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EXISTENCE OF GLOBAL SOLUTIONS FOR SOME HIGHER ORDER DIFFERENTIAL AND INTEGRODIFFERENTIAL EQUATIONS

In this paper we study the existence of global solutions for some higher order differential and integrodifferential equations. The main tool employed in our analysis is based on a simple and classical application of the Leray-Schauder alternative.

Key words: existence theorems, differential and integrodifferential equations, Leray-Schauder alternative, Wintner's theorem.

1. Introduction

This paper is concerned with the existence of global solutions for higher order differential equations of the forms:

$$(P_1) \qquad y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}),$$

and

$$(P_2) \qquad y^{(n)} + \sum_{i=1}^{n} a_i(t) y^{(n-i)} = f(t, y, y', \ldots, y^{(n-1)}),$$

considered as a perturbation of the linear equation

$$(P_3) \qquad y^{(n)} + \sum_{i=1}^{n} a_i(t) y^{(n-i)} = 0,$$

and the higher order integrodifferential equations of the forms:

$$(P_4) \qquad y^{(n)} = g(t, y, y', \ldots, y^{(n-1)},\ Sy),$$

and

$$(P_5) \qquad y^{(n)} + \sum_{i=1}^{n} a_i(t) y^{(n-1)} = g(t, y, y', \ldots, y^{(n-1)},\ Sy),$$

considered as a perturbation of the linear equation (P_5), where

$$(P_6) \qquad S\, y(t) = \int_{t_o}^{t} h(t, s, y(s), y'(s), \ldots, y^{(n-1)}(s))ds.$$

Along with each of the above equations we also consider the initial condition

(C) $y^{(k)}(t_0) = c_k$, $k = 0,1,2,\ldots,n-1$.

In the following when mention the term „solution of the equation (P_i), $i = 1,\ldots,6$", or „solution of the problem (P_i), $i = 1,\ldots,6$" we will mean a solution of the corresponding initial value problem $(P_i)-(C)$, $i = 1,\ldots,6$. In equations $(P_1)-(P_6)$, (C) we will assume that $a_i \in C(I,R)$, $f \in C(I \times R^n, R)$, $H \in C(I^2 \times R^n, R)$, $g \in C(I \times R^{n+1}, R)$, c_k are given real constants, where $I = [t_0, T]$, $t_0 \geq 0$, $T > 0$ and R denotes the set of real numbers. We define $B = C^{n-1}(I) = C^{(n-1)}(I,R)$ to be the Banach space of functions u such that $u^{(n-1)}$ is continuous on I endowed with norm $\|u\| = \max\left\{ |u|_0, |u'|_0, \ldots, |u^{(n-1)}|_0 \right\}$, where $|u|_0 = \max\left\{ |u(t)| : t \in I \right\}$.

The problems of existence and other properties of the solutions of the equations (P_1), (P_2), (P_4), (P_5) and their special versions have been studied by many authors by using different techniques, see [2,4,6,7,12,13] and the references cited therein. Our main objective here is to study the existence of global solutions of equations (P_1), (P_2), (P_4), (P_5) by using a simple and classical application of the topological transversality theorem of Granas [3, p. 61], known as Leray-Schauder alternative. An important feature of this method, is that this yields simultaneously the existence of a solution and the maximal interval of existence. In fact, the results obtained here are motivated by a remarkable theorem given by Wintner in [14] concerning the nonlocal existence of solutions of ordinary differential equations and its extensions recently given by various investigators in $[1,8-11]$ by using topological arguments based on the Leray-Schauder alternative.

2. Statement of results

Our existence theorems are based on the following theorem, which is a version of the topolgical tranversality theorem given by A. Granas in [3, p. 61].

Theorem G. Let B be a convex subset of a normed linear space E and assume $O \in B$. Let $F : B \to B$ be a completely continuous operator and let

$$U(F) = \{x \in B : x = \lambda F x \quad \textit{for some} \quad 0 < \lambda < 1\}$$

Then either $U(F)$ is bounded or F has a fixed point.

