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POINTWISE TRANSFORMATIONS OF LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE n-TH ORDER

In the paper there is derived the most general from of pointwise transformation for linear functional-differential equations of the n-th ($n \geq 1$) order of mixed or neutral type, respectively.

Key words: pointwise transformation, canonicals forms, linear functional differential equation.

1. Introduction

Pointwise transformation and canonical forms of ordinary linear differential equations, systems of differential equations, linear functional differential equations, were studied by many authors (see e.g. the survey paper [5]).

The most general form of pointwise transformation converting differential equations globally into another equation of the same form and order is used for equivalence and canonical forms of differential equations (see [7]). Using the same arguments as in [8], [11], we obtain the form of pointwise transformation of the class C^n for an equation $(1, I, \xi_j)$. We use results from [9], [10], [4]. New methods of proofs without regularity conditoins are derived in [1], [2], [3].

2. Definitions and results

On an interval $I = (a,b)$, $-\infty \leq a < b \leq \infty$ ($a, b \in I$ is not excluding), we consider an equation of the form

$$(1, I, \xi_j) \qquad \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}(x) y^{(i)}(\xi_j(x)) = 0,$$

m, n being integer, $' = d/dx$, $\xi_0 = id_I$.Let us suppose that $a_{ij} \in C^0(I)$, $a_{n0} \equiv 1$ (without loss of generality for $a_{n0} \neq 0$), deviating arguments $\xi_j : I \to I$, $\xi_j \in C^1(I), \xi_j' \neq 0$ on I, $\xi_j(x) \neq \xi_k(x)$ on I if $j \neq k$, $j, k \in \{0, 1, \ldots, m\}$.

Let us remark that the equation $(1,I,\xi_j)$ is for

$$\sum_{j=1}^{n} a_{ij}^2 = 0, \qquad \sum_{i=0}^{n-1}\sum_{j=0}^{m} a_{ij}^2 \neq 0$$

a retarded (an advanced) equation if $\xi_j(x) < x$, $(\xi_j(x) > x)$ for $j = 1,2,\dots,m$; an equation of the mixed type if there exists a pair (k,l), $k \neq l$, such that $\xi_k(x) < x$, $\xi_l(x) > x$ on I and $a_{i_1k} \cdot a_{i_2l} \neq 0$ for some i_1, $i_2 \in \{0,1,\dots,n-1\}$. The equation $(1,I,\xi_j)$ is of a neutral type if

$$\sum_{j=1}^{n} a_{ij}^2 \neq 0$$

on I.

There was derived the most general form $x = f(t)$, $y = g(t)u$ of pointwise transformation for retarded or advanced equations (see [[6], [9], [10], [3]). A general form of such transformation for all equation of the form $(1,I,\xi_j)$ is the same, becouse $(1,I,\xi_j)$ contains retarded equations. A form of pointwise transformation may be generally different for subsets of mixed or neutral equations, respectively. However, we prove that the most general transformation for mixed or neutral equations is again the same, i.e of the form $x = f(t)$, $y = g(t)u$.

In this paper we derive a form of pointwise transformation B which converts any linear differential equation $(1,I,\xi_j)$ of the mixed resp. neutral type into another equation of the same type and form

$$(2,J,\eta_j) \qquad \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}(t)\, u^{(i)}(\eta_j(t)) = 0$$

defined on an interval J.

We assume that for arbitrary $x_0 \in I$, $y_0 \in R$, there exists a nontrivial solution y of $(1,I,\xi_j)$ defined on the whole interval I such that $y(x_0) = y_0$.

Poinwise transformations: We suppose that B is a C^n – differomorphism of $I \times R$ onto $J \times R$ and for every equation $(1,I,\xi_j)$ there exists an equation $(2,J,\eta_j)$ such that B pointwisely converts:

every nontrivial solution $y(x)$ of $(1,I,\xi_j)$ into a nontrivial solution $u(t)$ of $(2,J,\eta_j)$;

every function $y \circ \xi_j(x)$, where $y(x)$ is a solution of $(1, I, \xi_j)$, into a function $u \circ \eta_j(t)$, where $u(t)$ is a solution of $(2, J, \eta_j)$; $j = 1, 2, \ldots, m$.

