FASCICULI MATHEMATICI
Nr 27 1997

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ON SOME CLASS OF THE PARABOLIC POLYNOMIALS

The subject of the paper is the construction of the polynomials satisfying some parabolic equations of the second and or the fourth order. The approximative formula to the solution of the fourth order parabolic equation is given. The modal lines of the heat polymials are examined.

Key words: parabolic polymials, degree of the homogennity, approximative formula, the Fourier problem.

1. Introduction

In the present paper we shall construct the parabolic polynomials with different method with respect to those given in [1]. We shall construct also the polymials solutions to some more general equations of parabolic type homogeneous and non homogeneous of the order two and four. We shall introduce the degree of the homogenity to the parabolic polymials. We shall give also the approximative formula to the solutions of the parabolic equations of the fourth order and an example of the polymial solution to the Fourier problem. The nodal lines for some parabolic polymials are given.

2. The heat polynomials

In the paper [1] the parabolic polynomials only to the heat equation

(1) $$(D_x^2 - D_t)\, u(x,t) = 0$$

are given. This polynomials $P_n(x,t)$ are of the form

(2) $$P_n(x,t) = n! \sum_{j=0}^{C(n/2)} \frac{x^{n-2i}}{(n-2j)!} \frac{t^j}{j}$$

$C(x)$ being the most integer non greater then x.

In [1] the polynomials are defined by the generating formula

$$\exp(z\,x + z^2 t) = \sum_{j=0}^{\infty} P_n(x,t) \frac{z^n}{n!}.$$

In the present paper we shall give the another construction to the polynomials P_n. Moreover we shall construct the parabolic polynomials to the equations

(1a) $(D_x^2 - D_t)\, u(x,t) = f(x,t), \qquad (x,t) \in R^2,$

(3) $(D_x^2 - (a(t))^{-1} D_t)\, u(x,t) = 0, \qquad (x,t) \in R^1 \cap R^+,$

(4) $(D_x^2 + W(t)D_x + c(t) - D_t)u(x,t) = 0\,, \qquad x \in R^1, \quad t > 0\,,$

(5) $(\Delta - D_t)\, u(x,t) = 0,\ x = (x_1, x_2),\ \Delta = D_{x_1}^2 D_{x_2}^2,\ (x_1, x_2, t) \in R^3,$

(6) $(\Delta - D_t)u(x,t) = f(x,t), \quad x_2 = (x_1, x_2),$

(7) $(D_x^4 - D_t)u(x,t) = 0, \quad (x,t) \in R^2,$

(8) $(D_x^4 - D_t)\, u(x,t) = f(x,t), \quad (x,t) \in R^2,$

(9) $(D_{x_1}^4 + D_{x_2}^4 - D_t)u(x,t) = 0\,, \quad x = (x_1, x_2), \quad (x,t) \in R^3,$

(10) $(D_{x_1}^4 + D_{x_2}^4 - D_t)u(x,t) = f(x,t), \quad x = (x_1, x_2), \quad (x,t) \in R^3$

where the functions f are given polynomials, the function $a(t) = t^{m-1}$, $m > 1$, $W(t)$ is a polynomial and the function $c(t) = Q'(t)/Q(t)$, $Q(t)$ being a positive polynomial.

All polynomial solutions of the equations (1a), (3) – (10) here given are not known in the literature.

3. The another construction for the polynomials P_n

Let us consider the formulas

$$(n!)^{-1} K_{2n}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{2j}\, W_{2n}(x)\,, \qquad W_{2n}(x) = x^{2n}\,,$$

$$(n!)^{-1} K_{2n+1}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{2j}\, W_{2n+1}(x)\,, \qquad W_{2n+1}(x) = x^{2n+1}\,.$$

Lemma 1. The functions $(n!)^{-1} K_{2n}(x,t)$, $(n!)^{-1} K_{2n+1}(x,t)$ satisfy the equation (1).

