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THE BIHARMONIC POLYNOMIALS OF THE TWO INDEPENDENT VARIABLES

ABSTRACT. In the paper the system of all linearly independent homogeneous biharmonic polynomials is constructed.

KEY WORDS: biharmonic polynomials, Kronecker matrix, the maximal system of the biharmonic polynomials linearly independent.

1. INTRODUCTION

The subject of the paper is the construction of the homogeneous linearly independent biharmonic polynomials W_n of the degree n, satisfying the iterated Laplace equation $\Delta^2 u(x) = 0$ of the two independent variables.

In [1] the similar problem with application to the equation of the plate $\Delta^2 u = f$ is treated. In the present paper the maximal system of the linear independent homogeneous biharmonic polynomials W_n is constructed on the contrary to the results in [1].

2. SOME EXAMPLES OF THE BIHARMONIC HOMOGENEOUS POLYNOMIALS OF THE DEGREE $n < 3$

Let us consider the homogeneous polynomials

$$W_n(x) = \sum_{j=0}^{n} a_{n-j,j} x_2^{n-j} x_1^j .$$

We shall give the example of the biharmonic polynomials of the degree zero, one, two and three:

$$W_0^0, W_1^0, W_1^1, W_2^0, W_2^1, W_2^2, W_3^0, W_3^1, W_3^2, W_3^3,$$

1° there exists one biharmonic polynomial $W_0^0(x)$ of the degree zero linearly independent $W_0(x) = 1$, Indeed. We have $\Delta^2(a_{0,0}, 1) = 0$ for every $a_{0,0} \neq 0$ and the function $I(x)$ is the unique linearly independent homogeneous biharmonic polynomial of the degree zero;

2° for the polynomial $W_1(x) = a_{1,0}x_2 + A_{0,1}x_1$ we obtain $\Delta^2 W_1(x) = 0$ for arbitrary $a_{1,0}$, $a_{0,1}$. Consequently there exist exactly two linearly independent polynomials $W_1^0(x) = x_2$, $W_1^1(x) = x_1$, biharmonic homogeneous of the degree one. They are linearly independent, Indeed. Let us consider its linear combination $C_1 W_1^0(x) + C_2 W_1^1(x)$, and the identity $C_1 x_2 + C_2 x_1 = 0$, C_1, C_2 are constants. Differentiating the last identity with respect to x_2, we obtain $C_1 = 0$, which implies that $C_2 = 0$ because by $C_2 x_1 = 0$, we obtain $C_2 = 0$;

3° for the polynomial $W_2(x) = a_{2,0}x_2^2 + a_{1,1}x_2x_1 + a_{0,2}x_1^2$, we obtain $\Delta^2 W_2(x) = 0$ for three arbitrary coefficients $a_{2,0}$, $a_{1,1}$, $a_{0,2}$, and there exist exactly three polynomials $W_2^0(x) = x_2^2$, $W_2^1(x) = x_2x_1$, and $W_2^2(x) = x_1^2$, linearly indepent, biharmonic and homogeneous of the degree two. They are linearly independent. Indeed. Similarly as for W_1 let us consider the identity $C_1x_2^2 + C_2x_2x_1 + C_3x_1^2 = 0$ with arbitrary constants C_1, C_2, C_3. Differentiating the last identity two times with respect to x_2, x_1 respectively, we obtain $C_1 = 0$, $C_2 = 0$, $C_3 = 0$;

4° similarly as for W_2 we can prove that the polynomials $W_3(x)$ linearly independent are of the form: $W_3^0(x) = x_2^3$, $W_3^1(x) = x_2^2x_1$, $W_3^2(x) = x_2x_1^2$, $W_3^3(x) = x_1^3$.