Theorem. For each $n \geq 1$, *the most general pointwise transformation converting any mixed or neutral equation* $(1, I, \xi_j)$ *into some equation* $(2, J, \eta_j)$ *of the same type is of the form*

$$x = f(t)\,, \quad y = g(t)u$$

where $f \in C^n(J)$ *is a diffeomorphism of* J *onto* $I, g \in C^n(J)$, $f'(t)\,g(t) \neq 0$, *and* $\xi_j(f(t)) = f(\eta_j(t))$ *on* $J, j = 1, 2, \ldots, m$.

Proof. If y is a nontrivial solution of $(1, I, \xi_j)$ then a mapping $A: I \to R^2$ defined by $A(x) = (x, y(x))$, $x \in I$, is a homeomorphism of I onto graph of the given solution y. Conversely, in accordance with the assumption to any point $(x_0, y_0) \in R^2$, $x_0 \in I$, there exists a solution y of $(1, I, \xi_j)$ such that a graph of y contains the point (x_0, y_0).

Consider a diffeomrphism $B: I \times R \to J \times R$ such that $B \in C^n(I \times R)$ and Jacobian $|B'(p)| \neq 0$ for all $p \in I \times R$. The inverse diffeomorphism denote $B^{-1} = (f, g)$. Then for all $q \in J \times R$ is $|(B^{-1})'(q)| \neq 0$. The mapping B^{-1} define a pointwise transformation and for any nontrivial solution y of $(1, I, \xi_j)$ and an arbitrary fixed $x \in I$ there is unique set of at least two mutually disjoint points in $J \times R$ such that

(1) $$(t, u) = B(A(x)), \qquad (t_j, u_j) = B(A(\xi_j(x)))$$

for $j = 1, 2, \ldots, m$. Using $(x, y(x)) = A(x) = B^{-1}(t, u) = (f(t, u), g(t, u))$ we get

(2) $$x = f(t, u), \qquad y = g(t, u)$$
$$\xi_j(x) = f(t_j, u_j), \qquad y(\xi_j(x)) = g(t_j, u_j).$$

The set of mixed or neutral equations of the form $(1, I, \xi_j)$ contains equations with

(3) $$a_{00}(x)\, a_{10}(x)\, a_{0j}(x) \neq 0$$

on I, $j \in \{1, \ldots, m\}$. Thus

$$a_{00}(x) y(x) + a_{10}(x) y'(x) = a_{00}(f) g + a_{10}(f)(g_1 + g_2 u') / (f_1 + f_2 u')$$

is linear for u, u' if and only if $f_2 \equiv 0$, i.e.

(4) $$x = f(t), \qquad f'(t) \neq 0$$

is a diffeomorphism of J onto I according to

(5) $$|(B^{-1})'(t,u)| = (f_1g_2 - f_2g_1)(t,u) = (f_1g_2)(t,u) \neq 0,$$

$(t,u) \in J\times R$; f_k, g_k denotes the partial derivatives of f, g with respect to the k-th variable, $k = 1,2$. Thus (2) becomes to

(6) $$x = f(t), \qquad y = g(t,u(t))$$
$$\xi_j(x) = f(\eta_j(t)), \qquad y(\xi_j(x)) = g(\eta_j(t),u(\eta_j(t))),$$

where $\eta_j : J \to R$ is defined by means of $\eta_j(t) = t_j$ and $f(\eta_j(t)) = \xi_j(f(t))$. Moreover, $\xi_j(x) \neq \xi_k(x) \Leftrightarrow \eta_j(x) \neq \eta_k(x)$, $j \neq k$; $j,k \in \{0,1,\dots,m\}$.

