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OSCILLATION CRITERIA FOR A CLASS OF NEUTRAL DIFFERENCE EQUATIONS

This paper is concerned with the oscillatory behavior of a class of neutral difference equations. Sufficient conditions for all solutions to be oscillatory are derived.

Key words: oscillation criteria, neutral difference equation with delay.

1. Introduction

Qualitative behavior of solutions of delay difference equations has received a considerable attention recently (see for example [1–11]). This paper is concerned with the oscillation of solutions of a class of neutral difference equations of the form

$$\Delta(v(j)+d(j)v(j-\tau))=B(j)v(j)+c(j)v(j-\sigma), \quad j\geq 0, \tag{1.1}$$

where τ is a positive integer and σ a non-negative integer, and $\{d(j)\},\{B(j)\}$ and $\{c(j)\}$are real sequences. The forward difference operator is defined as usual, i.e., $\Delta x(k)=x(k+1)-x(k)$.

Let $\mu=\max\{\sigma,\tau\}$. Then by a solution of equation (1.1), we mean a real sequence $\{v(j)\}$ which is defined for $j\geq -\mu$ and which satisfies equation (1.1) for $j\geq 0$. By writting (1.1) in the form of a recurrence relation, it is clear that if

$$v(k)=v_k, \qquad k=-\mu,-\mu+1,\dots,0$$

are given, then equation (1.1) has a unique solution satisfying these initial conditions.

A nontrivial solution $\{v(j)\}$ of (1.1) is said to be eventually positive if $v(j)>0$ for all large j, and eventually negative if $v(j)<0$ for all large j. It is said to be oscillatory if it is neither eventually positive nor eventually negative, and non-oscillatory otherwise. It is clear that the behavior of the solutions of (1.1) depends on the conditions imposed on the integers σ and τ, as well as the coefficient sequences $\{B(j)\},\{c(j)\}$ and $\{d(j)\}$. We will be concerned with conditions which are sufficient for all nontrivial solutions of (1.1) to be oscillatory. Much of our results will differ from those established in the literature for the special case

(1.2) $$\Delta(v(j)+dv(j-\tau))=c(j)v(j-\sigma), \qquad j\geq 0$$

(see for example Lalli [8]) since we do not assume that the sequence $\{d(j)\}$ is identically constant, neither do we assume that $\{B(j)\}$ is identically zero.

To facilitate discussions, we shall use the notation

$$\overline{B}(j) = \prod_{i=0}^{j-1}(1+B(i))^{-1}, \qquad j\geq 1,$$

when $\{B(j)\}_{i=0}^{\infty}$ is a real sequence which satisfies $B(j)\neq -1$ for $j\geq 0$. A helpful result related to the sequence $\{\overline{B}(j)\}$ is first established.

Lemma 1.1 Let $\{B(i)\}_{i=0}^{\infty}$ *be a real sequence such that* $B(i)\neq -1$ *for* $i\geq 0$. *Then*

$$\Delta(\overline{B}(j)v(j))=\overline{B}(j+1)(\Delta v(j)-B(j)v(j)), \quad j\geq 1.$$

Proof. Note that

$$\Delta\overline{B}(j)=\prod_{i=0}^{j}(1+B(i))^{-1}-\prod_{i=0}^{j-1}(1+B(i))^{-1}=$$
$$=\prod_{i=0}^{j-1}(1+B(i))^{-1}[(1+B(j))^{-1}-1]=\frac{-B(j)\overline{B}(j)}{1+B(j)},$$

thus

$$\Delta(\overline{B}(j)v(j))=\overline{B}(j+1)\Delta v(j)+v(j)\Delta\overline{B}(j)=$$
$$=\overline{B}(j+1)\Delta v(j)-\frac{v(j)B(j)\overline{B}(j)}{1+B(j)}=$$
$$=\overline{B}(j+1)\left(\Delta v(j)-\frac{B(j)\overline{B}(j)v(j)}{\overline{B}(j+1)(1+B(j))}\right)=\overline{B}(j+1)(\Delta v(j)-B(j)v(j))$$

as required. ■

There are several standard results (see for example [4,Chapter 7]) related to the oscillations of solutions of difference equation two of which will be used in later discussions.

Lemma 1.2. Assume that $\{q(j)\}$ *is a real sequence such that* $\{1-q(j)\}$ *is not eventually positive, then*

(1.3) $$\Delta x(j)+q(j)x(j)\leq 0, \qquad n=0,1,2,\ldots$$

cannot have an eventually positive solution.

Lemma 1.3 Assume that $\{q(j)\}$ *is a non-negative real sequence and let* μ *be a positive integer. Suppose that*

$$\lim_{i\to\infty}\left\{\frac{1}{\mu}\sum_{i=j-\mu}^{j-1}q(i)\right\}>\frac{\mu^{\mu}}{(\mu+1)^{\mu+1}}.$$

Then the recurrence relation

$$\Delta x(j)+q(j)x(j-\mu)\le 0, \qquad n=0,1,2,\ldots$$

cannot have an eventually positive solution.

2. The case $d(j)<-1$

We first consider the case where $d(j)<-1$ for $j\ge 0$. Assuming $B(j)\le 0$ and $c(j)<0$ for $j\ge 0$, note that for any eventually positive solution $\{v(j)\}$ of (1.1), the sequence $\{z(j)\}$ defined by

(2.1) $$z(j)=v(j)+d(j)v(j-\tau), \qquad j\ge 0$$

satisfies $\Delta z(j)<0$ for all large j. We assert further that $z(j)<0$ for all large j provided $d(j)\le d_0<-1$ for $j\ge 0$.