3. THE BIHARMONIC POLYNOMIALS W_n, $n \geq 4$

Next let us consider the polynomials

$$W_n(x) = \sum_{j=0}^{n} a_{n-j,j}\, x_2^{n-j} x_1^j, \qquad n \geq 4$$

and let us consider the equation

$$\Delta^2 W_n(x) = 0.$$

We have

$$\Delta^2 W_n(x) = \sum_{j=0}^{n} a_{n-j,j}(D_{x_2}^4 x_2^{n-j})x_1^j + \sum_{j=0}^{n} a_{n-j,j} 2(D_{x_2}^2 x_2^{n-j})(D_{x_1}^2 x_1^j) +$$
$$+\sum_{j=0}^{n} a_{n-j,j} x_2^{n-j}(D_{x_1}^4 x_1^j) = \sum_{j=0}^{n-4} (C_{n-j,j}a_{n-j,j} + C_{n-2-j,j+2}a_{n-2-j,j+2} +$$
$$+C_{n-4-j,j+4}a_{n-4-j,j+4})x_2^{n-4-j}x_1^j,$$

with

$$C_{n-j,j} = (n-j)(n-1-j)(n-2-j)(n-3-j),$$

$$C_{n-2-j,j+2} = 2(n-2-j)(n-3-j)(j+2)(j+1),$$

$$C_{n-4-j,j+4} = (j+4)(j+3)(j+2)(j+1), \qquad j = 0,1,\dots,n-4.$$

Hence $\Delta^2 W_n(x) = 0$ for arbitrary $x \in R^2$ if and only if

(1) $$C_{n-j,j}a_{n-j,j} + C_{n-2-j,j+2}a_{n-2-j,j+2} + C_{n-4-j,j+4}a_{n-4-j,j+4} = 0,$$

$$j = 0,1,\dots,n-4.$$

By the formulas (1) for the unknown $a_{n-j,j}$, $j = 0,1,\dots,n$, we obtain the system of the linear equations

$$C_{n,0}a_{n,0} + C_{n-2,2}a_{n-2,2} + C_{n-4,4}a_{n-4,4} = 0,$$

$$C_{n-1,1}a_{n-1,1} + C_{n-3,3}a_{n-3,3} + C_{n-5,5}a_{n-5,5} = 0,$$

$$C_{n-2,2}a_{n-2,2} + C_{n-4,4}a_{n-4,4} + C_{n-6,6}a_{n-6,6} = 0,$$

$$C_{n-3,3}a_{n-3,3} + C_{n-5,5}a_{n-5,5} + C_{n-7,7}a_{n-7,7} = 0,$$

(2) $$\dots$$

$$C_{n-2p,2p}a_{n-2p,2p} + C_{n-2p-2,2p+2}a_{n-2p-2,2p+2} + C_{n-2p-4,2p+4}a_{n-2p-4,2p+4} = 0,$$

$$C_{n-(2p+1),2p+1}a_{n-(2p+1),2p+1} + C_{n-(2p+3),2p+3}a_{n-(2p+3),2p+3} +$$

$$+ C_{n-(2p+5),2p+5}a_{n-(2p+5),2p+5} = 0,$$

$$\dots$$

$$C_{5,n-5}a_{5,n-5} + C_{3,n-3}a_{3,n-3} + C_{1,n-1}a_{1,n-1} = 0,$$

$$C_{4,n-4}a_{4,n-4} + C_{2,n-2}a_{2,n-2} + C_{0,n}a_{0,n} = 0.$$

We assume that $a_{n,0}$, $a_{n-1,1}$, $a_{n-2,2}$, $a_{n-3,3}$ are given arbitrary coefficients.

4. THE RECURRENT FORMULAS FOR THE COEFFICIENTS $a_{n-2k,2k}$, $a_{n-(2k+1),2k+1}$

Next we shall determine the coefficients $a_{n-2k,2k}$, $a_{n-(2k+1),2k+1}$ as linear combinations of the form $a_{n-2k,2k} = B_n^{2k,0}a_{n,0} + B_n^{2k,2}a_{n-2,2}$, $a_{n-(2k+1),2k+1} = B_n^{2k+1,1}a_{n-1,1} + B_n^{2k+1,3}a_{n-3,3}$ for n even and n odd respectively, where $B_n^{2p,0}$, $B_n^{2p,2}$, $B_n^{2p+1,1}$, $B_n^{2p+1,3}$ are given by suitable recurrent formulas.

