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ON THE EXISTENCE OF ASYMPTOTICALLY CONSTANT SOLUTIONS OF A SYSTEM LINEAR DIFFERENCE EQUATIONS

ABSTRACT. We consider linear system of difference equations and give sufficient condition for this equation has solution which tend to a constant vector. Applying these conditions two scalar m-th order difference equations are studied in relation to the problem: when one of them possesses a solution which could be approximated by linear combinations of the solutions of the second equation.

KEY WORDS. difference equation, system, asymptotically constant solution, comparison theorem.

In this paper we provide sufficient conditions for the solution of any linear system of difference equations to tend to a constant vector. Applying these conditions we study two particular scalar difference equations and present conditions under which the solutions of one of them approximates linear combinations of the solutions of the second equation.

1. NOTATIONS

We shall use the following notations :

N - set of positive integers,

R - set of real numbers,

$(\mathbf{V}, \|\,\|_v)$ - normed space of m-dimensional vectors with real components,

$x, y, \ldots$ - elements of $\mathbf{V}$,

$x = [x^1, \ldots, x^m]^T$,

$\|x\|_v = \max_{1 \le i \le m} |x^i|$

$(\mathbf{M}, \|\,\|_{\mathbf{M}})$ - normed space of $m \times m$ matrices,

for $\mathbf{A} = [a^{ij}]_{m \times m} \in \mathbf{M}$, $a^{ij} \in R$,

$A^{i,j}$ - (i, j)-minor of the matrix **A**,

$\|\mathbf{A}\|_{\mathbf{M}} = \max_{1 \le i \le m} \sum_{j=1}^{m} |a^{ij}|$,

$(l_m^\infty, \| \|_l)$ - Banach space of bounded sequences of elements from **V**,

$\mathbf{x},\mathbf{y},\ldots$ elements of l_m^∞,

for $\mathbf{x} = \{x_n\}_{n=1}^\infty = \left\{[x_n^1,\ldots,x_n^m]^{\mathrm{T}}\right\}_{n=1}^\infty$

$$\|\mathbf{x}\|_l = \sup_{n\geq 1}(\|x_n\|_v) = \sup_{n\geq 1}(\max_{1\leq i\leq m}|x_n^i|).$$

For any function $z:\mathbf{N}\to\mathbf{R}$ the difference operators are defined in the usual way

$$\Delta z_n = z_{n+1} - z_n, \qquad n\in\mathbf{N},$$

$$\Delta^k z_n = \Delta(\Delta^{k-1} z_n), \qquad n\in\mathbf{N}.$$

Similarly for $z:\mathbf{N}\to\mathbf{V}$

$$\Delta z_n = [z_{n+1}^1,\ldots,z_{n+1}^m]^{\mathrm{T}} - [z_{n+1}^1,\ldots,z_{n+1}^m]^{\mathrm{T}}, \qquad n\in\mathbf{N}.$$

If $\|x_n - c\|_v \to 0$ as $n\to\infty$, where $c,x_n\in\mathbf{V}$ $(n=1,2,\ldots)$ then we write $x_n \xrightarrow[v]{} c$.

Suppose we have the difference equation (we do not make precise here the nature of the elements x_n)

$$(*) \qquad \Delta^k x_n = F(n,\ldots), \qquad n\in N.$$

By a solution of $(*)$ we mean any sequence $\{x_n\}_{n=1}^\infty$ which satisfies equation $(*)$ for all $n\in\mathbf{N}$, whereas by a generalised solution of $(*)$ we mean such a sequence which satisfies the relation $(*)$ for all n sufficiently large.

2. LINEAR SYSTEM OF DIFFERENCE EQUATIONS

Theorem 1. Let $\mathrm{A}:\mathbf{N}\to\mathbf{M}$ *be such that*

$$(1) \qquad \sum_{j=1}^\infty \|\mathrm{A}_j\|_{\mathrm{M}} < \infty.$$

Then for arbitrary $\mathrm{c}\in\mathbf{V}$, *there exists a generalised solution* $\mathbf{x}$ *of the equation*

$$(\mathrm{E}) \qquad \Delta x_n = \mathrm{A}_n x_{n+k}, \qquad n\in\mathbf{N},$$