We list the folowing hypotheses used in our discussion.

(H_1) There exists a function $p \in C(I, R_+)$, $R_+ = [0, \infty)$, such that

$$|f(t, y, y', \ldots, y^{(n-1)})| \leq p(t)\, H\left(\sum_{i=0}^{n-1} |y^{(i)}|\right),$$

where $H : R_+ \to (0, \infty)$ is a continuous nondecreasing function.

(H_2) There exists a function $q \in C(I, R_+)$ such that

$$|g(t, y, y', \ldots, y^{(n-1)}, Sy)| \leq q(t)\left(\sum_{i=0}^{n-1} |y^{(i)}| + |Sy|\right),$$

for $t \in I$.

(H_3) There exists a function $r \in C(I, R_+)$ such that

$$|h(t, s, y, y', \ldots, y^{n-1)})| \leq r(s)\, H\left(\sum_{i=0}^{n-1} |y^{(i)}|\right),$$

where $H : R_+ \to (0, \infty)$ is a continuous nondecreasing function.

Our first results deals with the existence of global solutions of the problem (P_1).

Theorem 1. Suppose that the hypothesis (H_1) is satisfied. Then the problem (P_1) has a solution y in B provided that T satisfies

$$N \int_{t_0}^{T} p(s)ds < \int_{M}^{\infty} \frac{ds}{H(s)}, \tag{2.1}$$

where

$$N = \sum_{j=0}^{n-1} \frac{1}{(n-j-1)!}(T - t_0)^{n-j-1}, \tag{2.2}$$

$$M = \sum_{j=0}^{n-1}\left[\sum_{i=j}^{n-1} \frac{|c_i|(T - t_0)^{i-j}}{(i-j)!}\right]. \tag{2.3}$$

In the following theorem we establish the existence of a global solution of problem (P_2). In what follows, $\phi_1, \ldots, \phi_n$ denote the n lineary independent solutions of (P_3) and

$$\sigma_j(t) = \frac{W_j(\phi_1,\ldots,\phi_n)(t)}{W(\phi_1,\ldots,\phi_n)(t)}, \qquad t \in I, \quad \text{where } W_j(\phi_1,\ldots,\phi_n) \text{ is the}$$

determinat obtained from the Wronskian $W(\phi_1,\ldots,\phi_n)$ by replacing the j-th column by $(0,0,\ldots,0,1)$.

Theorem 2. Let $a_1 \in C(I,R)$, $\phi_1,\ldots,\phi_n$ *and* σ_1 *be as explained above. Suppose that the hypothesis* (H_1) *is satisfied. Then the problem* (P_2) *has a solution y in B provided that T satisfies*

$$\overline{A}\int_{t_0}^{T} p(s)\,ds < \int_{A}^{\infty} \frac{ds}{H(s)}, \tag{2.4}$$

where

$$A = \sum_{i=0}^{n-1}\sum_{j=1}^{n} |d_j|\, A_i\,, \tag{2.5}$$

$$\overline{A} = \sum_{i=0}^{n-1}\sum_{j=1}^{n} A_i\, B_j\,, \tag{2.6}$$

in which d_j *are real constants,*

$$A_i = \max\left\{|\Phi_j^{(i)}(t)| : t \in I\right\}, \qquad B_j = \max\left\{|\sigma_j(t)| : t \in I\right\}.$$

Remark 1. We note that our Theorem 1 is a further generalization of the well known theorem of Wintner given in [14], to the higher order differential equations of the form (P_1). In Wintner's theorem $n=1$ in (P_1) and $p(t)=1$ in hypothesis (H_1) and the integral on the right side in (2.1) is assumed to diverge. Thus a solution exists for every $T<\infty$. For further details about extensions of Wintner's theorem for differential equations, see [1,8]. The result given in Theorem 2 can be considered as a further generalization of the Wintner's theorem given [14] to the problem (P_2).

Next we will establish the following theorems which deal with the global existence of solutions of problems (P_4) and (P_5).