Let us introduce some denotations. Let $C_{p,q,r,s} = -(C_{p,q})^{-1}C_{r,s}$, p, q, r, s are the nonnegative integers. Applying the system (2) we obtain

$$a_{n-4,4} = B_n^{4,0}a_{n,0} + B_n^{4,2}a_{n-2,2}, \qquad B_n^{4,0} = C_{n-4,4,n,0}, \qquad B_n^{4,2} = C_{n-4,4,n-2,2},$$

$$a_{n-6,6} = B_n^{6,0}a_{n,0} + B_n^{6,2}a_{n-2,2}, \qquad B_n^{6,0} = C_{n-6,6,n-4,4}B_n^{4,0},$$

$$B_n^{6,2} = C_{n-6,6,n-2,2} + C_{n-6,6,n-4,4} B_n^{4,2},$$

$$a_{n-8,8} = B_n^{8,0} a_{n,0} + B_n^{8,2} a_{n-2,2} \qquad B_n^{8,0} = C_{n-8,8,n-4,4} B_n^{4,0} + C_{n-8,8,n-6,6} B_n^{6,0},$$

$$B_n^{8,2} = C_{n-8,8,n-4,4} B^{4,2} + C_{n-8,8,n-6,6} B_n^{6,2}.$$

Similarly we obtain

$$a_{n-5,5} = B_n^{5,1} a_{n-1,1} + B_n^{5,3} a_{n-3,3}, \qquad B_n^{5,1} = C_{n-3,5,n-1,1}, \quad B_n^{5,3} = C_{n-5,5,n-3,3},$$

$$a_{n-7,7} = B_n^{7,1} a_{n-1,1} + B_n^{7,3} a_{n-3,3}, \qquad B_n^{7,1} = C_{n-7,7,n-5,5} B_n^{5,1},$$

$$B_n^{7,3} = C_{n-7,7,n-5,5} B_n^{5,3} + C_{n-7,7,n-3,3},$$

$$a_{n-9,9} = B_n^{9,1} a_{n-1,1} + B_n^{9,3} a_{n-3,3}, \qquad B_n^{9,1} = C_{n-9,9,n-5,5} B_n^{5,1} + C_{n-9,9,n-7,7} B_n^{7,1},$$

$$B_n^{9,3} = C_{n-9,9,n-5,5} B_n^{5,3} + C_{n-9,9,n-7,7} B_n^{7,3}.$$

Next we shall give two lemmas:

Lemma 1. If

$$a_{n-2p,2p} = B_n^{2p,0} a_{n,0} + B_n^{2p,2} a_{n-2,2}$$

for n even, $p = 5, \ldots, \frac{n}{2}$, *then*

$$a_{n-2p-2,2p+2} = B_n^{2p+2,0} a_{n,0} + B_n^{2p+2,2} a_{n-2,2}, \tag{3}$$

with

$$B_n^{2p+2,0} = C_{n-2p-2,2p+2,n-(2p-2),2p-2} B_n^{2p-2,0} + C_{n-2p-2,2p+2,n-2p,2p} B_n^{2p,0},$$

$$B_n^{2p+2,2} = C_{n-2p-2,2p+2,n-(2p-2),2p-2} B_n^{2p-2,1} + C_{n-2p-2,2p+2,n-2p,2p} B_n^{2p,2}.$$

