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ASYMPTOTIC EQUIVALENCE BETWEEN A NONLINEAR SYSTEM AND ITS NONLINEAR PERTURBATION

In this paper, we shall give some general results on the asymptotic relationship between the solutions of a linear (nonlinear) system and its perturbed linear (nonlinear) system.

Key words: difference systems, asymptotic equivalence.

The problem of asymptotic equivalence for systems of ordinary differential equations has been studied by many authors e.q. see [1, 2, 3] and progress has been made in extending this work to systems of difference equations. The paper [4] deals with the property of asymptotic relationship between the solution of linear difference equation

$$x(n+1) = A(n)\, x(n)$$

and its perturbed nonlinear equation

$$y(n+1) = A(n)y(n) + F(n, y(n)) + G(n, y(n), Ty(n)).$$

In this paper we deal with the problem of the asymptotically equivalence of two systems

(I) $\Delta x(n) = A(n, x(n)) + B(n, x(n))$

and

(II) $\Delta y(n) = A(n, y(n))$.

Denote by $N^+_{n_0} = \{n_0, n_0+1, \ldots\}$, where n_0 is a fixed monnegative integer, R^k the – k-dimensional real euclidean space with norm $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, \ldots, x_k)$; M^k – is the space of all $k \times k$ matrices $A = (a_{ij})$ with norm $|A| = \max_j \sum_{i=1}^{k} |a_{ij}|$. We denote by $B(n_0) = B(N^+_{n_0}, R^k)$ the Banach space of all bounded functions from $N^+_{n_0}$ to R^k. The norm in $B(n_0)$ is defined by $|x|_B = |x(n)|_B = \sup\{|x(n)| : n \in N^+_{n_0}\}$. We will be interested in establishing asymptotic relationships, between the solutions of unperturbed equation (II) and its perturbed equation (I) and

(III) $x(n+1) = C(n)x(n)+D(n)x(n)$

(IV) $y(n+1)=C(n)\,y(n)$

where x,y are k-dimensional vectors; $C,D: N_{n_0}^{+} \to M^{k}$ are such that $C(n)$, $D(n)$ are nonsingular for all $n \in N_{n_0}^{+}$; $A,B: N_{n_0}^{+} \times Q \to R^{k}$ (Q – a region in R^{k}), are for any $n \in N_{n_0}^{+}$ continuous with respect to the second argument.

In this paper we consider the notion of asymptotic equivalence given by

Definition 1. We say that the equations (III) and (IV) are asymptotically equivalent if, corresponding to each solution $x = x(n)$ of (III), there exists a solution $y = y(n)$ of (IV) with property

(1) $\lim \left[y(n)-x(n)\right] = 0$ *as* $n \to \infty$,

and conversely.

Definition 2. Let $q(n)$ be a positive function on $N_{n_0}^{+}$. We say that the equations (I) and (II) are q-asymptotically equivalent if corresponding to each solution $x = x(n)$ of (1), there exists a solution $y = y(n)$ of (II) with the property

(2) $q^{-1}(n)|x(n)-y(n)| \to 0$ *as* $n \to \infty$

and conversely.

Definition 3. We say that the equations (I) and (II) are (q,p)--asymptotically equivalent if

(3) $q^{-1}(n)|x(n)-y(n)| \in l^{p}$.

For $r > 0$ and $n_1 \in N_{n_0}^{+}$ we give two conditions:

Condition $C_1(h,r,n_1)$. The vector $R(n,u)$ satisfies Condition $C_1(h,r,n_1)$ if there exists a nonnegative function $h(n)$ such that for every $n \in N_{n_1}^{+}$ and $|u| \le r$ imply

$$|R(n,u+y)| \le h(n), \qquad y \in R^{k}$$

and

$$\sum_{s=i}^{\infty} h(s) \in l^p .$$

Condition $C_2(h,g,r,n_1)$. *The vector* $R(n,u)$ *satisfies condition* $C_2(h,g,r,n_1)$, *if there exist nonnegative functions* h,g *such that for every* $n \in N^+_{n_1}$, $|u| \le r$, $|v| \le r$ *imply*

$$|R(n,u+y) - R(n,v+y)| \le h(n)\, g(|u-v|), \qquad y \in R^k$$

and

$$\sum_{s=i}^{\infty} h(s) \in l^p ,$$

$$\sup q(|w|) < \infty \quad on \quad |w| \le b = const. \; for \; every \; b .$$