Lemma 2.1. Suppose $B(j)\le 0$, $c(j)<0$ *and* $d(j)\le d_0<-1$ *for* $j\ge 0$. *Then for any eventually positive solution* $\{v(j)\}$ *of (1.1), the sequence* $\{z(j)\}$ *defined by (2.1) satisfies* $\Delta z(j)<0$ *and* $z(j)<0$ *for all large* j.

Proof. We have already noted that $\Delta z(j)<0$ for all large j. Thus $z(j)$ is eventually positive or eventually negative. If $z(j)>0$ for j larger than or equal to N, then

$$v(j)>-d(j)v(j-\tau)\ge -d_0v(j-\tau), \qquad j\ge N$$

so that

$$\frac{1}{v(j)}<\frac{-1}{d_0}\frac{1}{v(j-\tau)}, \qquad j\ge N.$$

As a consequence,

$$\frac{1}{v(j)}<\frac{-1}{d_0}\sup_{N-\tau\le s\le j}\frac{1}{v(s)}, \qquad j\ge N,$$

which implies

$$\sup_{N-\tau\le j\le i}\frac{1}{v(j)}\le\frac{-1}{d_0}\sup_{N-\tau\le j\le i}\left\{\sup_{N-\tau\le s\le j}\frac{1}{v(s)}\right\}=\frac{-1}{d_0}\sup_{N-\tau\le s\le i}\frac{1}{v(s)},\qquad i\ge j\ge N.$$

But then

$$0<\left(1+\frac{1}{d_0}\right)\sup_{N-\tau\le s\le i}\frac{1}{v(s)}\le 0,\qquad i\ge j\ge N,$$

which is a contradiction. ■

Theorem 2.1. Suppose the following conditions hold:
(H1) $-1<B(j)\le 0,\ \ c(j)<0,\ \ d(j)\le d_0<-1$ *for all large* j,
(H2) $B(j+\tau)c(j)d(j+\tau)\ge B(j)c(j+\tau)d(j+\tau-\sigma)$ *for all large* j,
(H3) $\tau>\sigma$,

(H4)
$$\varliminf_{j\to\infty}\left[\frac{-\overline{B}(j+1)c(j)}{\overline{B}(j+\tau+1)d(j+\tau-\sigma)c(j+\tau)}\right]>0,$$

(H5)
$$\varliminf_{j\to\infty}\left[\frac{\overline{B}(j+1)c(j)}{\overline{B}(j+\tau-\sigma)d(j+\tau-\sigma)}\right]>0$$

and
(H6)
$$\varliminf_{j\to\infty}\left[\frac{\overline{B}(j+1)c(j)(\tau-\sigma)}{\overline{B}(j+\tau-\sigma)d(j+\tau-\sigma)}-\frac{\overline{B}(j+1)c(j)}{\overline{B}(j+\tau+1)d(j+\tau-\sigma)c(j+\tau)}\right]>1$$

Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). Then by Lemma 2.1, the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j)<0$ and $z(j)<0$ for all large j. We now rewrite equation (1.1) in the form

(2.2) $\quad \Delta z(j)=B(j)z(j)-d(j)B(j)v(j-\tau)+c(j)v(j-\sigma),\qquad j\ge 0.$

In view of the assumption $-1<B(j)$ for $j\ge 0$, we may then infer from Lemma 1.1 that

(2.3) $\quad \Delta(\overline{B}(j)z(j))=-\overline{B}(j+1)d(j)B(j)v(j-\tau)+\overline{B}(j+1)c(j)v(j-\sigma)$

for $j\ge 0$. By means of (2.1), we see that

$$v(j-\tau)=\frac{z(j)}{d(j)}-\frac{v(j)}{d(j)},$$

which implies

$$v(j-\sigma)=\frac{z(j+\tau-\sigma)}{d(j+\tau-\sigma)}-\frac{v(j+\tau-\sigma)}{d(j+\tau-\sigma)}. \tag{2.4}$$

Note that from (2.3), we have

$$v(j+\tau-\sigma)=\frac{\Delta(\overline{B}(j+\tau)z(j+\tau))}{\overline{B}(j+1+\tau)c(j+\tau)}+\frac{d(j+\tau)B(j+\tau)v(j)}{c(j+\tau)}.$$

Thus multiplying (2.4) by $\overline{B}(j+1)c(j)$, we have

$$\begin{aligned}\overline{B}(j+1)c(j)v(j-\sigma)=&\frac{\overline{B}(j+1)c(j)z(j+\tau-\sigma)}{d(j+\tau-\sigma)}-\\&-\frac{\overline{B}(j+1)c(j)\Delta(\overline{B}(j+\tau)z(j+\tau))}{d(j+\tau-\sigma)\overline{B}(j+1+\tau)c(j+\tau)}-\\&-\frac{\overline{B}(j+1)c(j)d(j+\tau)B(j+\tau)v(j)}{d(j+\tau-\sigma)c(j+\tau)}.\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.3), we obtain

$$\begin{aligned}\Delta(\overline{B}(j)z(j))=&\frac{\overline{B}(j+1)c(j)z(j+\tau-\sigma)}{d(j+\tau-\sigma)}-\\&-\frac{\overline{B}(j+1)c(j)\Delta(\overline{B}(j+\tau)z(j+\tau))}{d(j+\tau-\sigma)\overline{B}(j+1+\tau)c(j+\tau)}-\\&-\frac{\overline{B}(j+1)c(j)d(j+\tau)B(j+\tau)z(j)}{d(j+\tau-\sigma)c(j+\tau)}+\\&+\left[\frac{\overline{B}(j+1)c(j)d(j+\tau)B(j+\tau)d(j)}{d(j+\tau-\sigma)c(j+\tau)}-\overline{B}(j+1)d(j)B(j)\right]v(j-\tau).\end{aligned}$$

