$$= A_{m-1}\int_0^t\int_a^b (t,s)^{m-1}\exp(B(t,s)(r+2b-p))^2\,L(p,s)\,dp\,ds\ ,$$

$$w_m^4(r,t) = A_{m-1}\int_0^t\int_a^b (t-s)^{m-1}\,H^1(r,t;p,s)\,L_1(p,s)\,dp\,ds\ ,$$

where

$$H^1(r,t;p,s) = H(r,t;p,s) + U_1^1(r,t;p,s)+U_1^2(r,t;p,s),$$

By [7], p.128, we have

Lemma 6. If the function L belong to the class of all functions $L(p,s)$ such that the functions $D_p^i D_s^j L(p,s)$, $0\le i\le m-1$, $0\le j\le m-1$, are continuous and bounded in $\overline{D}_1$ and

$$D_s^i\,L(a,s) = D^i\,L(b,s) = 0\ ,\quad D_s^1\,L(p,0) = 0\ ,\quad i=0,1,\ldots,m-1$$

then

1° $Q^m w_m(r,t) = L(r,t)$ for $(r,t)\in D_1$,

2° $Q^i w_m(r,0)=0$, $i=0,1,\ldots,m-1$, for $r\in(a,b)$,

3° $Q^i w_m(a,t)=0$, $i=0,1,\ldots,m-1$, $t\in(0,T]$,

4° $Q^1 w_m(b,t)=0$, $i=0,1,\ldots,m-1$, $t\in(0,T]$.

By [7], p.130, we have

Lemma 7. If the function $\overline{H}_i,\overline{K}_i$, $i=0,1,\ldots,m-1$, belong to the class of all functions such that $\overline{H}_i,\overline{K}_i\in C^i([0,T])$ and $D_t^i\,\overline{H}_i(t)=D_t^i\,\overline{K}_1(t)=0$, $i=0,1,\ldots,m-1$ then

1° $Q^m w_j^i(r,t)=0$, $j=0,1,\ldots,m-1$, $i=1,2$ *for* $(r,t)\in D_1$,

2° $Q^k w_j^i(r,t)=0$, $i=1,2$, $k,j=0,1,\ldots,m-1$,

3° $Q^1 w_j^2(r,t)\to 0$ *as* $r\to a$, *if* $i\ne j$, $Q^i w_j^1(r,t)\to\overline{H}_1(t)$ *as* $r\to a$, $i=0,1,2,\ldots,m-1$, $j=0,1,\ldots,m-1$, $t\in(0,T]$,

4° $Q^i w_j^2(r,t)\to 0$ *as* $r\to b$, *if* $i\ne j$, $Q^i w_i^2(r,t)\to\overline{K}_i(t)$ *as* $r\to b$, $i=0,1,\ldots,m-1$, $j=0,1,\ldots,m-1$, $t\in(0,T]$.

5. Theorem on the existence of the solution

Theorem 2. If the assumptions of Lemmas 1 – 7 are satisfied, then the function

$$w(r,t) = w_m(r,t) + \sum_{j=0}^{m-1} (w_j^1(r,t) + w_j^2(r,t))$$

is the solution of the problem (1b) – (4b) and the function

$$u(x,t) = U(r,t)\Big|_{r=|x|} = |x|^{-1}(w|x|,t) + r(|x|,t) =$$

$$= |x|^{-1}\left\{w_m(|x|,t) + \sum_{j=0}^{m-1} (w_j^1(|x|,t) + w_j^2(|x|,t) + r(|x|,t)\right\}$$

is the solution of the problem (1) – (4).

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