Proof. We shall give the proof only to the function $(n!)^{-1} K_{2n}(x,t)$. The proof to the function $(n!)^{-1} K_{2n+1}(x,t)$ is similar. We have

$$D_x^2 K_{2n}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{2j+2}\, W_{2n}(x) = \sum_{j=1}^{n} \frac{t^{j-1}}{(j-1)!} D_x^j\, W_{2n}(x).$$

We have also

$$D_t K_{2n}(x,t) = \sum_{j=1}^{n} \frac{t^{j-1}}{(j-1)!} D_x^2 W_{2n}(x).$$

By the last formula we obtain the assertion of the Lemma 1.

4. The explicit formulas to K_{2n+1}, K_{2n}

Next we shall give the other formulas for $(n!)^{-1} K_{2n}(x,t)$, and $(n!)^{-1} K_{2n+1}(x,t)$ for $m = 2n$ and $m = 2n+1$.

Theorem 1. The formula

(11) $$(m!)^{-1} K_m(x,t) = P_m(x,t), \qquad m = 0,1,...$$

holds.

Proof. We have

$$D_x^{2j} x^m = m(m-1)...(m-2j+1)x^{m-2j} = \frac{m}{(m-2j)!} x^{m-2j},$$

$$j = 0,1,..., C(n/2).$$

By the last formula we obtain the formula (11).

5. Examples of the polynomials P_n

By the formula (11) we obtain:

$$P_0(x,t) = 1, \qquad P_1(x,t) = x, \qquad P_2(x,t) = \frac{x^2}{2!} + t,$$

$$P_3(x,t) = \frac{x^3}{3!} + xt, \qquad P_4(x,t) = \frac{x^4}{4!} + \frac{x^2}{2!} t + \frac{t^2}{2!},$$

$$P_5(x,t) = \frac{x^5}{5!} + \frac{x^3}{3!} t + \frac{t^2}{2!},$$

$$P_{2k}(x,t) = \frac{x^{2k}}{(2k)!} + \frac{x^{2k-2}}{(2k-2)!} t + \cdots + \frac{t^k}{(k)!},$$

$$P_{2k+1}(x,t) = \frac{x^{2k+1}}{(2k+1)!} + \frac{x^{2k-1}}{(2k-1)!} t + \cdots + x \frac{t^k}{(k)!}$$

6. Degree of the homogenitx to the polynomials P_n

Let us consider the monomials

$$m_j(x,t) = C_{n,j}\, t^j\, x^{n-2j}, \qquad j = 0,1,\dots,C(n/2)\,.$$

The monomial m_j is called (1,2) homogenous if for every $q>0$, $(x,t)\neq(0,0)$ the identity

$m_j(qx,q^2t) = q^n m_j(x,t), \qquad j = 0,1,\dots,C(n/2)\,, \qquad n\in N$, holds.

The polynomial P_n is colled (1,2) homogenous if all the its summands are (1,2) homogenous.

Theorem 2. The polynomials $P_n(x,t)$ are (1,2) homogeneous.

Proof. To the proof it is sufficient to verify that m_j are (1,2) homogeneous. Indeed. We have

$$m_j(qx,q^2t) = q^n\, m_j(x,t), \qquad j = 0,1,\dots,C(n/2).$$

7. Parabolic polynomials to the equation (1a)

Let $F(x,t) = (ax+b)\int_0^t V(s)\,ds,\ V(s) = \sum_{k=0}^{n} a_k\, s^k,\ n\in N\,.$

Theorem 3. The functions

$$R_n(x,t) \;=\; P_n(x,t) \;+\; F(x,t),$$

satisfy the equation (1a), with $f(x,t) \;=\; -(ax+b)V(t)$.

Proof. We have

$$(D_x^2 - D_t)R_n(x,t) \;=$$

$$= (D_x^2(ax+b))\int_0^t V(s)\,ds - (ax+b)\,V(t) \;=\; f(x,t)\,.$$

8. The parabolic polynomials to the equation (3)

Let us consider the transformation

(12) $$s = F(t) = \int_0^t a(v)\,dv ,$$

where $a(v)$ is continuous and positive function. Hence the function F is inversible. Let $t = F^{-1}(s)$ denote the inverse function with respect to F. We have

(13) $$u(x,t)\big|_{t=F^{-1}(s)} = p(x,s), \quad s>0, \quad u(x,t) = p(x,t)\big|_{s=F(t)} .$$