By means of (H2), we see that

$$\begin{aligned}\Delta(\overline{B}(j)z(j))\leq&\frac{\overline{B}(j+1)c(j)z(j+\tau-\sigma)}{d(j+\tau-\sigma)}-\\&-\frac{\overline{B}(j+1)c(j)\Delta(\overline{B}(j+\tau)z(j+\tau))}{d(j+\tau-\sigma)\overline{B}(j+1+\tau)c(j+\tau)}.\end{aligned} \tag{2.6}$$

Denoting $\overline{B}(j)z(j)$ by $p(j)$, we can write (2.6) as

$$\Delta p(j) \le \frac{\overline{B}(j+1)c(j)p(j+\tau-\sigma)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)} - \\ -\frac{\overline{B}(j+1)c(j)\Delta p(j+\tau)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}. \tag{2.7}$$

Since $z(j)<0$ for all large j, we see that $p(j)<0$ and

$$\frac{\Delta p(j)}{p(j)} \ge \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}\frac{p(j+\tau-\sigma)}{p(j)} - \\ -\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}\frac{p(j+\tau)}{p(j)}\frac{\Delta p(j+\tau)}{p(j+\tau)}$$

for all large j. Denoting $\Delta p(j)/p(j)$ by $q(j)$, then $p(j+1)/p(j) = 1+q(j)$ and $q(j)>0$ (which holds eventually in view of (2.3)). Furthermore, we have

$$q(j) \ge \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}\prod_{i=j}^{j+\tau-\sigma-1}(1+q(i)) - \\ -\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}\prod_{i=j}^{j+\tau-1}(1+q(i))q(j+\tau), \tag{2.8}$$

for all large j, and thus

$$q(j) \ge \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)} \tag{2.9}$$

and

$$q(j) \ge \frac{-B(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}\prod_{i=j}^{j+\tau-1}(1+q(i))q(j+\tau), \tag{2.10}$$

for all large j. In view of (2.9) and (H5), we see that $\underline{\lim}_{j\to\infty} q(j)>0$. Furthermore, in view of (2.10) and (H4), it is not difficult to see that $\overline{\lim}_{j\to\infty} q(j)<+\infty$. Denoting $\underline{\lim}_{j\to\infty} q(j)$ by q_0, we may now infer from (2.8) that

$$(2.11)\qquad q(j) \geq \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)} \prod_{i=j}^{j+\tau-\sigma-1}(1+q(i)) -$$

$$-\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)} q(j+\tau) \geq$$

$$\geq \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}(1+q_0)^{\tau-\sigma} -$$

$$-\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)} q_0 .$$

But then

$$\frac{q(j)}{q_0} \geq \frac{\overline{B}(j+1)c(j)(1+q_0)^{\tau-\sigma}}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)q_0} - \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)} \geq$$

$$\geq \frac{\overline{B}(j+1)c(j)(\tau-\sigma)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)} - \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}$$

so that

$$1 = \underline{\lim}_{j\to\infty}\left(\frac{q(j)}{q_0}\right) \geq$$

$$\geq \underline{\lim}_{j\to\infty}\left(\frac{\overline{B}(j+1)c(j)(\tau-\sigma)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)} - \frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}\right).$$

This is contrary to our hypothesis (H6). ■

We remark that if $B(j)=0$ for all $j \geq 0$, then the assumption (H2) in the above Theorem is automatically true. Under the additional assumption that $d(j)=d_0<-1$ for all j, the assumption (H5) is reduced to

$$\underline{\lim}_{j\to\infty} \frac{c(j)}{d_0} > 0 ,$$

(H4) to

$$\underline{\lim}_{j\to\infty}\left(\frac{-c(j)}{d_0 c(j+\tau)}\right) > 0 ,$$

and (H6) to

$$\frac{-1}{d_0}\underline{\lim}_{j\to\infty}\left((\sigma-\tau)c(j) + \frac{c(j)}{c(j+\tau)}\right) > 1 .$$

In case $B(j)=0$, $d(j)=-1.5$, $c(j)=c_0<-1/4$ for all j, and $\tau-\sigma=2$, every solution of (1.2) is oscillatory. The same conclusion cannot be obtained from Theorem 4.2 in [5].

We remark further that when $\tau-\sigma=\omega>1$, a slight modification of the last few steps of the proof of Theorem 2.1 leads to the following result.

Theorem 2.2 *Suppose the following conditions hold:*

(H1) $-1<B(j)\le 0,\quad c(j)<0,\quad d(j)\le d_0<-1$ *for all large* j,

(H2) $B(j+\tau)c(j)d(j+\tau)\ge B(j)c(j+\tau)d(j+\tau-\sigma)$ *for all* j,

(H3) $\tau-\sigma=\omega>1$,

(H4) $$\underline{\lim}_{j\to\infty}\left[\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}\right]>0$$

(H5) $$\underline{\lim}_{j\to\infty}\left[\frac{\overline{B}(j+1)c(j)(\omega-1)^{1-\omega}\omega^{\omega}}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}-\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}\right]>1 .$$

Then every solution of (1.1) is oscillatory.