Let $\psi(n,k)$ *and* $\varphi(n,k)$ *denote the fundamental matrices of (III) and (IV) respectively.*

Now, we present the following theorems:

Theorem 1. Suppose that the following conditions are satisfied:

1° $|\phi(n,k)| \le \phi = const$ *for all* $n,k \in N^+_{n_0}$,

2° *for all* $k \in N^+_{n_0}$, $\phi(n,k)$ *has limit as* $n \to \infty$, *and* $\lim \phi(n,n_0) = P_1$ *as* $n \to \infty$, $\det P_1 \ne 0$,

3° $\sum_{s=0}^{\infty} |D(s)| < \infty$,

4° *if* $\lim_{n\to\infty} \psi(n,n_0)$ *exists, then we assume that* $\det P_2 \ne 0$, *where* $P_2 = \lim_{n\to\infty} \psi(n,n_0)$.

Then the systems (III) and (IV) are asymptotically equivalent.

Proof. Because ψ and ϕ are fundamental matrices of (III) and (IV) respectively, then $\phi(n,k)$ and $\psi(n,k)$ satisfy

$$\psi(n,n_0) = \phi(n,n_0) + \sum_{k=n_0}^{n-1} \phi(n,k+1)\, D(k)\, \psi(k,n_0) . \tag{4}$$

Taking norms of both sides of (4) and considering n_0 fixed, we have using the discrete version of Gronwall inequality

(5) $$|\psi(n,n_0)| \le \phi \exp\left\{\phi \sum_{k=0}^{\infty} |D(k)|\right\}.$$

Since $\sum_{k=0}^{\infty} |D(k)| < \infty$, then we have that $|\psi(n,n_0)|$ is bounded.

Moreover, for given any $m \in N_{n_0}^+$ and any $n_3 > n_2 > m$ we have

$$\psi(n_3,n_0) - \psi(n_2,n_0) = \phi(n_3,n_0) - \phi(n_2,n_0) +$$
$$+\sum_{k=n_0}^{n_3-1} \phi(n_3,k+1) D(k)\, \psi(k,n_0) - \sum_{k=n_0}^{n_2-1} \phi(n_2,k+1) D(k)\, \psi(k,n_0).$$

Hence

$$|\psi(n_3,n_0) - \psi(n_2,n_0)| = |\phi(n_3,n_0) - \phi(n_2,n_0)| +$$
$$+\left|\sum_{k=n_0}^{m-1} \phi(n_3,k+1)\, D(k)\, \psi(k,n_0) + \sum_{k=m}^{n_3-1} \phi(n_3,k+1) D(k)\, \psi(k,n_0) -\right.$$
$$\left. -\sum_{k=n_0}^{m-1} \phi(n_2,k+1)\, D(k)\, \psi(k,n_0) - \sum_{k=m}^{n_2-1} \phi(n_2,k+1) D(k)\, \psi(k,n_0)\right| \le$$
$$\le |\phi(n_3,n_0) - \phi(n_2,n_0)| +$$
$$+\sum_{k=n_0}^{m-1} |\phi(n_3,k+1) - \phi(n_2,k+1)||D(k)||\psi(k,n_0)| +$$
$$+\sum_{k=m}^{n_3-1} |\phi(n_3,k+1)||D(k)||\psi(k,n_0)| + \sum_{k=m}^{n_2-1} |\phi(n_2,k+1)||D(k)||\psi(k,n_0)|.$$