The transformation (6) implies the following relations

(7) $$y(x) = g(t,u(t))$$
$$y'(x) = [g_2(t,u(t))u'(t) + g_1(t,u(t))]/f'(t),$$
$$y^{(k)}(x) = g_2(t,u(t))u^{(k)}(t)/f'^k(t) + R_k(t)$$

and

(8) $$y(\xi_j(x)) = g(\eta_j(t),u(\eta_j(t))),$$
$$y'(\xi_j(x)) =$$
$$= [g_2(\eta_j(t),u(\eta_j(t)))u'(\eta_j(t)) + g_1(\eta_j(t),u(\eta_j(t)))\eta_j'(t)]/f'(t),$$
$$y^{(k)}(\xi_j(x)) = g_2(\eta_j(t),u(\eta_j(t)))u^{(k)}(\eta_j(t))/f'^k(t) + S_{kj}(t),$$

where $' = d/dx$, $' = d/dt$, respecytively, and $R_k(t)$, $S_{kj}(t)$ obtain terms $u(t)$, $u(\eta_j(t))$, $u'(t)$, $u'(\eta_j(t))$, ... of order lower than k, $k = 2,3,\dots,n$, $j = 1,2,\dots,m$. Hence the transformation (6) converts any equation $(1,I,\xi_j)$ into

(9) $$a_{00}(f(t))f'^n(t)\frac{g(t,u(t))}{g_2(t,u(t))} + a_{10}(f(t))f'^{n-1}(t)\left[u'(t) + \frac{g_1(t,u(t))}{g_2(t,u(t))}\right] +$$
$$+\sum_{i=2}^{n-1} a_{i0}(f(t))f'^{n-i}(t)u^{(i)}(t) + a_{n0}(t)u^{(n)}(t) +$$
$$+\sum_{j=1}^{m} a_{0j}(f(t))f'^n(t)\frac{g(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))} +$$
$$+\sum_{i=2}^{n}\sum_{j=1}^{m} a_{ij}(f(t))f'^{n-i}(t)u^{(i)}(\eta_j(t))\frac{g_2(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))} +$$

$$+\sum_{j=1}^{m} a_{1j}(f(t))f'^{n-1}(t)\eta_j'(t)\frac{g_1(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))}+$$

$$+\sum_{i=2}^{n}\left[a_{i0}(f(t))R_i(t)+\sum_{j=1}^{m} a_{ij}(f(t))S_{ij}(t)\right]\frac{f'^{n}(t)}{g_2(t,u(t))}=0$$

and in case $n=1$ into

$$(10)\quad a_{00}(f(t))f'(t)\frac{g(t,u(t))}{g_2(t,u(t))}+\sum_{j=1}^{m} a_{0j}(f(t))f'(t)\frac{g(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))}+$$

$$+a_{10}(f(t))\left[u'(t)+\frac{g_1(t,u(t))}{g_2(t,u(t))}\right]+$$

$$+\sum_{j=1}^{m} a_{1j}(f(t))\left[\frac{g_2(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))}u'(\eta_j(t))+\right.$$

$$\left.+\frac{g_1(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))}\eta_j'(t)\right]=0$$

Hence, from (3) we obtain that for arbitrary fixed $n\geq 1$ the following relations

$$(11)\quad \frac{g(t,u(t))}{g_2(t,u(t))}=a(t)u(t),\qquad \frac{g_1(t,u(t))}{g_2(t,u(t))}=b(t)u(t),$$

$$\frac{g_1(\eta_j(t),u(\eta_j(t)))}{g_2(t,u(t))}=c_j(t)u(\eta_j(t)),$$

must be valid for suitable functions $a(t)$, $b(t)$, $c_j(t)$ to obtain the linearity of

$$a_{00}(f)\frac{g(t,u)}{g_2(t,u)},\quad a_{10}(f)\frac{g_1(t,u)}{g_2(t,u)},\quad a_{0j}(f)\frac{g_1(\eta_j,u(\eta_j))}{g_2(t,u)}$$

on J. Then (11) implies that

$$(12)\qquad g(t,u)=g(t)u$$

and $g(t)\neq 0$ on J (see [9], [10]).

The remainder part of the proof follows from [4].

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