Indeed, note that in view of (2.8) and (H4) (the condition (H4) in Theorem 2.1 is not needed), it follows again (since $\tau-\sigma>1$) that

$$0<\underline{\lim}_{j\to\infty}q(j)=q_0<\infty.$$

Note further that when $\omega>1$, the function

$$f(x)=\frac{(1+x)^{\omega}}{x},\qquad x>0$$

has a minimum

$$f\left(\frac{1}{\omega-1}\right)=(\omega-1)^{1-\omega}\omega^{\omega}.$$

Thus, under our new hypotheses, instead of (2.11), we will obtain

$$\frac{q(j)}{q_0}\ge\frac{\overline{B}(j+1)c(j)(\omega-1)^{1-\omega}\omega^{\omega}}{d(j+\tau-\sigma)\overline{B}(j+\tau-\sigma)}-\frac{\overline{B}(j+1)c(j)}{d(j+\tau-\sigma)c(j+\tau)\overline{B}(j+\tau+1)}.$$

A contradiction will be arrived again by taking lower limits on both sides of this inequality.

In the next result, we shall remove the assumption that $\tau - \sigma > 0$

Theorem 2.3. Suppose the following conditions hold:

(G1) $-1 < B(j) \le 0, \quad c(j) < 0, \quad d(j) \le d_0 < -1$ *for all large* j,

(G2) $B(j-\tau)c(j)d(j-\sigma) \le B(j)c(j-\tau)d(j)$ *for all* j,

(G3) $$\overline{\lim_{j\to\infty}} \left[\frac{-\overline{B}(j+1)c(j)d(j-\sigma)}{c(j-\tau)\overline{B}(j-\tau+1)} \right] < 1.$$

Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). Then by Lemma 2.1, the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j) < 0$ and $z(j) < 0$ for all large j. As we have seen in the proof of Theorem 2.1, the equality (2.3) holds for all large j. Since

$$v(j-\sigma) = z(j-\sigma) - d(j-\sigma)v(j-\sigma-\tau) \tag{2.12}$$

we see from (2.3) that

$$v(j-\sigma-\tau) = \frac{\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)} + \frac{B(j-\tau)d(j-\tau)v(j-2\tau)}{c(j-\tau)}. \tag{2.13}$$

Substituting (2.13) into (2.12), we obtain

$$v(j-\sigma) = z(j-\sigma) - \frac{d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)} - \frac{B(j-\tau)d(j-\tau)d(j-\sigma)v(j-2\tau)}{c(j-\tau)}. \tag{2.14}$$

Substituting (2.14) into (2.3), we obtain

$$\begin{aligned} \Delta(\overline{B}(j)z(j)) = \\ &= -\overline{B}(j+1)d(j)B(j)v(j-\tau) + \overline{B}(j+1)c(j)z(j-\sigma) - \\ &- \frac{\overline{B}(j+1)c(j)d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)} - \\ &- \frac{\overline{B}(j+1)c(j)B(j-\tau)d(j-\tau)d(j-\sigma)v(j-2\tau)}{c(j-\tau)} = \end{aligned} \tag{2.15}$$

$$=\left[\frac{\overline{B}(j+1)c(j)B(j-\tau)d(j-\sigma)}{c(j-\tau)}-\overline{B}(j+1)d(j)B(j)\right]v(j-\tau)-$$

$$-\frac{\overline{B}(j+1)c(j)B(j-\tau)d(j-\sigma)}{c(j-\tau)}z(j-\tau)+\overline{B}(j+1)c(j)z(j-\sigma)-$$

$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)}\geq$$

$$\geq-\frac{\overline{B}(j+1)c(j)B(j-\tau)d(j-\sigma)}{c(j-\tau)}z(j-\tau)+\overline{B}(j+1)c(j)z(j-\sigma)-$$

$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)},$$

where we have used the assumption (G2) for obtaining the last inequality. Denoting $\Delta(\overline{B}(j)z(j))/(\overline{B}(j)z(j))$ by $q(j)$ (which is eventually positive in view of (2.12)), we obtain from (2.16) that

$$q(j)\leq\frac{-d(j-\sigma)\overline{B}(j+1)c(j)}{\overline{B}(j-\tau+1)c(j-\tau)}\frac{\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j)z(j)}=$$

$$=\frac{-d(j-\sigma)\overline{B}(j+1)c(j)}{\overline{B}(j-\tau+1)c(j-\tau)}\left[\prod_{i=j-\tau}^{j-1}(1+q(i))\right]^{-1}q(j-\tau),$$

or

$$q(j)\prod_{i=j-\tau}^{j-1}(1+q(i))\leq\frac{-d(j-\sigma)\overline{B}(j+1)c(j)q(j-\tau)}{\overline{B}(j-\tau+1)c(j-\tau)}$$

for all large j. In view of the assumption (G3), we see from the above inequality that

$$0<\overline{\lim_{j\to\infty}}\,q(j)=q_0<\infty.$$

Thus

$$q(j)\leq q(j)\prod_{i=j-\tau}^{j-1}(1+q(i))\leq\frac{-d(j-\sigma)\overline{B}(j+1)c(j)q_0}{\overline{B}(j-\tau+1)c(j-\tau)}$$

for all large j. This implies

$$\frac{q(j)}{q_0}\leq\frac{-d(j-\sigma)\overline{B}(j+1)c(j)}{\overline{B}(j-\tau+1)c(j-\tau)}$$

and

$$1 < \overline{\lim_{j\to\infty}} \left[\frac{-d(j-\sigma)\overline{B}(j+1)c(j)}{\overline{B}(j-\tau+1)c(j-\tau)} \right]$$

which is contrary to our assumption (G4). ■

In case $B(j)=0$,$c(j)<0$ and $d(j)=d_0<-1$ for all $j\geq 0$, (G2) is automatically true, and (G3) is reduced to

$$\overline{\lim_{j\to\infty}} \left(\frac{-c(j)\,d_0}{c(j-\tau)} \right) < 1 .$$

Under these conditions, every solution of (1.2) will be oscillatory.