Theorem 4. The transformations (12), (13) transforme the equation (3) into the equation

$$D_x^2\, p(x,s) - D_s\, p(x,s) = 0, \qquad s>0 .$$

Proof. By (12), (12a) we have

$$D_x^2\, u(x,t)\big|_{t=F^{-1}(s)} = D_x^2\, p(x,s) , \qquad s>0,$$

and

$$(a(t))^{-1} D_t\, u(x,t)\big|_{t=F^{-1}(s)} =$$
$$= (a(F^{-1}(s)))^{-1} a(F^{-1}(s))\, D_s\, p(x,s) = D_s\, p(x,s) .$$

Consequently we obtain (13).

By Theorem 4, we get

Theorem 5. If $a(v) = v^{m-1}$, $m>1$, $s = F(t) = m^{-1}t^m$, *then the polynomials*

$$P_n(x,s) = \sum_{j=0}^{C(n/2)} \frac{s^j}{j!}\frac{x^{n-2j}}{(n-2j)!} , \qquad s>0$$

are the parabolic polynomials to the equations (14) and the polynomials

$$Q_n(x,t) = P_n(x,t)\big|_{s=m^{-1}t^m} = \sum_{j=0}^{C(n/2)} C_{n,j}\, x^{n-2j}\, t^{mj} ,$$

$$C_{n,j} = \frac{m^{-j}}{j!}\frac{1}{(n-2j)!} ,$$

are the parabolic polynomials to the equation (3).

Remark 1. The polynomials $Q_n(x,t)$ are (l,k) homogegeneous for $k = 2(m)^{-1}$.

9. The parabolic polynomials to the equations (4)

Next we shall construct the polynomials $u_n(x,t)$ satisfying the equation (4). We assume that $w(t)$ is a polynomials and $c(t) = Q'(t)/Q(t)$, $Q(t)$ is a positive polynomial. Applying the transformation of the unknown function

$$u(x,t) = V(x,t)\,Q(t),$$

we obtain the equation

(15) $$D_x^2 V(x,t) + w(t) D_x V(x,t) - D_t V(x,t) = 0.$$

Using in (15) tha change of independent variable

$$x = X - \int_0^t w(s)ds\,, \quad t = t,$$

and putting

$$V(x,t)\big|_{x = X - \int_0^t w(s)ds} = Z(X,t),$$

we obtain

$$D_x V(x,t)\big|_{x = X - \int_0^t w(s)ds} = D_x Z(X,t),$$

$$D_x^2 V(x,t)\big|_{x = X - \int_0^t w(s)ds} = D_x^2 Z(X,t),$$

and $$D_t V(x,t)\big|_{x = X - \int_0^t w(s)ds} = w(t)\, D_x Z(X,t) + D_t Z(X,t).$$

By the last formulas and by (15), we obtain the equation

(16) $$D_x^2 Z(X,t) - D_t Z(X,t) = 0.$$

Denote by $Z_n(X,t)$ the parabolic polynomials to the equation (16).

We obtain

$$V_n(x,t) = Z_n\left(X - \int_0^t w(s)\,ds, t\right) = \sum_{j=0}^{C(n/2)} \frac{t^j \left(X - \int_0^t w(s)ds\right)^{n-j}}{j!(n-2j)!}.$$

Hence we obtain

Theorem 6 . *The functions*

$$u_n(x,t) = V_n(x,t)Q(t)$$

are the parabolic polynomials to the equation (4).

10. The multiplicative polynomials $P_{m,n}$ to the equation (5)

Lemma 2. *If the functions* $u_i = u_i(x_i, t)$, $i = 1,2$, *satisfy the heat equations*

$$(D_{x_i}^2 - D_t)\, u_i(x_i, t) = 0\,, \qquad i = 1,2,$$

respectively, then the function

$$u(x,t) = u_1(x_1,t)\, u_2(x_2,t), \quad x = (x_1, x_2),$$

satisfies the equation (5).