Lemma 2. If

$$a_{n-(2p+1),2p+1} = B_n^{2p+1,1} a_{n-1,1} + B_n^{2p+1,3} a_{n-3,3}$$

for n *even,* $p = 5, \ldots, \frac{n-1}{2}$, *then*

$$a_{n-(2p+3),2p+3} = B_n^{2p+3,1} a_{n-1,1} + B_n^{2p+3,3} a_{n-3,3},$$

with

$$B_n^{2p+3,1} = C_{n-(2p+3),2p+3,n-(2p-1),2p-1} B_n^{2p-1,1} + C_{n-(2p+3),2p+3,n-(2p+1),2p+1} B_n^{2p+1,1},$$

$$B_n^{2p+3,3} = C_{n-(2p+3),2p+3,n-(2p-1),2p-1} B_n^{2p-1,3} + C_{n-(2p+3),2p+3,n-(2p+1),2p+1} B_n^{2p+1,3}.$$

Proof. We shall give the proof only for the Lemma 1, because the proof for the Lemma 2 is similar. by the equation

$$C_{n-2p+2,2p-2} a_{n-2p+2,2p-2} + C_{n-2p,2p} a_{n-2p,2p} + C_{n-2p-2,2p+2} a_{n-2p-2,2p+2} = 0$$

we obtain

$$a_{n-2p-2,2p+2} = C_{n-2p-2,2p+2,n-2p+2,2p-2}a_{n-2p+2,2p-2} +$$
$$+C_{n-2p-2,2p+2,n-2p,2p}a_{n-2p,2p} =$$
$$= C_{n-2p-2,2p+2,n-2p+2,2p-2}(B_n^{2p-2,0}a_{n,0} + B_n^{2p-2,2}a_{n-2,2}) +$$
$$+C_{n-2p-2,2p+2,n-2p,2p}(B_n^{2p,0}a_{n,0} + B_n^{2p,2}a_{n-2,2}).$$

By the last formula we obtain the assertion of the Lemma 1.

5. Construction of the biharmonic polynomials W_n, $n \geq 4$

Applying the formulas (3), (4) and the Kronecker matrix $[\delta_{i,k}]$, $i,k = 0,1,2,3$, we shall prove

Theorem 1. There exist exactly four biharmonic polynomials $W_n(x)$, linearly independent homogeneous of the degree n.

To the proof we shall give: 1° the form of the polynomials W_n, 2° we shall prove its linear independence.

Proof. Ad 1°. We shall consider two cases: $n = 2m$, $n = 2m+1$. For $n = 2m$ by (3), (4) we have

$$a_{n-4,4}x_2^{n-4}x_1^4 = (B_n^{4,0}a_{n,0} + B_n^{4,2}a_{n-2,2})x_2^{n-4}x_1^4,$$
$$a_{n-5,5}x_2^{n-5}x_1^5 = (B_n^{5,1}a_{n-1,1} + B_n^{5,3}a_{n-3,3})x_2^{n-5}x_1^5,$$
$$\ldots$$
$$a_{1,n-1}x_2x_1^{n-1} = (B_n^{n-1,1}a_{n-1,1} + B_n^{n-1,3}a_{n-3,3})x_2x_1^{n-1},$$
$$a_{0,n}x_1^n = (B_n^{n,0}a_{n,0} + B_n^{n,2}a_{n-2,2})x_1^n.$$

Let us consider the polynomials

$$(5) \quad W_n^i(x) = \sum_{j=0}^{3} \delta_{i,j}x_2^{n-j}x_1^j + (B_n^{4,0}\delta_{i,0} + B_n^{4,2}\delta_{i,2})x_2^{n-4}x_1^4 +$$
$$+(B_n^{5,1}\delta_{i,1} + B_n^{5,3}\delta_{i,3})x_2^{n-5}x_1^5 + (B_n^{6,0}\delta_{i,0} + B_n^{6,2}\delta_{i,2})x_2^{n-6}x_1^6 +$$
$$+(B_n^{7,1}\delta_{i,1} + B_n^{7,3}\delta_{i,3})x_2^{n-7}x_1^7 + \ldots + (B_n^{n-1,1}\delta_{i,1} + B_n^{n-1,3}\delta_{i,3})x_2x_1^{n-1} +$$
$$+(B_n^{n,0}\delta_{i,0} + B_n^{n,2}\delta_{i,2})x_1^n, \qquad i = 0,1,2,3.$$

