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ON RANDOM LIPSCHITZ CONDITION AND ITS APLICATION IN APPROXIMATION THEORY

ABSTRACT. There are estimated moduli of continuity of functions satisfying Lipschitz condition with random exponents both in the sense of convergence in probability and convergence in mean. The results are applied to extend Jackson's direct approximation theorem to the case of Lipschitz condition with a random exponent.

KEY WORDS: modulus of continuity, Lipschitz condition, convergence in probability, convergence in mean, modular space, approximation by trigonometric polynomials.

1. Let G be an arbitrary non-empty set and let an operation "$\cdot$" be defined on $G \times G$ with values in G. Let (G,Σ,μ) be a measure space with a σ-finite measure, and let U be a filter of subsets of G possessing a base $U_0 \subset \Sigma$. By $L^0(G)$ we denote the space of Σ-measurable, extended real-valued functions on G, finite μ-almost everywhere on G, with equality μ-a.e. on G. Let $A \subset G$ and $t \in G$ be arbitrary. We put $A_t^l = \{s \in G: ts \in A, s \notin A \text{ or } ts \notin A, s \in A\}$ and $A_t^r = \{s \in G: st \in A, s \notin A \text{ or } st \notin A, s \in A\}$. In [1], the system $\boldsymbol{G} = \{G,U,\Sigma,\mu\}$ was called *left-correctly filtered* [resp. *right-correctly filtered*] if the following conditions are satisfied: (1) if $A \in \Sigma$ and $\mu(A) < +\infty$ then $A_t^l \in \Sigma$ [resp. $A_t^r \in \Sigma$] for every $t \in \Sigma$ and $\mu(A_t^l) \xrightarrow{U} 0$ [resp. $\mu(A_t^r) \xrightarrow{U} 0$], (2) if $f \in L^0(G)$ then $f(t\cdot) \in L^0(G)$ and $f(\cdot\, t) \in L^0(G)$ for all $t \in G$. Let us remark, that if G is a locally compact Hausdorff topological group, U is the family of neighbourhoods of the neutral element in G, μ_l and μ_r are the left-invariant and the right-invariant. Haar measure in G, then the system $\{G,U,\Sigma,\mu_l\}$ [resp. $\{G,U,\Sigma,\mu_r\}$] is left-correctly filtered [resp. right-correctly filtered].

Let $\eta: L^0(G) \to \overline{R}_0^+ = [0,+\infty\}$ be a modular in $L^0(G)$, i.e. $\eta(f) = 0$ if and only if $f = 0$, $\eta(-f) = \eta(f)$ and $\eta(\alpha f + \beta g) \le \eta(f) + \eta(g)$ for all $f,g \in L^0(G)$, $\alpha,\beta \ge 0$, $\alpha+\beta = 1$. The vector space $L_\eta^0(G) = \{f \in L^0(G): \eta(\alpha f) \to 0$ as $\lambda \to 0_+\}$ is called the *modular space generated* by η (see e.g. [2]). The modular η is called 1° *convex*, if $\eta(\alpha f + \beta g) \le \alpha\eta(f) + \beta\eta(g)$ for $f,g \in L^0(G)$, $\alpha,\beta \ge 0$, $\alpha+\beta = 1$; 2° *monotone*, if $f,g \in L^0(G)$, $|f| \le |g|$ imply

$\eta(f)\le\eta(g)$; 3° *absolutely finite*, if $A\in\Sigma$, $\mu(A)<+\infty$ imply $\chi_A\in L^0(G)$ (χ_A - the characteristic function of the set A) and for every $\varepsilon>0$ and every $\lambda_0>0$ there exists a $\delta>0$ such that for every $B\in\Sigma$ with $\mu(B)<\delta$ there holds $\eta(\lambda_0\chi_B)<\varepsilon$; 4° *absolutely continuous*, if there exists an $\alpha>0$ such that for any $f\in L^0(G)$ with $\eta(f)<+\infty$ there hold the following conditions: (a) for every $\varepsilon>0$ there exists a set $A\in\Sigma$ with $\mu(A)<+\infty$ such that $\eta(\alpha f\chi_{G\setminus A})<\varepsilon$, (b) for every $\varepsilon>0$ there exists a $\delta>0$ such that for every $B\in\Sigma$ with $\mu(B)<\delta$ there holds $\eta(\alpha f\chi_B)<\varepsilon$; 5° *left* τ*-bounded* [resp. *right* τ*-bounded*] if there exists a number $C\ge 1$ and a Σ-measurable function $h:G\to\overline{R}_0^+$ with $h(t)\xrightarrow{U}0$ such that for every $f\in L^0(G)$ with $\eta(f)<+\infty$ there holds the inequality $\eta(f(t\cdot))\le\eta(Cf)+h(t)$ [resp. $\eta(f(\cdot t))\le\eta(Cf)\ +h(t)$] for all $t\in G$.