(where k *is some fixed nonnegative integer) such that*

$$(2) \qquad x_n \xrightarrow[v]{} c \quad as \quad n\to\infty.$$

Proof. Let us take any $c\in\mathbf{V}$, $c\neq 0\in\mathbf{V}$ and $\varepsilon>0$, $\varepsilon\in\mathbf{R}$. For $c=0$ the zero solution is the required one. Denote

$$\gamma = \max_{1\leq i\leq m}\left\{\max\{|c^j+\varepsilon|,|c^j-\varepsilon|\}\right\}$$

and

(3) $$\alpha_n = \gamma \sum_{j=n}^{\infty} \|A_j\|_M, \qquad n = 1,2,\ldots .$$

From (1) it follows that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is nonincreasing and $\lim_{n\to\infty}\alpha_n = 0$, so there exists $n_1 \in \mathbf{N}$ such that $\alpha_n \le \varepsilon$ for all $n \ge n_1$. Let us take the smallest n_1 with this property, that is

$$n_1 = \min\left\{ n \in \mathbf{N}: \gamma \sum_{j=n}^{\infty} \|A_j\|_M \le \varepsilon \right\}.$$

Let $P \subset l_m^{\infty}$ be the set of all sequences such that

$$\mathbf{p} = \{p_n\}_{n=1}^{\infty} = \left\{ \left[p_n^1,\ldots,p_n^m\right]^T \right\}_{n=1}^{\infty} \in P$$

if
$$\begin{cases} p_n = c & \text{for } \quad n = 1,\ldots,n_1 - 1 \\ p_n^i \in D_n^i & \text{for } \quad n \ge n_1, i - 1,\ldots,m \end{cases}$$

where

$$D_n^i = [c^i - \alpha_n, c^i + \alpha_n] \subset \mathbf{R}.$$

The set P is bounded, convex, and closed in l_m^{∞}. Furthermore, since $diam\, D_m^i = 2\alpha_n \to 0$ as $n \to \infty$, so for arbitrary $\varepsilon_1 > 0$ we can build finite ε_1 - net for the set P. Hence by the Hausdorff theorem P is compact. Define now some operator $\mathcal{A}$ by the formula

$$\mathcal{A}\mathbf{x} = \mathbf{y} = \left\{ \left[y_n^1,\ldots,y_n^m\right]^T \right\}_{n=1}^{\infty}$$

where

(4) $$\begin{cases} y_n = c & \text{for } n = 1,\ldots,n_1 - 1, \\ y_n = c - \sum_{j=n}^{\infty} A_j x_{j+k} & \text{for } \quad n \ge n_1, \end{cases}$$

for $\mathbf{x} = \{x_n\}_{n=1}^{\infty} = \left\{ \left[x_n^1,\ldots,x_n^m\right]^T \right\}_{n=1}^{\infty} \in l_m^{\infty}$.

Observe that $\|\mathbf{x}\|_l \le K$ for some real constant K, yields $\|x_n\|_v \le K$ for all $n \in \mathbf{N}$, and

$$\sum_{j=n}^{\infty} \|A_j\|_M \le (\alpha_1)/\gamma.$$

Therefore

$$\|y_n\|_v \le \|c\|_v + \left\| \sum_{j=n}^{\infty} A_j x_{j+k} \right\|_v \le \|c\|_v + (K\alpha_1)/\gamma \quad \text{for all } n \in \mathbb{N}.$$

Hence, the operator $\mathcal{A}$ well defined on l_m^∞.

Now let $\mathbf{x} \in P$ and $\mathbf{y} = \mathcal{A}\mathbf{x}$. By (4) we get for $n \ge n_1$ and $i \in \{1,\dots,m\}$

$$\left| y_n^i - c \right| \le \left\| y_n - c \right\|_v = \left\| \sum_{j=n}^{\infty} \mathrm{A}_j x_{j+k} \right\|_v \le \sum_{j=n}^{\infty} \left\| \mathrm{A}_j \right\|_{\mathrm{M}} \left\| x_{j+k} \right\|_v \le \left\| \mathbf{x} \right\|_l \sum_{j=n}^{\infty} \left\| \mathrm{A}_j \right\|_{\mathrm{M}}$$

$$\le \gamma \sum_{j=n}^{\infty} \left\| \mathrm{A}_j \right\|_{\mathrm{M}} = \alpha_n .$$

Therefore $y_n^i \in D_n^i$ for all $n \ge n_1$, $i \in \{1,\dots,m\}$, and so $\mathcal{A}(P) \subset P$.
We now prove that $\mathcal{A}$ is continuous on P.