By assumptions 1°, 2°, 3° we obtain that

$$\lim \psi(n,n_0) \quad \text{as} \quad n \to \infty \quad \text{exists.}$$

Since

$$y(n) = \phi(n,n_0)\, y(n_0) \quad \text{and} \quad x(n) = \psi(n,n_0)\, x(n_0)$$

we see that

(6) $$\lim\,[x(n) - y(n)] = \lim\, \psi(n,n_0)\, x(n_0) - \lim\, \phi(n,n_0)\, y(n_0) =$$
$$= P_2\, x(n_0) - P_1\, y(n_0) \quad \text{as} \quad n \to \infty.$$

The conditions 2° and 4° imply that there exist P_1^{-1} and P_2^{-1}. This shows that for every solution $y(n)$ of (IV) there exist a solution $x(n)$ of (III) with $x(n_0) = P_2^{-1} P_1\, y(n_0)$ such that (1) hold, and conversely. Hence systems (III), (IV) are asymptotically equivalent.

Example. Consider the equations

$$x(n+1) = e^{-n} x(n), \qquad x(0) = x_0 ;$$

$$y(n+1) = e^{-n} y(n) - (1/2) e^{-n} y(n), \qquad y(0) = y_0$$

and

$$x(n+1) = x(n) , \qquad x(0) = x_0 ,$$

$$y(n+1) = e^{-n} y(n) , \qquad y(0) = y_0 .$$

The above examples show that the conditions 3°, 5° are necessary.

Theorem 2. Let the conditions 1° and 3° of Theorem 1 hold and

$$\overline{\lim_{n\to\infty}} \{\det \psi(n,n_0)\} \neq 0,$$

then the systems (III) and (IV) are asymptotically equivalent.

Proof. The solution of (III) can now be written as

$$x(n) = \phi\,(n,n_1)\,x(n_1) + \sum_{k=n_1}^{n-1} \phi\,(n,k+1)\,D(k)\,x(k) \tag{7}$$

for each fixed $n_1 \in N^+_{n_0}$

By conditions of Theorem and (4) it follows that $|\psi(n,k)| \leq \psi = \text{const}$ for every $n \geq k$, every $k \in N^+_{n_0}$. Besides

$$\lim_{n\to\infty} \sum_{k=n_1}^{n-1} |\phi\,(n,k+1)||D(k)||x(k)| \leq$$

$$\leq \lim_{n\to\infty} \sum_{k=n_1}^{n-1} \phi\,|D(k)||\psi(k,n_0)||x(n_0)| \leq$$

$$\leq \phi\,\psi |x(n_0)| \lim_{n\to\infty} \sum_{k=n_1}^{n-1} |D(k)| \leq \infty .$$

Applying properties of the matrix ϕ

$$\phi\,(n,s) = \phi\,(n,n_1)\,\phi\,(n_1,s)$$

to (7) gives

$$x(n) = \phi\,(n,n_1)\,x(n_1) + \sum_{k=n_1}^{\infty} \phi\,(n,k+1)\,D(k)\,x(k) - \tag{8}$$

$$-\sum_{k=n}^{\infty} \phi\,(n,k+1)\,D(k)\,x(k) =$$

$$= \phi\,(n,n_1)\left[x(n_1)+\sum_{k=n_1}^{\infty}\phi\,(n_1,k+1)\,D(k)\,x(k)\right]-$$

$$-\sum_{k=n}^{\infty}\phi\,(n,k+1)\,D(k)\,x(k)\;.$$

Set $x(n)=\psi\,(n,n_0)\,x(n_0)$ in (8). Then we obtain

$$\text{(9)}\quad x(n)=\phi\,(n,n_1)\left[\psi\,(n_1,n_0)+\sum_{k=n_1}^{\infty}\phi\,(n_1,k+1)\,D(k)\,\psi\,(k,n_0)\right]x(n_0)-$$

$$-\sum_{k=n}^{\infty}\phi\,(n,k+1)\,D(k)\,x(k)\;.$$

To every solution $x(n)$ of the equation (III) we can attach suitable solution $y(n)$ of the equation (IV) with the initial condition