Let $f\in L^0(G)$, $U\in\mathcal{U}$. The *left* η*-modulus* [resp. *right* η*-modulus*] *of continuity* of the function f with respect to the set U is defined as $\Lambda_\eta^l(f,U)=\sup_{t\in U}\eta(f(t\cdot)-(f(\cdot))$ [resp. $\Lambda_\eta^r(f,U)=\sup_{t\in U}\eta(f(\cdot t)-(f(\cdot))$]. The following theorem was proved in [1].

Theorem A. Let $\mathbf{G}=\{G,\mathbf{U},\Sigma,\mu\}$ *be a left-correctly filtered system. Let* η *be a monotone, absolutely finite, absolutely continuous, and left* τ*-bounded modular in* $f\in L^0(G)$. *Then for every function* $f\in L^0(G)$ *there exists a number* $\lambda>0$ *such that* $\Lambda_\eta^l(\lambda f,U)\xrightarrow{U}0$.

Remark A. In particular, if G is a metric group and $\mathbf{U}$-the set of all neighbourhoods of the neutral element $e\in G$ and $\mathbf{U}_0$ is the set of all balls U_n with centre at e and radius $\frac{1}{n}$, where $n=1,2,\ldots$, then we shall write $\Lambda_\eta^l(f,\frac{1}{n})$ in place of $\Lambda_\eta^l(f,U_n)$. Under the assumptions of Theorem A, we have $\Lambda_\eta^l(\lambda f,\ \frac{1}{n})\to 0$ as $n\to\infty$ for every $f\in L^0(G)$ with some $\lambda>0$, depending on f.

2. The purpose of this paper is to investigate the more general case of functions $f:\Omega\times G\to\overline{R}$, where $(\Omega,\mathbf{B},P)$ is a probability space, and $f(\omega,\cdot)\in L_\eta^0(G)$ for all $\omega\in\Omega$, $f(\cdot,x)$ is a randem variable on Ω for all $x\in G$. The space of all such functions f will be denoted by $L^0(\Omega,G)$. Supposing the assumptions of Theorem A and Remark A to be satisfied, we have $\Delta_\eta^l(f(\omega,\cdot),\frac{1}{n})\to 0$ as $n\to\infty$ for all $\omega\in\Omega$. Let $\alpha:\Omega\to[0,1]$ be a random variable, $\alpha(\omega)>0$ P-a.e. in Ω. We denote by $\mathrm{Lip}_\eta(A,\alpha)$ the class of all functions $f\in L^0(\Omega,G)$ such that

$$\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)\leq\frac{A}{n^{\alpha(\omega)}}$$

for $\omega\in\Omega$ and $n=1,2,\ldots$. It is obvious that if $f\in\mathrm{Lip}_\eta(A,\alpha)$, then $\Delta_\eta^l(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}})\to 0$ as $n\to\infty$ pointwise in Ω. We are going to estimate the order of convergence to zero of the above expresion, taking convergence in probability and convergence in mean in place of pointwise convergence.

In the sequel, we denote by F_α the distribution function of the random variable α. Also the symbol log will mean a natural logarithm.

Theorem 1. If $f\in\mathrm{Lip}_\eta(A,\alpha)$ and $0<\varepsilon<A$, then

$$P\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)>\varepsilon\right)\leq F_\alpha\left(\frac{\log A/\varepsilon}{\log n}\right)$$

for $n=1,2,\ldots$ and the right-hand side of this inequality tends w zero as $n\to\infty$.

Proof. Since $f\in\mathrm{Lip}_\eta(A,\alpha)$, we have for all n

$$P\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)>\varepsilon\right)\leq P\left(\frac{1}{n^{\alpha(\omega)}}>\varepsilon\right)=F_\alpha\left(\frac{\log A/\varepsilon}{\log n}\right)\to F_\alpha(0+)$$

as $n\to+\infty$.