3. The case $-1 \leq d(j) \leq 0$

We now consider the case where $-1\leq d(j)\leq 0$ for $j\geq 0$. Assuming $B(j)\leq 0$ and $c(j)<0$ for $j\geq 0$, note that for any eventually positive solution $\{v(j)\}$ of (1.1), the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j)<0$ for all large j. Under the assumption that $-1\leq d(j)$, we may show that $z(j)>0$ for all large j.

Lemma 3.1. Suppose $B(j)\leq 0$, $c(j)<0$ and $-1\leq d(j)$ for $j\geq 0$. Then for any eventually positive solution $\{v(j)\}$of (1.1), the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j)<0$ and $z(j)>0$ for all large j.

Proof. It is clear from our assumptions and (1.1) that $\Delta z(j)<0$ for all large j. Thus $z(j)$ is eventually positive or eventually negative. If $z(j)<0$ for j larger than or equal to some integer N, then $z(j)<z(N)<0$ for $j\geq N$. Thus

$$\begin{aligned} v(N+n\tau) &= z(N+n\tau)-d(N+n\tau)v(N+(n-1)\tau)= \\ &= Z(N)+v(N+(n-1)\tau\,)= \\ &= z(N)+z(N+(n-1)\tau)-d(N+(n-1)\tau)v(N+(n-2)\tau)\leq \\ &\leq 2z(N)+v(N+(n-2)\tau)=\ldots\leq nz(N)+v(N). \end{aligned}$$

The left hand side is positive while the right hand side is negative for all large n. This contradiction completes our proof. ■

The proof of the above Lemma motivetes the following result which will be useful in later discussions.

Lemma 3.2 Suppose $-1 \le d(j)$ *for* $j \ge 0$. *If* $\{v(j)\}$ *is an eventually positive sequence such that the sequence* $\{z(j)\}$ *defined by (2.1) is not identically constant for all large* j, *and satisfies* $\Delta z(j) \le 0$ *for all large* j, *then* $z(j) > 0$ *for all large* j.

Theorem 3.1 *Suppose the following conditions hold:*

(H1) $-1 < B(j) \le 0,\ c(j) < 0,\ -1 \le d(j) \le 0$ *for all large j,*

(H2) $B(j)c(j-\tau)d(j) \le B(j-\tau)c(j)d(j-\tau)$ *for all large j,*

(H3) $\tau - \sigma \le 0$,

(H4) $$\lim_{j\to\infty}\left[\frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\tau)}\right] > 0$$

and

(H5) $$\lim_{j\to\infty}\left[\frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}\right] > 1.$$

Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). Then by Lemma 3.1, the sequence $\{z(j)\}$ defined by (1.1) satisfies $\Delta z(j) < 0$ and $z(j) > 0$ for all large j. As we have seen in the proof of Theorem 2.1, the equality (2.3) holds for all large j. Moreover, in view of (2.12) and (2.13), we see that

(3.1) $$\Delta(\overline{B}(j)z(j)) = -\overline{B}(j+1)d(j)B(j)z(j-\tau) + \overline{B}(j+1)c(j)z(j-\sigma) - \frac{\overline{B}(j+1)c(j)d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)} +$$
$$+\overline{B}(j+1)\left[d(j)B(j)d(j-\tau) - \frac{c(j)B(j-\tau)d(j-\sigma)d(j-\tau)}{c(j-\tau)}\right]v(j-2\tau) \le$$
$$\le -\overline{B}(j+1)d(j)B(j)z(j-\tau) + \overline{B}(j+1)c(j)z(j-\sigma) - \frac{\overline{B}(j+1)c(j)d(j-\sigma)\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau+1)c(j-\tau)},$$

where we have used (H2) for deriving the last inequality.

As a consequence,

$$\frac{\Delta(\overline{B}(j-\tau)z(j))}{\overline{B}(j)z(j)} \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)}\frac{\overline{B}(j-\tau)z(j-\tau)}{\overline{B}(j)z(j)} - \tag{3.2}$$

$$-\frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}\frac{\overline{B}(j-\sigma)z(j-\sigma)}{\overline{B}(j)z(j)} -$$

$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}\frac{\overline{B}(j-\tau)z(j-\tau)}{\overline{B}(j)z(j)}\left[\frac{\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j-\tau)z(j-\tau)}\right].$$

holds for all large j. Denoting $-\Delta(\overline{B}(j)z(j))/(\overline{B}(j)z(j))$ by $h(j)$, then

$$1-h(j) = \frac{\overline{B}(j+1)z(j+1)}{\overline{B}(j)z(j)} > 0$$

and $h(j)>0$ in view of (2.3) and (H1). Furthermore, from (3.2), we obtain

$$h(j) \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)}\left[\prod_{i=j-\tau}^{j-1}(1-h(i))\right]^{-1} - \tag{3.3}$$

$$-\frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}\left[\prod_{i=j-\tau}^{j-1}(1-h(i))\right]^{-1} -$$

$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)h(j-\tau)}{\overline{B}(j-\tau+1)c(j-\tau)}\left[\prod_{i=j-\tau}^{j-1}(1-h(i))\right]^{-1},$$

which implies

$$h(j) \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}. \tag{3.4}$$

Since $0<h(j)<1$ for all large j, we see that $0 \leq \varliminf_{j\to\infty} h(j) < \infty$. Furthermore, in view of our assumption (H4) and (3.4),

$$\varliminf_{j\to\infty} h(j) = h_0 > 0.$$

We may now infer from (3.3) that

$$h(j) \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)}(1-h_0)^{-\tau} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}(1-h_0)^{-\delta} -$$