Proof. We have

$$(\Delta - D_t)\, u(x,t) =$$
$$= u_1(x_1,t)\,(D_{x_2}^2 - D_t)\, u_2(x_2,t) + u_2(x_2,t)\,(D_{x_1}^2 - D_t)\, u_1(x_1,t) = 0$$

By Lemma 2 and (2), we obtain

Theorem 7. *If* $u_1(x_1,t) = P_n(x_1,t)$, $u_2(x_2,t) = P_m(x_2,t)$, *then the polynomials*

$$P_{n,m}(x,t) = P_n(x_1,t) P_m(x_2,t),$$

satisfy the equation (5).

Remark 2. The polynomials $P_{n,m}$ are not $(1,2)$ homogeneous. For example the polynomial

$$P_{2,2}(x,t) = (1/2\, x_1^2 + t)(1/2\, x_2^2 + t) = 1/4\, x_1^2 x_2^2 + 1/2\, t(x_1^2 + x_2^2) + t^2\ ,$$

is obviously non $(1,2)$ homogeneous and all its summands are of the different degree of the homogenity.

11. The polynomials to the equation (6)

Let us consider the polynomials

$$F(x,t) = h(x)\int_0^t V(s)ds\,, \qquad R_{n,m}(x,t) = P_{n,m}(x,t) + F(x,t),$$

$$f(x,t) = -h(x)\,V(t),$$

where $h(x)$ is a harmonic polynomials of the form

$$h(x) = \sum_{j=0}^{n} (C_{1,j}\,\mathrm{Re}(x_1 + ix_2)^j + C_{2,j}\,\mathrm{Im}(x_1 + ix_2)^j),$$

$C_{1,j}, C_{2,j}$ are arbitrary constants.

Theorem 8. *The functions $R_{m,n}$ satisfy the equation*

$$(\Delta - D_t)\,R_{m,n}(x,t) = f(x,t).$$

Proof. We have

$$(\Delta - D_t)\,R_{m,n}(x,t) = (\Delta h(x))\int_0^t V(s)\,ds - h(x)\,V(t) = f(x,t).$$

12. The parabolic polynomials to the equations (7), (8)

Let us consider the polynomials

(17)

$$S_{4n}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{4j}\,W_{4n}(x), \qquad \text{with } W_{4n}(x) = x^{4n},$$

$$S_{4n+1}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{4j}\,W_{4n+1}(x), \qquad \text{with } W_{4n+1}(x) = x^{4n+1},$$

$$S_{4n+2}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{4j}\,W_{4n+2}(x), \qquad W_{4n+2}(x) = x^{4n+2},$$

$$S_{4n+3}(x,t) = \sum_{j=0}^{n} \frac{t^j}{j!} D_x^{4j}\,W_{4n+3}(x), \qquad W_{4n+3}(x) = x^{4n+3}, \quad n \in N$$

Let

(18)

$$Z_n(x,t) = m!\sum_{j=0}^{n} \frac{t^j}{j!}\frac{x^{m-4j}}{(m-4j)!}.$$

Theorem 9. The polynomials $S_{4n+i}(x,t)$, $i=0,1,2,3$, *satisfy the equation* (7).

Proof. We shall give the proof only for S_{4n}. The proof for S_{4n+1}, $i=0,1,2,3$, is similar.

We have

$$D_x^4 S_{4n}(x,t)=\sum_{j=0}^{n}\frac{t^j}{j!}D_x^{4(j+1)}x^{4n}=\sum_{j=0}^{n-1}\frac{t^j}{j!}D_x^{4(j+1)}x^{4n}.$$

Changing in the last sum the index of the summation, we obtain

$$D_x^4 S_{4n}(x,t)=\sum_{j=1}^{n}\frac{t^{j-1}}{(j-1)!}D_x^{4j}x^{4n}$$

and
$$D_t S_{4n}(x,t)=\sum_{j=1}^{n}\frac{t^{j-1}}{(j-1)!}D_x^{4j}x^{4n}.$$

By the last formula we obtain the assertion of the Theorem 10.

Remark 3. It is easy to verify that the polynomials W_{4n+1}, $i=0,1,2,3$, are (1,4) homogeneous.