Theorem 3. Suppose that the hypotheses (H_2) *and* (H_3) *are satisfied. Then the problem* (P_4) *has a solution y in B provided that T satisfies*

$$\int_{t_0}^{T} Q(s)\,ds < \int_{M}^{\infty} \frac{ds}{s+H(s)}, \tag{2.7}$$

where $Q(t)=\max\{Nq(t),r(t)\}$, $t\in I$, N *and* M *are defined an in (2.2) and (2.3).*

Theorem 4. Let a_i, $\phi_1,\dots,\phi_n$ *and* σ *be as in Theorem 2. Suppose that the hypotheses* (H_2) *and* (H_3) *are satisfied. Then the problem* (P_3) *has a solution* y *in* B *provided that* T *satisfies*

$$\int_{t_0}^{T} G(s)\,ds < \int_{A}^{\infty}\frac{ds}{s+H(s)}, \tag{2.8}$$

where $G(t)=\max\{\bar{A}q(t),r(t)\}$, $t\in I$, A *and* $\bar{A}$ *are defined as in (2.5) and (2.6).*

Remark 2. If we replace the hypothesis (H_2) by:
(H_2') There exists a function $q\in C(I,R_+)$ such that

$$|g(t,y,y',\dots,y^{(n-1)},Sy)|\le q(t)\left(H\sum_{i=o}^{n-1}|y^{(i)}|+|Sy|\right),$$

where H is the same function defined as in (H_2), and the condition (2.7) by

$$N\int_{t_o}^{T} q(s)\left(1+\int_{t_0}^{s} r(\tau)d\tau\right)ds < \int_{M}^{\infty}\frac{ds}{H(s)}, \tag{2.9}$$

then the conclusion of Theorem 3 remains valid. In fact the function H appearing in (H_2') and (H_3) can be considered as two different functions Ω_1 and Ω_2 respectively satisfying the same conditions as those of the function H, then by taking $H(t)=\max\{\Omega_1(t),\Omega_2(t)\}$, we see that the conclusion of Theorem 3 holds provided that T satisfies the condition (2.9). Similar remark holds in case of Theorem 4.

3. Proofs of theorems 1 – 4

Since the proofs of Theorems 1 – 4 resemble one another, we only prove Theorems 1 and 4, the proofs of Theorems 2 and 3 can be completed by following the proofs of Theorems 1 and 4.

In order to prove Theorem 1 we apply Theorem G. First we establish the a-priori bounds for the initial value problem $(P_1)_\lambda$, $\lambda\in(0,1)$, where

$(\mathrm{P}_1)_\lambda \qquad y^{(n)} = \lambda f(t, y, y', \ldots, y^{(n-1)}), \quad y^{(k)}(t_0) = c_k, \quad k = 0, 1, \ldots, n-1.$

Let y(t) be solution of $(\mathrm{P}_1)_\lambda$. Then this solution of $(\mathrm{P}_1)_\lambda$ and its derivatives can be writen as

$$(3.1) \qquad y^{(j)}(t) = \sum_{i=j}^{n-1} \frac{c_1 (t-t_0)^{i-j}}{(i-j)!} + \lambda \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} \bar{f}(y(s))\, ds ,$$

for $0 \le j \le n-1$. Here we have used the notation $\bar{f}(y(s))$ for $f(s, y(s) y'(s), \ldots, y^{(n-1)}(s))$. From (3.1) and using the hypothesis (H_1) we have

$$(3.2) \qquad \sum_{j=0}^{n-1} |y^{(j)}(t)| \le \sum_{j=0}^{n-1} \left[\sum_{i=j}^{n-1} \frac{|c_1| (t-t_0)^{i-j}}{(i-j)!} \right] + $$

$$+ \sum_{j=0}^{n-1} \int_{t_0}^{t} \frac{(t-s)^{n-j-1}}{(n-j-1)!} |\bar{f}(y(s))| ds \le$$

$$\le M + N \int_{t_0}^{t} p(s) H\left(\sum_{j=0}^{n-1} |y^{(j)}(s)| \right) ds .$$