$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}h_0(1-h_0)^{-\tau}$$

for all large j. Thus

$$\frac{h(j)}{h_0} \ge \frac{h(j)}{h_0}(1-h_0)^{\tau} \ge \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)}\frac{1}{h_0} - \tag{3.5}$$

$$-\frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}\frac{(1-h_0)^{\tau-\sigma}}{h_0} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)} \ge$$

$$\ge \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}$$

where we have used (H3) in deriving the last inequality. Finally,

$$1 \ge \varliminf_{j\to\infty}\left[\frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}\right]$$

which is contrary to our assumption (H5). ■

We have assumed that $\tau-\sigma\le 0$ in the above Theorem. A slight modification of the proof of the above Theorem shows that this assumption can be changed to $\tau-\sigma<0$.

Theorem 3.2. Suppose the following conditions hold:

(G1) $-1<B(j)\le 0,\ c(j)<0,\ -1\le d(j)\le 0$ *for all large* j,

(G2) $B(j)c(j-\tau)d(j)\ge B(j-\tau)c(j)d(j-\sigma)$ *for all* j,

(G3) $\tau-\sigma=\omega<0$,

(G4)
$$\varliminf_{j\to\infty}\left[\frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}\right] > 0,$$

and

(G5)
$$\varliminf_{j\to\infty}\left[\frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)(-\omega)^{\omega}}{\overline{B}(j-\sigma)(1-\omega)^{\omega-1}} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}\right] > 1$$

Then every solution of (1.1) is oscillatory.

Indeed, by following the procedures in deriving (3.5), we will obtain

$$\frac{h(j)}{h_0} \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)}\frac{1}{h_0} - \frac{\overline{B}(j+1)c(j)}{\overline{B}(j-\sigma)}\frac{(1-h_0)^{\tau-\sigma}}{h_0} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)}. \tag{3.6}$$

Note that the function

$$f(x) = \frac{1}{(1-x)^{\sigma-\tau}x}, \qquad 0 < x < 1,$$

has a minimum

$$f\left(\frac{1}{1-\omega}\right) = (-\omega)^{\omega}(1-\omega)^{1-\omega}.$$

Thus

$$\frac{h(j)}{h_0} \geq \frac{\overline{B}(j+1)d(j)B(j)}{\overline{B}(j-\tau)} - \frac{\overline{B}(j+1)c(j)(-\omega)^{\omega}}{\overline{B}(j-\sigma)(1-\omega)^{\omega-1}} - \frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j-\tau+1)c(j-\tau)},$$

so that a contradiction will be arrived as in the proof of Theorem 3.1.

Theorem 3.3. Suppose the following conditions hold:

(K1) $-1 < B(j) \leq 0,\ c(j) < 0,\ -1 \leq d(j) \leq 0$ *for all large* j,

(K2) $B(j-\tau)d(j+1-\sigma) \leq B(j)d(j)$,
$d(j+1-\sigma)c(j-\tau) \leq c(j)d(j-\sigma)$ *for all large* j,

(K3) $\Delta d(j-\sigma) - B(j-\tau)d(j+1-\sigma) - B(j)d(j) + c(j) \leq 0$
but not identically zero for all large j,

(K4) $\tau - \sigma \leq 0$, *and*

(K5) $$1 + \frac{\overline{B}(j+1)\{Q(j) - B(j)d(j-\sigma) + c(j)\}}{\overline{B}(j)(1+d(j-\sigma))},$$

is not eventually positive, where

$$Q(j) = \Delta d(j-\sigma) + d(j+1-\sigma)B(j-\tau) - B(j)d(j).$$

Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). Then by Lemma 3.1, the the sequence $\{z(j)\}$ defined by

(2.1) satisfies $\Delta z(j)<0$ and $z(j)>0$ for all large j. As we have seen in the proof of Theorem 2.1, the equatlity (2.3) holds for all large j. Thus $\bar{B}(j)z(j)>0$ and $\Delta(\bar{B}(j)z(j))<0$ for all large j. Let us write

(3.7) $$h(j)=z(j)+d(j-\sigma)z(j-\tau), \qquad j\geq 0.$$

Then in view of (2.3), we see that

(3.8) $$\begin{aligned}\Delta h(j) &= \Delta z(j)+d(j+1-\sigma)\Delta z(j-\tau)+z(j-\tau)\Delta d(j-\sigma)=\\ &= B(j)v(j)+c(j)v(j-\sigma)+d(j+1-\sigma)B(j-\tau)v(j-\tau)+\\ &\quad +d(j+1-\sigma)c(j-\tau)v(j-\tau-\sigma)+z(j-\tau)\Delta d(j-\sigma)=\\ &= B(j)z(j)+(d(j+1-\sigma)B(j-\tau)-B(j)d(j))\,v(j-\tau)+c(j)z(j-\sigma)+\\ &\quad +(d(j+1-\sigma)c(j-\tau)-c(j)d(j-\sigma))v(j-\tau-\sigma)+z(j-\tau)\Delta d(j-\sigma).\end{aligned}$$

In view of the assumption (K2), we see that

$$\begin{aligned}&(d(j+1-\sigma)B(j-\tau)-B(j)d(j))\,(v(j-\tau)+z(j-\tau))=\\ &=(d(j+1-\sigma)B(j-\tau)-B(j)d(j))\,(-d(j-\tau)v(j-2\tau))\leq 0.\end{aligned}$$