$$\text{(10)}\qquad y(n_1)\;=\;\left[\psi\,(n_1,n_0)+Z(n_1)\right]x(n_0)$$

where

$$Z(n_1)=\sum_{k=n_1}^{\infty}\phi\,(n_1,k+1)\,D(k)\,\psi\,(k,n_0)\;.$$

Hence $Z(n_1)\to 0$ as $n_1\to\infty$. The condition 3° yields that there exists $n_1\in N_{n_0}^{+}$ such that

$$\det\,\psi\;(n_1,n_0)\;\neq\;0\;.$$

Since

$$|Z(n_1)|\;\leq\;\phi\psi\sum_{k=n_1}^{\infty}|D(k)|\,,$$

then we can choose $n_1\in N_{n_0}^{+}$ so large that

$$\text{(11)}\qquad \det\left[\psi(n_1,n_0)+Z(n_1)\right]\;\neq\;0\;.$$

Now, we choose $n_1\in N_{n_0}^{+}$ such that condition (11) will be satisfied. Existence of such n_1 yields existence of the inverse matrix

$$\left[\psi\,(n_1,n_0)+Z(n_1)\right]^{-1}\;=\;Q(n_1,n_0)\;.$$

Then by (10), we have

$$\text{(12)}\qquad x(n_0)\;=\;Q(n_1,n_0)\,y(n_1)\;.$$

Conditions (10) and (12) give one to one relations between solutions of the systems (III) and (IV). From (9) and (10) one obtains

$$|x(n)-y(n)| = \left|-\sum_{k=n}^{\infty} \phi\,(n,k+1)\,D(k)\,x(k)\right| \leq \phi\,\psi|x(n_0)|\sum_{k=n}^{\infty}|D(k)|\,.$$

This shows that

$$|x(n)-y(n)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and the theorem is proved.}$$

Theorem 3. Suppose that the vectors $A(n,u)$, $B(n,u)$ *satisfy condition* $C_2(h_1,g,r,n_0)$ *and* $B(n,u)$ *satisfies* $C_1(h_2,r,n_0)$. *Then for each solution* $y(n)$ *of the system (II) there exists a solution* $x(n)$ *of (I) such that (1) – (3) holds.*

Proof. Let $n_1 \in N_{n_0}^+$ be such that

$$\text{(13)} \qquad \sum_{s=n_1}^{\infty}(Kh_1(s)+h_2(s)) \leq r < \infty \quad \text{where } K = \sup_{|u|\leq 2r} g(|u|)\,.$$

Define operator T on

$$\mathcal{A} \equiv \left\{f \in B(n_1) : |f|_B \leq r,\ n_1 \in N_{n_0}^+\right\}$$

by

$$\text{(14)} \qquad (Tf)(n) = -\sum_{s=n}^{\infty}\left[A(s,f(s)+y(s))-A(s,y(s))\right]- -\sum_{s=n}^{\infty} B(s,f(s)+y(s)), \qquad n \in N_{n_1}^+\,.$$

We shall use the Schauder theorem to prove that T has a fixed point in $\mathcal{A}$. Since for $f \in \mathcal{A}$ and $y \in R^k$

$$\sum_{s=n}^{\infty}|A(s,f(s)+y(s))-A(s,y(s))|+\sum_{s=n}^{\infty}|B(s,f(s)+y(s))| \leq$$

$$\leq \sum_{s=n}^{\infty} h_1(s)\,g(|f|)+\sum_{s=n}^{\infty} h_2(s) \leq \sum_{s=n_1}^{\infty}(Kh_1(s)+h_2(s)) \leq r < +\infty\,,$$

T is indeed defined on $\mathcal{A}$ and maps $\mathcal{A}$ into $\mathcal{A}$.

It is easily see in a similar manner that T is continuous. Let $\varepsilon > 0$, and choose $n_2 \in N_{n_1}^+$ so large that $\sum_{s=n_2}^{\infty} h_1(s) \leq \varepsilon$.