Denoting by $u(t)$ the right-hand side of (3.2), we obtain

$$\sum_{j=0}^{n-1} |y^{(j)}(t)| \le u(t) , \qquad u(t_0) = M$$

$$(3.3) \qquad u'(t) \le N\, p(t) H(u(t))$$

From (3.3) follows that

$$(3.4) \qquad \frac{u'(t)}{H(u(t))} \le N\, p(t) .$$

The integration of (3.4) from t_0 to t and the use of the change of variable and the condition (2.1) give

$$(3.5) \qquad \int_{M}^{u(t)} \frac{ds}{H(s)} = \int_{t_0}^{t} \frac{u'(s)}{H(u(s))} ds \le N \int_{t_0}^{t} p(s)\, ds \le N \int_{t_0}^{T} p(s)\, ds < \int_{M}^{\infty} \frac{ds}{H(s)} .$$

From (3.5) follows that $u(t)$ must be bounded on I, i.e. there is a positive constant α independent of $\lambda \in (0,1)$ such that $u(t) \le \alpha$ and hence $\sum_{j=0}^{n-1} |y^{(j)}(t)| \le \alpha$ for $t \in I$. Thus we have $|y^{(j)}(t)| \le \alpha$, $t \in I$ for $0 \le j \le n-1$, and consequently $\|y\| \le \alpha$.

In the next step we rewrite the problem (P_1) as follows. If $z \in B$ and $y(t) = z(t) + e(t)$, where $e(t) = \sum_{i=0}^{n-1} c_i (t-t_0)^i / i!$, $t \in I$, it is easy to see that $z(t)$ satisfies $z(t_0) = 0$

$$z(t) = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f^*(z(s))\, ds \;;$$

if and only if $y(t)$ satisfies (P_1) or its equivalent integral equation

$$y(t) = \sum_{i=0}^{n-1} \frac{c_i (t-t_0)^i}{i!} + \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \bar{f}(y(s)) ds \;.$$

Here for convenience we have used the notation $f^*(z(s))$ for

$$f(s, z(s)+e(s), z'(s)+e'(s), \ldots, z^{(n-1)}(s)+e^{(n-1)}(s)) \;.$$

use this notation in the later part of the proof without further mentioning Define : $F : B_0 \to B_0$, $B_0 = \{z \in B : z(t_0) = 0\}$ by

$$F\, z(t) = \frac{1}{(n-1)!} \int_{t_o}^{t} (t-s)^{n-1} f^*(z(s))\, ds \tag{3.6}$$

for $t \in I$. Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0 , i.e. $\|w_k\| \le b$ for all k, where b is positive constant. From (3.6) and using hypothesis (H_1) and letting $p^* = \sup\{(t) : t \in I\}$ and $e^* = \sup\{|e^{(j)}(t)| : t \in I, 0 \le j \le n-1\}$, we have

$$|(Fw_k(t))^{(j)}| \le \frac{1}{(n-j-1)!} \int_{t_0}^{t} (t-s)^{n-j-1} |f^*(w_k(s))|\, ds \le \tag{3.7}$$

$$\le \frac{1}{(n-j-1)} p^* H(n(b+e^*))(T-t_0)^{n-j} = L_j \,,$$

for $0 \le j \le n-1$. Hence, from (3.7) we obtain $\|F w_k\| \le L$, where $L = \max\{L_j : 0 \le j \le n-1\}$. This means that $\{Fw_k\}$ is uniformly bounded.

Now we shall show that the sequence $\{Fw_k\}$ is equicontinuous. Let $t_0 \le t_1 \le t_2 \le T$. Then from (3.6) and using the hypothesis (H_1), and the elementary inequality (see, [5, p.39]) $x^r - y^r \le r x^{r-1}(x-y)$ for $r \ge 1$ and x, y nonnegative reals, and letting $\{w_k\}$, p^*, e^* as defined above, we observe the following cases.