Lemma 3. The polynomials (17) are of the form (18) for $m=4n+i$, $i=0,1,2,3$.

Proof. We shall give the proof only for S_{4n}. The proof for S_{4n+1}, $i=0,1,2,3$ is similar. By the formula

$$D_x^{4j}x^m=m(m-1)...(m-4j+1)x^{m-4j}=(m!)\frac{1}{(m-4j)!},\qquad 4j\le m$$

and from (17) we obtain (18).

Let us consider the functions

$$F(x,t)=H(x)\int_0^t V(s)\,ds,\qquad H(x)=\sum_{j=0}^{3}b_j x^j,$$

$$r_n(x,t)=F(x,t)+S_n(x,t),\qquad f(x,t)=-H(x)V(t).$$

Theorem 10. The polynomials r_n *satisfy the equation (8).*

Proof. We have

$$(D_x^4-D_t)r_n(x,t)=(D_x^4H(x))\int_0^t V(s)\,ds-H(x)V(t)=f(x,t).$$

13. The parabolic polynomials to the equation (9)

Lemma 4. If the functions $u_i(x_i,t)$, $i=1,2$ *satisfy the equations*

$$(D_x^3 - D_t)\, u_i(x_i,t) = 0, \qquad i=1,2,$$

respectively, the function $u(x,t) = u_1(x_1,t)\, u_2(x_2,t)$ *satisfies the equation (9).*

Proof. We have

$$(D_{x_1}^4 + D_{x_2}^4 - D_t)\, u(x,t) =$$
$$= u_2(x_2,t)(D_{x_1}^4 - D_t)u_1(x_1,t) + u_1(x_1,t)(D_{x_2}^4 - D_t)u_2(x_2,t) = 0.$$

By the Lemma 4, we obtain

Theorem 11. The polynomials

$$S_m(x,t) \;=\; S_m(x_1,t)\, S_n(x_2,t)$$

satisfy the equation (9).

14. The parabolic polynomials to the equation (10)

Let

$$F(x,t) \;=\; B(x)\int_0^t V(s)\,ds, \qquad z_{m,n}(x,t) = F(x,t) + S_{m,n}(x,t),$$

$$f(x,t) = -B(x)\,V(t).$$

where is an arbitrary polynomial satisfying the equation

(19) $$(D_{x_1}^4 + D_{x_2 t}^4)\, B(x) = 0$$

for example the polynomials

$$B_5^0(x) \;=\; x_2^5 - 5x_2x_1^4, \qquad B_5^1(x) \;=\; x_2^4x_1 - \frac{1}{5}x_1^5,$$

$$B_5^2(x) \;=\; x_2^2\,x_1^3, \qquad B_5^3(x) \;=\; x_2^3\,x_1^2,$$

satisfy the equation (19).

Theorem 12. The functions $z_{m,n}(x,t)$, *satisfy the equation (10) with* $f(x,t) = -B(x)V(t)$.

Proof. We have

$$(D_{x_1}^4 + D_{x_2}^4 - D_t) z_{m,n}(x,t) = (D_{x_1}^4 + D_{x_2}^4) B(x) - B(x) V(t) = f(x,t).$$

15. Application of the parabolic polynomials S_{4n+i}, $i = 0,1,2,3$ to the approximation of the solutions to the equation (7)

Let $u(x,t) \in C^{N,N}(D)$, $D = \{(x,t) : x^2 + t^2 < R^2\}$, N is a positive integer, be a solution to the equation (7). Let us consider the Taylor formula to the function u in the neighbourhood of the point $(0,0)$, being of the form

(20) $$u(x,t) = \sum_{j=0}^{N} \sum_{i+k=j} (D^{i,k} u)(0,0) \frac{x^i}{i!} \frac{t^k}{k!} + O\left((x^2 + t^2)^{\frac{N+1}{2}} \right),$$

$$D^{i,k} = D_x^i D_t^k .$$

Let

$$S_j(x,t) = \sum_{i+4k=j} (D^{i,k} u)(0,0) \frac{x^i}{i!} \frac{t^k}{k!}, \qquad j < N.$$