Let us take $\varepsilon_1 > 0$ and $\delta = (\gamma\varepsilon_1)/\alpha_{n_1}$ and any two arbitrary $\mathbf{x}, \mathbf{y} \in P$ such that $\left\| \mathbf{x} - \mathbf{y} \right\|_l < \delta$. We shall prove that $\left\| \mathcal{A}\mathbf{x} - \mathcal{A}\mathbf{y} \right\|_l < \varepsilon_1$. Indeed, absolute convergence of the series

$$\sum_{j=n_1}^{\infty} \mathrm{A}_j x_{j+k}, \quad \sum_{j=n_1}^{\infty} \mathrm{A}_j y_{j+k}$$

yields

$$\left\| \mathcal{A}\mathbf{x} - \mathcal{A}\mathbf{y} \right\|_l = \sup_{n \ge n_1} \left\| -\sum_{j=n}^{\infty} \mathrm{A}_j x_{j+k} + \sum_{j=n}^{\infty} \mathrm{A}_j y_{j+k} \right\|_v$$

$$\le \sup_{n \ge n_1} \sum_{j=n}^{\infty} \left\| \mathrm{A}_j \right\|_{\mathrm{M}} \left\| x_{j+k} - y_{j+k} \right\|_v \le \left\| \mathbf{x} - \mathbf{y} \right\|_l \sup_{n \ge n_1} \sum_{j=n}^{\infty} \left\| \mathrm{A}_j \right\|_{\mathrm{M}}$$

$$\le \delta \sup_{n \ge n_1} (\alpha_n/\gamma) \le \left(\gamma\,\varepsilon_1/\alpha_{n_1}\right)\left(\alpha_{n_1}/\gamma\right) = \varepsilon_1 .$$

Therefore $\mathcal{A}$ is continuous on P, and since all assumptions of the Schauder fixed point theorem are satisfied, there exists a solution of the equation $\mathbf{x} = \mathcal{A}\mathbf{x}$ in the set P. Let $\mathbf{z} = \{z_n\}_{n=1}^{\infty}$ be this solution. Therefore by (4) we obtain

$$z_n = c - \sum_{j=n}^{\infty} \mathrm{A}_j z_{j+k} \quad \text{for } n \ge n_1$$

and applying operator Δ

$$\Delta z_n = \mathrm{A}_n z_{n+k} ,$$

that is $\mathbf{z}$ is a generalised solution of (E). Furthermore, by absolute convergence of the series $\sum_{j=n}^{\infty} A_j z_{j+k}$ we have

$$\left\| z_n - c \right\|_v = \left\| -\sum_{j=n}^{\infty} A_j z_{j+k} \right\|_v \to 0$$

and this means that $\mathbf{z}$ possesses property (2).

Remark 1. If k is a positive integer then transforming (E) to the form

$$x_n = x_{n+1} - A_n x_{n+k}, \quad n \in \mathbf{N}$$

and putting $x_j = z_j$ for $j \geq n_1$ we can find $x_{n_1-1}, \ldots, x_1$ successively in a step by step fashion. The sequence thus obtained forms an ordinary solution of (E) and because it coincides with $\mathbf{z}$ for $n \geq n_1$, possesses property (2).

If $k = 0$ then from (E) we obtain

$$(I + A_n)\, x_n = x_{n+1}$$

where I is the identity matrix. Therefore to employ the above procedure we require that the matrices $I + A_n$ be non-singular. For second order non-linear scalar equations the argument can be found in e.g. [1].

3. APPLICATION TO AN *M*-TH ORDER DIFFERENCE EQUATION

In this section we show how Theorem 1 can be used to demonstrate that the solutions of two scalar difference equations are asymptotically close in a certain sense.

We shall study the following equations

(E1) $$\Delta^m y_n = a_n y_{n+1}, \qquad n \in \mathbf{N}$$

and

(E2) $$\Delta^m z_n = b_n z_{n+1}, \qquad n \in \mathbf{N}.$$

Let C denote determinant of the Casorati matrix for the solutions of the equation (E1), and let C_n denote this determinant (Casoratian) at the point n, that is

$$\mathsf{C}_n = \begin{vmatrix} y_n^1 & \ldots & y_n^m \\ \ldots & \ldots & \ldots \\ \Delta^{m-1} y_n^1 & \ldots & \Delta^{m-1} y_n^m \end{vmatrix}.$$

The following lemma can be easily proved using the standard properties of determinants and applying relation (E1).