Case I. If $j = 0, 1, \ldots, n-2$ then $n-j-1 \ge 1$, and

$$|(F w_k(t_2))^{(j)} - (F w_k(t_1))^{j)}| = \tag{3.8}$$

$$= \frac{1}{(n-j-1)!}\left| \int_{t_1}^{t_2} (t_2 - s)^{n-j-1} f^*(w_k(s))\, ds + \right.$$

$$\left. + \int_{t_0}^{t_1} [(t_2 - s)^{n-j-1} - (t_1 - s)^{n-j-1]}]f^*(w_k(s))ds \right| \le$$

$$\le \frac{1}{(n-j-1)!}\left[\int_{t_1}^{t_2} (t_2 - s)^{n-j-1} |f^*(w_k(s))| ds + \right.$$

$$\left. + \int_{t_0}^{T} [(t_2 - s)^{n-j-1} - (t_1 - s)^{n-j-1}] |f^*(w_k(s))| ds \right] \le$$

$$\le \frac{1}{(n-j-1)!}\left[(T - t_0)^{n-j-1} \int_{t_1}^{t_2} p^* H(n(b+e^*))\, ds + \right.$$

$$\left. + \int_{t_0}^{T} (n-j-1)(t_2 - s)^{n-j-2}[t_2 - t_1] p^* H(n(b+e^*))\, ds \right] \le$$

$$\le \frac{1}{(n-j-1)!}\left[(T - t_0)^{n-j-1} \int_{t_1}^{t_2} p^* H(n(b+e^*))\, ds + \right.$$

$$\left. + (n-j-1)(T-t_0)^{n-j-2}(T-t_0) \int_{t_1}^{t_2} p^* H(n(b+e^*))\, ds \right] =$$

$$= \frac{1}{(n-j-1)!}(T - t_0)^{n-j-1}(n-j) \int_{t_1}^{t_2} p^*(H(n(b+e^*))\, ds \; .$$

Case II. If $j = n-1$, then $n-j-1=0$ and

$$|(F w_k(t_2))^{(n-1)} - (F w_k(t_1))^{(n-1)}| = \left| \int_{t_1}^{t_2} f^*(w_k(s))\, ds \right| \leq \tag{3.9}$$

$$\leq \int_{t_1}^{t_2} p^* H(n(b+e^*))\, ds .$$

From the above estimates, we conclude that $\{F w_k\}$ is equicontinuous and hence by the Arzela-Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F) = \{z \in B_0 : z = \lambda F z; \lambda \in (0,1)\}$is bounded, since for every z in $U(F)$ the function $y(t) = z(t) + e(t)$ is a solution of $(P_1)_\lambda$, for which we have proved that $||y|| \leq \alpha$ and hence $||z|| \leq \alpha + e^*$. By applying the Theorem G, the operator F has a fixed point in B_0. This means that the problem (P_1) has a solution $y(t)$ in B. This completes the proof of Theorem 1.

To prove Theorem 4 we apply Theorem G. First, we shall establish the a-priori bounds for the solutions of problem $(P_5)_\lambda$, $\lambda \in (0,1)$, where

$$(P_5)_\lambda \qquad y^{(n)} + \sum_{i=1}^{n} a_i(t) y^{(n-1)} = \lambda g(t, y, y', \ldots, y^{(n-1)}, S y),$$

which is considered as a perturbation of the linear equation (P_3). Let $y(t)$ be a solution of $(P_5)_\lambda$. By the variation of constants formula [2, 4], the general solution of $(P_5)_\lambda$ and its derivatives can be written as

$$y^{(i)}(t) = \sum_{j=1}^{n} d_j \phi_j^{(i)}(t) + \lambda \sum_{j=1}^{n} \phi_j^{(i)}(t) \int_{t_0}^{t} \sigma_j(s) \bar{g}(y(s))\, ds , \tag{3.10}$$