Furthermore, in view of (K4) and the fact that $\Delta z(j)<0$ for all large j, we see that $c(j)z(j-\sigma)\leq c(j)z(j-\tau)$. Thus from (3.8), we have

(3.9) $$\Delta h(j)\leq B(j)z(j)+Q(j)z(j-\tau)+c(j)z(j-\sigma)$$

and

(3.10) $$\Delta h(j)\leq B(j)z(j)+(Q(j)+c(j))z(j-\tau),$$

where

(3.11) $$Q(j)=\Delta d(j-\sigma)+d(j+1-\sigma)B(j-\tau)-B(j)d(j).$$

In view of the assumption (K3), we have from Lemma 3.2 that $\{h(j)\}$ is eventually positive. If we now rewrite (3.10) in the following form

$$\Delta h(j)\leq B(j)h(j)+(Q(j)-B(j)d(j-\sigma)+c(j))z(j-\tau),$$

then in view of Lemma 1.1, we have

$$\Delta(\overline{B}(j)h(j))\leq \overline{B}(j+1)(Q(j)-B(j)d(j-\delta)+c(j))z(j-\tau),$$

which holds for all large j. Since $h(j)\leq(1+d(j-\sigma))z(j-\tau)$ eventually, we see that

$$\Delta(\overline{B}(j)h(j))\leq \frac{\overline{B}(j+1)(Q(j)-B(j)d(j-\sigma)+c(j))}{\overline{B}(j)(1+d(j-\sigma))}\overline{B}(j)h(j)$$

holds for all large j. But this, in view of assumption (K5), contradicts Lemma 1.2. ■

4. The case $d(j) \geq 0$

We first consider the case $\tau - \sigma < 0$.

Theorem 4.1. Suppose the following conditions hold:

(H1) $B(j) \leq 0$, $c(j) < 0$ *and* $d(j) \geq 0$ *for all large* j,

(H2) $c(j)d(j-\sigma)/c(j-\tau) \leq \beta$ *for all large j,*

(H3) $\tau - \sigma < 0$ *and,*

(H4)
$$\varliminf_{j\to\infty} \left\{ \frac{1}{\sigma-\tau} \sum_{i=j-\sigma+\tau}^{j-1} -c(i) \right\} > \frac{(1+\beta)(\sigma-\tau)^{\sigma-\tau}}{(\sigma-\tau+1)^{\sigma-\tau+1}} .$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). In view of Lemma 3.1, the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j) < 0$ and $z(j) > 0$ eventually. We first deduce from (1.1) that

$$\Delta z(j) \leq c(j)v(j-\sigma) = c(j)z(j-\sigma) - c(j)d(j-\sigma)v(j-\sigma-\tau), \tag{4.1}$$

holds for all large j. Since

$$v(j-\sigma) = z(j-\sigma) - d(j-\sigma)v(j-\sigma-\tau)$$

and

$$v(j-\sigma-\tau) = \frac{\Delta z(j-\tau)}{c(j-\tau)} - \frac{B(j-\tau)v(j-\tau)}{c(j-\tau)},$$

thus we may deduce from (4.1) that

$$\Delta z(j) \leq c(j)z(j-\sigma) - \frac{c(j)d(j-\sigma)}{c(j-\tau)} \Delta z(j-\tau) +$$
$$+ \frac{B(j-\tau)c(j)d(j-\sigma)}{c(j-\tau)} v(j-\tau) \leq c(j)z(j-\sigma) - \beta \Delta z(j-\tau)$$

or

$$\Delta(z(j) + \beta z(j-\tau)) \leq c(j)z(j-\sigma) .$$

By our assumptions (and in view of the proof of Lemma 3.1), we see that the sequence $\{g(j)\}$ defined by

$$g(j) = z(j) + \beta z(j-\tau), \qquad j \geq 0$$

satisfies $\Delta g(j) < 0$ and $g(j) > 0$ for all large j. Furthermore, note that $g(j) < (1+\beta)z(j-\tau)$ and satisfies

$$\Delta g(j) \leq c(j)z(j-\sigma) \leq \frac{c(j)}{1+\beta}\, g(j+\tau-\sigma)$$

for all large j. In view of our hypothesis (H4), this is contrary to the assertion of Lemma 1.3. ∎

We remark that the condition $-1 < B(j)$ is not required in the above Theorem.

Theorem 4.2. Suppose the following conditions hold:

(G1) $-1 < B(j) \leq 0$, $c(j) < 0$ *and* $d(j) \geq 0$ *for all large* j,

(G2) $B(j-\tau)d(j+1-\sigma) \leq B(j)d(j)$, $c(j-\tau)d(j+1-\sigma) \leq c(j)d(j)$ *for all large j,*

(G3) $\Delta d(j-\sigma) - B(j)d(j) - B(j)d(j-\sigma) + B(j-\tau)d(j+1-\sigma) \leq 0$ *for all large j,*

(G4) $\tau - \sigma < 0$, *and*

(G5) $$\varliminf_{j\to\infty}\left\{\frac{1}{\sigma-\tau}\sum_{i=j-\sigma+\tau}^{j-1}\frac{-\overline{B}(i+1)c(i)}{(1+d(i+\tau-2\sigma))\overline{B}(i+\tau-\sigma)}\right\} > \frac{(\sigma-\tau)^{\sigma-\tau}}{(\sigma-\tau+1)^{\sigma-\tau+1}}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). As in the proof of Theorem 3.2, the sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j) < 0$ and $z(j) > 0$ for all large j. Furthermore, the sequence $\{h(j)\}$ defined by (3.7) satisfies (3.9) for all large j, where $Q(j)$ has been defined by (3.11). In view of (3.7), we may deduce from (3.9) and assumption (G3) that

$$\Delta h(j) \leq B(j)h(j) + (Q(j) - B(j)d(j-\sigma))z(j-\tau) + c(j)z(j-\sigma) \leq$$
$$\leq B(j)h(j) + c(j)\,z(j-\sigma).$$