By the formulas (5) we obtain the following formulas

$$W_n^0(x) = x_2^n + B_n^{4,0}x_2^{n-4}x_1^4 + B_n^{6,0}x_2^{n-6}x_1^6 + \ldots + B_n^{n,0}x_1^n,$$

(6) $$W_n^1(x) = x_2^{n-1}x_1 + B_n^{5,1}x_2^{n-5}x_1^5 + B_n^{7,1}x_2^{n-7}x_1^7 + \ldots + B_n^{n-1,1}x_2x_1^{n-1},$$

$$W_n^2(x) = x_2^{n-2}x_1^2 + B_n^{4,2}x_2^{n-4}x_1^4 + B_n^{6,2}x_2^{n-6}x_1^6 + \ldots + B_n^{n,2}x_1^n,$$

$$W_n^3(x) = x_2^{n-3}x_1^3 + B_n^{5,3}x_2^{n-5}x_1^5 + \ldots + B_n^{n-1,3}x_2x_1^{n-1}.$$

Similarly for $n = 2m+1$ we obtain

$$W_n^0(x) = x_2^n + B_n^{4,0}x_2^{n-4}x_1^4 + \ldots + B_n^{n-1,0}x_2x_1^{n-1},$$

(7) $$W_n^1(x) = x_2^{n-1}x_1 + B_n^{5,1}x_2^{n-5}x_1^5 + \ldots + B_n^{n,1}x_1^n,$$

$$W_n^2(x) = x_2^{n-2}x_1^2 + B_n^{4,2}x_2^{n-4}x_1^4 + \ldots + B_n^{n-1,2}x_2x_1^{n-1},$$

$$W_n^3(x) = x_2^{n-3}x_1^3 + B_n^{5,3}x_2^{n-5}x_1^5 + \ldots + B_n^{n-3,3}x_1^n.$$

Next we shall prove the linear independence of the systems (6), (7). We shall give the proof for the polynomials $W_{2m}(x)$ because the proof for the polynomials $W_{2m+1}(x)$ is similar.

Let us consider the identity

$$C_0W_n^0(x) + C_1W_n^1(x) + C_2W_n^2(x) + C_3W_n^3(x) = 0,$$

or

$$\begin{aligned} &C_0(x_2^n + B_n^{4,0}x_2^{n-4}x_1^4 + \ldots + B_n^{n,0}x_1^n) + \\ &\quad + C_1(x_2^{n-1}x_1 + B_n^{5,1}x_2^{n-5}x_1^5 + \ldots + B_n^{n-1,1}x_2x_1^{n-1}) + \\ &\quad + C_2(x_2^{n-2}x_1^2 + B_n^{4,2}x_2^{n-4}x_1^4 + \ldots + B_n^{n,2}x_1^n) + \\ &\quad + C_3(x_2^{n-3}x_1^3 + B_n^{5,3}x_2^{n-5}x_1^5 + \ldots + B_n^{n,3}x_1^n) = 0. \end{aligned}$$

Differentiating the last identity n-times with respect to the variable x_2, we obtain $C_0(n!) = 0$ and consequently $C_0 = 0$ and

(8) $$C_1W_n^1(x) + C_2W_n^2(x) + C_3W_n^3(x) = 0.$$

Differentiating the last identity (8), $(n-1)$ –times with respect to x_2, we obtain $C_1 = 0$ and

$$C_2W_n^2(x) + C_3W_n^3(x) = 0.$$

Differentiating the identity (9), $(n-2)$-times with respect to x_2 we get $C_2 = 0$ which implies that $C_3 = 0$.

Finally obtain the following result

Theorem 2. The maximal number of the biharmonic homogeneous polynomials W_n, linearly independent is equal four. Every biharmonic homogeneous polynomial $P_n(x)$ of the degree n, is a linear combination of the polynomials W_n^i, $i = 1,2,3$.

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