Lemma 5. If the function u satisfies the equation (7), then the polynomials S_j satisfy the equation

(21) $$(D_x^4 - D_t) S_j(x,t) = 0 \qquad \text{in} \qquad D,\ j < N.$$

Proof. We have

$$D_t S_j(x,t) = \sum_{i+4k=j} ((D^{i,k} u)(0,0)) \frac{x^i}{i!} \frac{t^{k-1}}{(k-1)!} =$$

$$= \sum_{i+4k+4=j} ((D^{i,(k+1)} u)(0,0)) \frac{x^i}{i!} \frac{t^k}{k!}$$

and

$$D_x^4 S_j(x,t) = \sum_{i+4k=j} ((D^{i,k} u)(0,0)) \frac{x^{i-4}}{(i-4)!} \frac{t^k}{k!}.$$

Changing in the last sum the index of the summation i, we obtain

$$D_x^4 S_j(x,t) = \sum_{i+4k+4=j} ((D^{i+4,k} u)(0,0)) \frac{x^i}{i!} \frac{t^k}{k!}.$$

Hence

$$(D_t - D_x^4)S_j(x,t) = \sum_{i+4+4k=j} (((D^{i,k+1} - D^{i+4,k})u)(0,0))\frac{x^i}{i!}\frac{t^k}{k!} =$$

$$= \sum_{i+4+4k=j} (((D^{i,k}(D_t - D_x^4)u)(0,0))\frac{x^i}{i!}\frac{t^k}{k!} = 0\ ,$$

because

$$(D_x^4 - D_t)\,u(0,0) = 0\ .$$

Hence we obtain (20).

Next let us consider the polynomials S_N, T_j, T_N being of the following form

$$S_N(x,t) = \sum_{j<N} S_j(x,t),$$

$$T_j^p(x,t) = \sum_{i+pk=j} ((D^{i,k}u)(0,0))\frac{x^i}{i!}\frac{t^k}{k!}, \qquad p = 1,2,3,$$

$$T_j(x,t) = T_j^1(x,t) + T_j^2(x,t) + T_j^3(x,t), \quad T_N(x,t) = \sum_{j<N} T_j(x,t).$$

By Lemma 5 we obtain the following result:

Theorem 13. If the function u satisfies the conditions (20), then there exists the polynomials S_N, T_N such that

$$u(x,t) - S_N(x,t) = T_N(x,t) + O((x^2+t^2)^{(N+1)/2}).$$

16. The example of the polynomial solution to the first Fourier problem to the equation (1a)

We shall construct the polynomial solution to the equation

(22) $$(D_x^2 - D_t)\,u(x,t) = -xV(t), \qquad V(t) = \sum_{j=0}^{m} a_j t^j$$

in the domain

$$D = \{(x,t) : x \in (0,1),\ t \in (0,T]\}\ ,$$

satisfying the initial condition

(23) $$u(x,0) = 1/2\,x^2 + 1/6\,x^3, \qquad x \in (0,1),$$

and the boundary-value conditions

$$(24) \qquad u(0,t) = t, \quad u(1,t) = 2/3 + 2t + \int_0^t V(s)\,ds, \qquad t \in (0,T],$$

with $V(t)$ given in (22).

By Theorem 3, we obtain

Theorem 14. The function

$$u(x,t) = x\int_0^t V(s)ds + P_2(x,t) + P_3(x,t),$$

is a polynomial solution to the problem (22) – (24).

17. The examples of the modal lines to the parabolic polynomials P_n

Let $N(P_i) = \{(x,t) : P_i(x,t) = 0\}$. The set $N(P_i)$ is called the modal set to the polynomial P_i. We give $N(P_i)$, $i = 1,2,3$.

We have

$N(P_2) = \{(x,t) : 1/2x^2 + t = 0\}$ is a parabol,

$N(P_3) = S_3^1(x,t) \cup S_3^2(x,t)$, with $S_3^1 = \{(x,t) : 1/6x^2 + t = 0\}$,

$S_3^2(P_3) = \{(0,t) : t \in (-\infty,\infty)\}$.

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Received on 23.01.1995 and, in revised form, on 22.04.1996.