Lemma 1. Let $y=\{y^1,\dots,y^m\}$ *be solutions of* (E1) *such that their Casoratian* $C_1=-1$ *then* $C_n=-1$ *for all* $n\in\mathbf{N}$.

Lemma 2.([2]) *Let* $y^1,\dots,y^m$ *be linearly independent solutions of the equation* (E1) *. Then*

$$\Delta C_n^{2,j}=C_n^{1,j} \qquad \text{for } j=1,\dots,m\ ,\ n\in\mathbf{N}$$

$$\Delta C_n^{i+1,j}=C_{n+1}^{i,j} \qquad \text{for } i=2,\dots,m-1\ ,\ j=1,\dots,m\ ,\ n\in\mathbf{N}.$$

Theorem 2. Let $y^1,\dots,y^m$ *be linearly independent solutions of the equation* (E1) *such that* $C_1=-1$ *. If*

$$\text{(5)} \qquad \sum_{n=1}^{\infty}\left\{\left[\left|a_n-b_n\right|\sum_{i=1}^{m}\left|y_{n+1}^i\right|\right]\max_{1\le j\le m}\left|C_{n+1}^{m,j}\right|\right\}<\infty .$$

Then equation (E2) *possesses a solution* z *which can be written in the form*

$$\text{(6)} \qquad z_n=\alpha_n^1y_n^1+\dots+\alpha_n^my_n^m \quad ,\ n\in\mathbf{N}$$

and

$$\lim_{n\to\infty}\alpha_n^i=\alpha_*^i\ ,\ i=1,\dots,m$$

where α_*^i *are constants.*

Proof. Let $y^1,\dots,y^m$ be linearly independent solutions of (E1) for which $C_1=-1$, then by Lemma 1 $C_n=-1$ for all $n\in\mathbf{N}$. Let us denote

$$\text{(7)} \qquad u_n^i=\sum_{j=1}^{m}(-1)^{j+i-1}C_n^{j,i}\Delta^{j-1}z_n, \quad i=1,\dots,m$$

where $z=\{z_n\}_{n=1}^{\infty}$ is any solution of the equation (E2).

Multiplying the i-th equation in (7) by y_n^i , and summing over i we get

$$\sum_{i=1}^{m}y_n^iu_n^i=\sum_{i=1}^{m}(-1)^iC_n^{1,i}z_ny_n^i+\sum_{i=1}^{m}y_n^i\left[\sum_{j=2}^{m}(-1)^{j+i-1}C_n^{j,i}\Delta^{j-1}z_n\right]=$$

$$= -\left[\sum_{i=1}^{m} y_n^i (-1)^{i+1} C_n^{1,i}\right] z_n - \sum_{j=2}^{m}\left[\sum_{i=1}^{m} y_n^i (-1)^{j+i} C_n^{j,i}\right]\Delta^{j-1} z_n \ .$$

The first term in the above sum is the first row expansion of the Casoratian and by Lemma 1 is equal to -1. With regard to the second sum, we multiply entries of the first row by cofactors from the other rows, so this sum is equal zero. Therefore

$$\sum_{i=1}^{m} y_n^i u_n^i = z_n \ , \quad n \in \mathbf{N} \ . \tag{8}$$

Differencing (7), using Lemma 2, the properties of the operator Δ, and replacing $\Delta^m z_n$ by the right hand side of (E2) we get

$$\begin{aligned}
\Delta u_n^i &= \sum_{j=1}^{m} (-1)^{j+i-1} \Delta\left[C_n^{j,i} \Delta^{j-1} z_n\right] \\
&= (-1)^i z_{n+1} \Delta C_n^{1,i} + (-1)^i \left[C_n^{1,i} - \Delta C_n^{2,i}\right] \Delta z_n \\
&\quad + \sum_{j=2}^{m-1} (-1)^{j+i-1}\left[C_{n+1}^{j,i} - \Delta C_n^{j+1,i}\right]\Delta^j z_n + (-1)^{m+i-1} C_{n+1}^{m,i} \Delta^m z_n \\
&= (-1)^i z_{n+1} \Delta C_n^{1,i} + (-1)^{m+i-1} C_{n+1}^{m,i} b_n z_{n+1} \\
&= (-1)^i \left[\Delta C_n^{1,i} + (-1)^{m-1} C_{n+1}^{m,i} b_n\right] z_{n+1} \ .
\end{aligned}$$