Let $\{u^i(n)\}_{i=1}^{\infty}$ be a sequence of elements of $\mathcal{A}$ such that $\lim |u^i - u| = 0$ as $i \to \infty$. Since $\mathcal{A}$ is closed, $u \in \mathcal{A}$. Then from (14) and Condition C_2 we get

$$|(Tu^i)(n)-(Tu)(n)| = \Big| -\sum_{s=n}^{\infty}\left[A(s,u^i(s)+y(s))\right] - A(s,y(s)) -$$

$$-\sum_{s=n}^{\infty} B(s,u^i(s)+y(s)) + \sum_{s=n}^{\infty}\left[A(s,u(s)+y(s)) - A(s,y(s))\right] +$$

$$+\sum_{s=n}^{\infty} B(s,u(s)+y(s))\Big| \le \sum_{s=n}^{\infty} |A(s,u^i(s)+y(s)) - A(s,u(s)+y(s))| +$$

$$+\sum_{s=n}^{\infty} |B(s,u^i(s)+y(s)) - B(s,u(s)+y(s))| \le$$

$$\le \sum_{s=n_1}^{n_2-1} |A(s,u^i(s)+y(s)) - A(s,u(s)+y(s))| +$$

$$+\sum_{s=n_2}^{\infty} |A(s,u^i(s)+y(s)) - A(s,u(s)+y(s))| +$$

$$+\sum_{s=n_1}^{n_2-1} |B(s,u^i(s)+y(s)) - B(s,u(s)+y(s))| +$$

$$+\sum_{s=n_2}^{\infty} |B(s,u^i(s)+y(s)) - B(s,u(s)+y(s))| \le$$

$$\le \sum_{s=n_1}^{n_2-1} |A(s,u^i(s)+y(s)) - A(s,u(s)+y(s))| + \sum_{s=n_2}^{\infty} h_1(s)\, g(|u^i(s)-u(s)|) +$$

$$+\sum_{s=n_1}^{n_2-1} |B(s,u^i(s)+y(s)) - B(s,u(s)+y(s))| + \sum_{s=n_2}^{\infty} h_1(s)\, g(|u^i(s)-u(s)|) \le$$

$$\le 2K \sum_{s=n_2}^{\infty} h_1(s) + \sum_{s=n_1}^{n_2-1} |A(s,u^i(s)+y(s)) - A(s,u(s)+y(s))| +$$

$$+ \sum_{s=n_1}^{n_2-1} |B(s,u^i(s)+y(s)) - B(s,u(s)+y(s))|,$$

form which, by the continuity of A and B, follows that

$$\lim_{i\to\infty} |(Tu^i)(n) - (Tu)(n)|_B = 0.$$

Hence T is continuous.

$T\mathcal{A}$ is precompact. Since $T\mathcal{A}\subset\mathcal{A}$, then $T\mathcal{A}$ is uniformly bounded. It sufficies to prove that elements of $T\mathcal{A}$ satisfy Cauchy's condition uniformly on $T\mathcal{A}$. In fact, let $f\in\mathcal{A}$ and $n>m\in N^+_{n_1}$. Then we have

$$(Tf)(n)-(Tf)(m) = -\sum_{s=n}^{\infty}\left[A(s,f(s)+y(s))-A(s,y(s))\right]-$$

$$-\sum_{s=n}^{\infty} B(s,f(s)+y(s))+\sum_{s=m}^{\infty}\left[A(s,f(s)+y(s))-A(s,y(s))\right]+$$

$$+\sum_{s=m}^{\infty} B(s,f(s)+y(s))$$

which implies

$$|(Tf)(n)-(Tf)(m)| \le 2\sum_{s=m}^{\infty}|A(s,f(s)+y(s))-A(s,y(s))|+$$

$$+\,2\sum_{s=m}^{\infty}|B(s,f(s)+y(s))| \le 2\sum_{s=m}^{\infty}(Kh_1(s)+h_2(s))\,.$$

By assumption (14) last sum tends to zero as $m\to\infty$, so given $\varepsilon>0$, there exists $n_4\in N^+_{n_1}$ such that for all $f\in\mathcal{A}$, $y\in R^k$, $|(Tf)(n)-(Tf)(m)|<\varepsilon$ for all $n,m\in N_{n_4}$.