for $0 \leq i \leq n-1$, where d_j are constants and $\phi_j(t)$ are n linearly independent solutions of (P_3). Here we have used the notation $\bar{g}(y(s))$ for $g(s, y(s), y'(s), \ldots, y^{(n-1)}(s), S y(s))$. From (3.10) and using the hypotheses $(H_2)-(H_3)$ we have

$$\sum_{i=0}^{n-1} |y^{(i)}(t)| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{n} |d_j| |\phi_j^{(i)}(t)| + \tag{3.11}$$

$$+\sum_{i=0}^{n-1}\sum_{j=1}^{n}|\phi_j^{(j)}(t)|\int_{t_0}^{t}|\sigma_j(s)||\overline{g}(y(s))|ds\le$$

$$\le A+\overline{A}\int_{t_0}^{t}q(s)\left[\sum_{i=0}^{n-1}|y^{(i)}(s)|+\int_{t_0}^{s}r(\tau)H\left(\sum_{i=o}^{n-1}|y^{(i)}(\tau)|\right)d\tau\right]ds\ .$$

Denoting by $u(t)$ the right-hand side of (3.11), we have

$$\sum_{i=0}^{n-1}|y^{(i)}(t)|\le u(t)\ ,\qquad u(t_0)=A\ ,$$

$$u'(t)\le \overline{A}q(t)\left[u(t)+\int_{t_0}^{t}r(\tau)+H(u(\tau))d\tau\right]\ .$$

Define a function v(t) by

$$v(t)=u(t)+\int_{t_0}^{t}r(\tau)H(u(\tau))d\tau\ ,\qquad t\in I\ ,$$

then $u(t)\le v(t)$, $T\in I$, $v(t_0)=u(t_0)=A$ and

(3.12) $$v'(t)\le \overline{A}\,q(t)v(t)+r(t)H(v(t))\le$$
$$\le G(t)\left[v(t)+H(v(t))\right].$$

From (3.12) we have

(3.13) $$\frac{v'(t)}{v(t)+H(v(t))}\le G(t)\ .$$

The integration of (3.13) from t_0 to t and the use of the change of variable and the condition (2.8) give

(3.14) $$\int_{A}^{v(t)}\frac{ds}{s+H(s)}=\int_{t_0}^{t}\frac{v'(s)}{v(s)+H(v(s))}ds\le\int_{t_0}^{t}G(s)ds\le$$
$$\le\int_{t_0}^{T}G(s)\,ds\le\int_{A}^{\infty}\frac{ds}{s+H(s)}\ .$$

From (3.14) follows that $v(t)$ must be bounded on I, i.e. there is a positive constant α independent of $\lambda\in(0,1)$ such that $v(t)\le\alpha$ and hence $u(t) \le \alpha$ and consequently $\sum_{i=0}^{n-1} |y^{(i)}(t)| \le \alpha$ for $t \in I$. Thus we have $|y^{(i)}(t)| \le \alpha$ for $0 \le i \le n-1$, $t \in I$ and consequently $\|y\| \le \alpha$.

In the next step we rewrite the solution $y(t)$ of (P_5) as follows. If $z \in B$ and $y(t) = z(t) + \sum_{j=1}^{n} d_j\, \phi_j(t)$, $t \in I$ it is easy to see that $z(t)$ satisfies

$$z(t_0) = 0,$$

$$z(t) = \sum_{j=1}^{n} \phi_j(t) \int_{t_0}^{t} \sigma_j(s)\, g^*(z(s))\, ds,$$

if and only if $y(t)$ satisfies

$$y(t) = \sum_{j=1}^{n} d_j\, \phi_j(t) + \sum_{j=1}^{n} \phi_j(t) \int_{t_0}^{t} \sigma_j(s)\, \bar{g}(y(s))\, ds.$$

Here we have used the notation $g^*(z(s))$ for

$$g\Bigg(s, z(s) + \sum_{j=1}^{n} d_j \phi_j(s), z'(s) + \sum_{j=1}^{n} d_j \phi'_j(s), \dots, z^{(n-1)}(s) + \\ + \sum_{j=1}^{n} d_j\, \phi_j^{(n-1)}(s), S(z(s) + \sum_{j=n}^{n} d_j \phi_j(s)) \Bigg).$$