It is straightforward to verify that

$$\Delta C_n^{1,i} = (-1)^m a_n C_{n+1}^{m,i} \ .$$

Hence

$$\Delta u_n^i = (-1)^{m+i}\left[a_n - b_n\right] C_{n+1}^{m,i} z_{n+1} \ . \tag{9}$$

Replacing z_{n+1} by $\sum_{j=1}^{m} y_{n+1}^j u_{n+1}^j$ in (9) we can rewrite the system (9) in the form

$$\Delta u_n = \left(a_n - b_n\right) A_n u_{n+1} \ , \ n \in \mathbf{N} \tag{10}$$

where $u_n = \left[u_n^1, \ldots, u_n^m\right]^{\mathrm{T}}$ and

$$A_n = \begin{bmatrix} (-1)^{m+1} C_{n+1}^{m,1} y_{n+1}^1 & \dots & (-1)^{m+1} C_{n+1}^{m,1} y_{n+1}^m \\ \dots & \dots & \dots \\ (-1)^{2m} C_{n+1}^{m,m} y_{n+1}^1 & \dots & (-1)^{2m} C_{n+1}^{m,m} y_{n+1}^m \end{bmatrix}$$

$$\Delta u_n^i = u_{n+1}^i - u_n^i = (-1)^{m+i} [a_n - b_n] C_{n+1}^{m,i} \left(y_{n+1}^1 u_{n+1}^1 + \dots + y_{n+1}^m u_{n+1}^m \right).$$

Equation (10) is of the form (E). By assumption (5) we see that condition (1) of Theorem 1 is satisfied. Hence for any constant vector $\alpha_* = \left[\alpha_*^1, \dots, \alpha_*^m\right]^T$ there exists a solution (in fact an ordinary solution by Remark 1) $\alpha_n = \left[\alpha_n^1, \dots, \alpha_n^m\right]^T$ of (10) such that $\alpha_n \underset{v}{\to} \alpha_*$ as $n \to \infty$. Comparing (8) (where α_n^i are in the place of u_n^i) with (6) we immediately see that the proof is complete. ■

Remark 2. If we have any m solutions $y^1, \dots, y^m$ of (E1) whose Casoratian $C_1 \neq 0$ (these solutions form a linearly independent system of solutions) then by linearity of the equation (E1) we can find some constants $\beta^1, \dots, \beta^m$ such that the sequences $\beta^1 y^1, \dots, \beta^m y^m$ build a new system of solutions for which $C_1 = -1$, and check condition (5) is satisfied for these solutions. If so, Theorem 2 holds and

$$z_n = \left(\beta^1 \alpha_n^1\right) y_n^1 + \dots + \left(\beta^m \alpha_n^m\right) y_n^m , \quad n \in \mathbf{N}$$

and

$$\beta^i \alpha_n^i \to \beta^i \alpha_*^i \text{ (constants) as } n \to \infty , \quad i = 1, \dots, m.$$

Corollary. If we take as (E1) *the equation*

(11) $$\Delta^m y_n = 0 , \quad n \in \mathbf{N}$$

and for the equation (E2) *we have*

$$\sum_{n=1}^{\infty} |b_n| n^{2m-2} < \infty$$

then (E2) *possesses solution* z *with the property*

$$z_n = \alpha_n^1 + \alpha_n^2 n + \dots + \alpha_n^m n^{m-1}$$

where α_n^i *tend to some constants.*

This follows from the fact that equation (11) possesses the linearly independent solutions $y_n^i = n^{(i-1)}$, $i = 1,\dots,m$.

REFERENCES

[1] A. Drozdowicz, J. Popenda, Asymptotic behavior of solutions of the second order difference equation, *Proc. Amer. Math. Soc.*, 99(1)(1987), 135-140.

[2] M. Migda, Asymptotic properties of solutions of higher order difference equations, *Radovi Matematicki*, 5(1989), 297-309.

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