In view of Lemma 3.2, we see that $h(j) > 0$ for all larg j. Furthermore, since $h(j) \leq (1+d(j-\sigma))z(j-\tau)$, thus we may conclude that

$$\Delta h(j) \leq B(j)h(j) + \frac{c(j)h(j+\tau-\sigma)}{1+d(j+\tau-2\sigma)}.$$

In view of Lemma 1.1, we see that

$$\Delta(\overline{B}(j)h(j)) \leq \frac{\overline{B}(j+1)c(j)}{\overline{B}(j+\tau-\sigma)(1+d(j+\tau-2\sigma))}\overline{B}(j+\tau-\sigma)h(j+\tau-\sigma).$$

But in view of our hypothesis (G5), this contradicts the assertion of Lemma 1.3. ■

We have assumed in the previous two results that $\tau-\sigma<0$. This assumption shall be removed in our final Theorem. However, this result will not apply to equation (1.2) in view of the condition (K4) below.

Theorem 4.3. Suppose the following conditions hold:

(K1) $B(j)\geq 0,\ c(j)<0,\ d(j)\geq 0$ *for all large* j,

(K2) the inequality

$$\frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j+1-\tau)c(j-\tau)} \leq \beta$$

holds for all large j,

(K3) the inequality

$$B(j)d(j)-\frac{B(j-\tau)c(j)d(j-\sigma)}{c(j-\tau)} \leq 0$$

holds for all large j, and

(K4)

$$1 - \frac{\overline{B}(j+1)B(j)d(j)}{(1+\beta)\overline{B}(j-\tau)}$$

is not eventually positive. Then every solution of (1.1) is oscillatory.

Proof. Assume to the contrary that $\{v(j)\}$ is an eventually positive solution of (1.1). The sequence $\{z(j)\}$ defined by (2.1) satisfies $\Delta z(j)<0$ and $z(j)>0$ for all large j. We now rewrite (1.1) in the form

$$\Delta z(j)=B(j)z(j)-d(j)B(j)v(j-\tau)+c(j)v(j-\sigma),$$

and then conclude from Lemma 1.2 that

$$\begin{aligned}(4.2)\quad \Delta(\bar{B}(j)z(j)) &= -\bar{B}(j+1)d(j)B(j)v(j-\tau)+\bar{B}(j+1)c(j)v(j-\sigma)=\\ &= -\overline{B}(j+1)d(j)B(j)v(j-\tau)+\overline{B}(j+1)c(j)z(j-\sigma)-\\ &\quad -\overline{B}(j+1)c(j)d(j-\sigma)v(j-\sigma-\tau)\end{aligned}$$

holds for all large j. Note that

$$v(j-\sigma)=z(j-\sigma)-d(j-\sigma)v(j-\sigma-\tau),$$

and

$$v(j-\tau-\sigma)=\frac{\Delta(\overline{B}(j-\tau)z(j-\tau))}{\overline{B}(j+1-\tau)c(j-\tau)}+$$
$$+\frac{\overline{B}(j+1-\tau)d(j-\tau)B(j-\tau)}{\overline{B}(j+1-\tau)c(j-\tau)}v(j-2\tau).$$

Thus we can write (4.2) in the following form

$$\Delta(\overline{B}(j)z(j))=\overline{B}(j+1)d(j)B(j)z(j-\tau)+\overline{B}(j+1)c(j)z(j-\sigma)-$$
$$-\frac{\overline{B}(j+1)c(j)d(j-\sigma)}{\overline{B}(j+1-\tau)c(j-\tau)}\Delta(\overline{B}(j-\tau)z(j-\tau))+$$
$$+\overline{B}(j+1)\left[B(j)d(j)d(j-\tau)-\frac{c(j)d(j-\sigma)d(j-\tau)B(j-\tau)}{c(j-\tau)}\right]v(j-2\tau).$$

Since $\Delta(\overline{B}(j)z(j))<0$ in view of (K1) and (4.2), thus by means of our assumptions (K2) and (K3), we see that

$$\Delta(\overline{B}(j)z(j))\le-\overline{B}(j+1)d(j)B(j)z(j-\tau)-\beta\Delta(\overline{B}(j-\tau)z(j-\tau))$$

or

$$\Delta(\overline{B}(j)z(j)+\beta\overline{B}(j-\tau)z(j-\tau))\le-\overline{B}(j+1)d(j)B(j)z(j-\tau)$$

holds for all large j. In view of our hypothesis (K1) and $\beta\ge0$ (and the proof of Lemma 3.2), we see that the sequence $\{g(j)\}$ defined by

$$g(j)=\overline{B}(j)z(j)+\beta\overline{B}(j-\tau)z(j-\tau)$$

satisfies $g(j)>0$ for all large j. Finally, since $g(j)\le(1+\beta)\overline{B}(j-\tau)\cdot$ $\cdot z(j-\tau)$, thus

$$\Delta g(j)\le-\frac{\overline{B}(j+1)d(j)B(j)}{(1+\beta)\overline{B}(j-\tau)}g(j)$$

for all large j. In view of our hypothesis (K4), we see that this is contrary to the assertion of Lemma 1.2. ■

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