Define $F : B_0 \to B_0$, $B_0 = \{z \in B : z(t_0) = 0\}$ by

$$F\, z(t) = \sum_{j=1}^{n} \phi_j(t) \int_{t_0}^{t} \sigma_j(s)\, g^*(z(s))\, ds, \tag{3.15}$$

for $t \in I$. Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0, i.e. $\|w_k\| \le b$, for all k, where b is a positive constant. From (3.15) and using hypotheses $(H_2)-(H_3)$ and defining $G^* = \sup\{G(t) : t \in I\}$ we have

$$|(Fw_k(t))^{(i)}| \le \sum_{j=1}^{n} |\phi_j^{(i)}(t)| \int_{t_0}^{t} |\sigma(s)||g^*(w_k(s))| ds \le \tag{3.16}$$

$$\le \sum_{j=1}^{n} A_i B_j G^* \left[\sum_{i=0}^{n-1} \left(b + \sum_{i=0}^{n} |d_j| A_i \right) + \right.$$

$$\left. + G^* H \left(\sum_{i=0}^{n-1} \left(b + \sum_{j=1}^{n} |d_j| A_i \right) \right) (T - t_0) \right] (T - t_0) =$$

$$= \sum_{j=1}^{n} A_i B_j \beta (T - t_0) = L_i ,$$

for $0 \le i \le n-1$, where

$$(3.17) \quad \beta = G^* \left[\sum_{i=0}^{n-1} \left(b + \sum_{j=1}^{n} |d_j| A_i \right) + G^* H \left(\sum_{i=0}^{n-1} \left(b + \sum_{j=1}^{n} |d_j| A_i \right) \right) (T - t_0) \right].$$

Hence from (3.17) we obtain $\|F w_k\| \le L$, where $L = \max \{ L_i : 0 \le i \le n-1 \}$. This means that $\{F w_k\}$ is uniformly bounded

Now we shall prove that the sequence $\{F w_k\}$ is equicontinuous. Let $t_o \le t_1 \le t_2 \le T$. Then from (3.15) and using the hypotheses (H_2), (H_3) we have, for $0 \le i \le n-1$,

$$(3.18) \quad |(Fw_k(t_2))^{(i)} - (Fw_k(t_1))^{(i)}| +$$

$$+ \left| \int_{t_j}^{t_2} \sum_{j=1}^{n} \phi_j^{(i)}(t_2) \sigma_j(s) g^*(w_k(s)) ds = \right.$$

$$= \int_{t_0}^{t} \sum_{j=1}^{n} \left[\phi_j^{(i)}(t_2) - \phi_j^{(i)}(t_1) \right] \sigma_j(s) g^*(w_k(s)) ds | \le$$

$$\le \int_{t_1}^{t_2} \sum_{j=1}^{n} A_i B_j \beta ds + \int_{t_o}^{T} \sum_{j=1}^{n} \left| \phi_j^{(i)}(t_2) - \phi_j^{(i)}(t_1) \right| B_j \beta ds ,$$

where β is defined by (3.17). From (3.18) and by the continuity of $\phi_j^{(i)}(t)$, $t \in I$, we conclude that $\{F w_k\}$ is equicontinuous and hence by Arzela--Ascoli theorem operator F is completely continuous.

The rest of the proof can be completed by following the similar arguments as in the proof of Theorem 1 given above with suitable modifications. This completes the proof of Theorem 4.

Remark 3. In a recent paper [9] the authors have established the existence of solutions for systems of differential-difference equations of delay type using topological arguments based on a nonlinear alternative. We note that one can also extend the ideas of this paper to the problems of the forms (P_1), (P_2), (P_4), (P_5) when the functions appearing on the right sides depend on the delay arguments, under appropriate initial conditions. For similar results, see [10, 11].

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