By Schauder's Fixed Point Theorem we conclude that there exists at least one fixed point in $\mathcal{A}$. This fixed point $f(n)$ has the property that the function

$$x(n) = f(n) + y(n)$$

satisfies (I).

Therefore we have to prove that

(15) $$|f(n)|\in l^p\,.$$

Using Minkowski's inequality we obtain

$$\left(\sum_{k=n_1}^{\infty}|f(n)|^p\right)^{1/p} \le \left(\sum_{k=n_1}^{\infty}\left(\sum_{s=k}^{\infty}|A(s,f(s)+y(s))-A(s,y(s))|+\right.\right.$$

$$\left.\left.+\sum_{s=k}^{\infty}|B(s,f(s)+y(s))|\right)^p\right)^{1/p} \le$$

$$\le \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} |A(s, f(s)+y(s)) - A(s, y(s))| \right)^p \right)^{1/p} +$$

$$+ \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} |B(s, f(s)+y(s))| \right)^p \right)^{1/p} \le$$

$$\le \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} h_1(s) g(|f(s)|) \right)^p \right)^{1/p} + \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} h_2(s) \right)^p \right)^{1/p} \le$$

$$\le K \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} h_1(s) \right)^p \right)^{1/p} + \left(\sum_{k=n_1}^{\infty} \left(\sum_{s=k}^{\infty} h_2(s) \right)^p \right)^{1/p} .$$

By the assumptions of theorem we get $|x(n) - y(n)| \in l^p$.

To verify that (1) holds, observe that we have

$$|f(n)| \le \sum_{s=n}^{\infty} |A(s, f(s)+y(s)) - A(s, y(s))| +$$

$$+ \sum_{s=n}^{\infty} |B(s, f(s)+y(s))| \le \sum_{s=n}^{\infty} h_1(s) g(|f(s)|) + \sum_{s=n}^{\infty} h_2(s) \le$$

$$\le \sum_{s=n}^{\infty} [K h_1(s) + h_2(s)] \to 0 \quad \text{as} \quad n \to \infty .$$

The proof of theorem is complete.

Theorem 4. Suppose that $x(n)$, $n \in N_{n_0}^+$ *is a fixed solution of (I) and that* $A(n,u)$ *satisfies Condition* $C_2(h_1, g, r, n_0)$ *with* $y(n)$ *replaced by* $x(n)$ *throughout. Moreover, assume that*

$$\sum_{n=n_0}^{\infty} |B(n, x(n))| < \infty .$$

Then there exists a solution $y(n)$ *of (II) such that*

$$|x(n) - y(n)| \in l^p .$$

Proof. We consider now the operator $T: \mathcal{A} \to \mathcal{A}$ such that

$$(Tf)(n) = -\sum_{k=n}^{\infty} \left[A(k,f(k)+x(k)) - A(k,x(k)) \right] - \sum_{k=n}^{\infty} B(k,x(k))$$

and the proof follows as in Theorem 3.

Theorem 5. Suppose that $A(n,u)$, $B(n,u)$ *satisfy Conditions* $C_2(l_1,g,r,n_0)$, $C_2(l_2,g,r,n_0)$ *respectively, where* g *is the identity function on* $<0,+\infty)$. *Moreover, assume that*

$$\sum_{k=n_0}^{\infty} |B(k,y(k))| < +\infty \quad \textit{for each solution } y(n) \textit{ of (II)}.$$

Then the solution $y(n)$ *whose existence is ensured by Theorem 3 is unique.*

Proof. It is easy to verify that the operator T in this case is a contraction mapping.

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