Since $\alpha(\omega)>0$, we have

$$\sum_{n=1}^{\infty}P\left(\frac{1}{n+1}\leq\alpha(\omega)<\frac{1}{n}\right)\leq P(\alpha(\omega)<1)\leq 1.$$

Hence

$$F_\alpha(0+)=\lim_{n\to\infty}F_\alpha\left(\frac{1}{n}\right)=\sum_{k=n}^{\infty}P\left(\frac{1}{k+1}\leq\alpha(\omega)<\frac{1}{k}\right)\to 0\quad\text{as}\quad n\to\infty.$$

Example 1. Let us take $\Omega=[0,1]$, $\boldsymbol{B}$-the family of Lebesgue measurable subsets of Ω, and let α be a random variable in Ω. We define $P(A)=K\int_A\alpha(\omega)d\omega$ for $\alpha\in\boldsymbol{B}$, where $K>0$ is chosen in such a manner that $P(\Omega)=1$.

(a) Take $\alpha(\omega)=\alpha_0=\text{const.}$, where $0<\alpha_0\leq 1$. Then $P(A)=K\alpha_0|A|$ for $A\in\boldsymbol{B}$, where $|A|$ is the Lebesgue measure of A, and $F_\alpha(x)=0$ for $x\leq\alpha_0$, $F_\alpha(x)=1$ for $x>\alpha_0$. Hence

$$P\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha_0}}\right)>\varepsilon\right)\le F_{\alpha_0}\left(\frac{\log A/\varepsilon}{\log n}\right)=0$$

for $n\ge (A/\varepsilon)^{1/\alpha_0}$.

(b) Let $\alpha(\omega)=2\omega$ for $0\le\omega\le\frac{1}{2}$, $\alpha(\omega)=-2\omega+2$ for $\frac{1}{2}<\omega\le 1$. Then $F_a(x)=2K\int_0^{x/2}\alpha(\omega)d\omega=\frac{1}{2}Kx^2$ for $0\le x\le 1$, $F_a(x)=0$ for $x<0$ and $F_a(x)=1$ for $x>1$. Hence, by Theorem 1,

$$P\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)>\varepsilon\right)\le\frac{1}{2}K\frac{\log^2 A/\varepsilon}{\log^2 n}\quad\text{for}\quad n>\frac{A}{\varepsilon}.$$

(c) Let $\alpha(\omega)=\exp\frac{(2\omega-1)^2}{4\omega(\omega-1)}$ for $0<\omega<1$, $\alpha(0)=\alpha(1)=0$, where we denote $\exp x=e^x$ for $x\in R$. Then α is an infinitely differentiable function with support $[0,1]$, symmetric with respect to the straight line $\omega=\frac{1}{2}$. Moreover, we have

$$F_\alpha(x)=2K\int_0^{\omega_x}\exp\frac{(2\omega-1)^2}{4\omega(\omega-1)}d\omega$$

for $0\le x\le 1$, where $\alpha(\omega_x)=x$, $0\le\omega_x\le\frac{1}{2}$. Easy calculation shows that

$$\omega_x=\frac{1}{2}-\frac{1}{2}\sqrt{\frac{\log 1/x}{1+\log 1/x}}.$$

Hence, assuming $n\ge A/\varepsilon$ and putting

$$\varepsilon_n=\frac{1}{2}-\frac{1}{2}\sqrt{\frac{\log(\log n)-\log(\log A/\varepsilon)}{1+\log(\log n)-\log(\log A/\varepsilon)}}$$

we obtain

$$P\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)>\varepsilon\right)\le 2K\int_0^{\varepsilon_n}\exp\frac{(2\omega-1)^2}{4\omega(\omega-1)}d\omega\le 2K\varepsilon_n.$$

3. There holds the following

Theorem 2. Let $f\in\mathrm{Lip}_\eta(A,\alpha)$, $0<\varepsilon<A$, *and let us suppose that* $\Lambda_\eta^l(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}})$ *is a random variable in* Ω *for* $n=1,2,\ldots$. *Then its expected value satisfies the inequality*

$$(*)\qquad E\left(\Lambda_\eta^l\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)\right)\le-\int_{1/n}^{1}x\,dF_\alpha\left(\frac{\log 1/x}{\log n}+0\right)$$

for $n=1,2,\ldots$. Moreover, the right-hand side of this inequality rends to zero as $n\to\infty$.

Proof. Since $f\in \mathrm{Lip}_\eta(A,\alpha)$, the tandom variable $\Lambda^l_\eta(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}})$ is bounded, and so has a finite expected value. Moreover,

$$E\left(\Lambda^l_\eta\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)\right)\le AE\left(\frac{1}{n^{\alpha(\omega)}}\right)=A\int_0^1 xdF_{\frac{1}{n^\alpha}}(x),$$

because $F_{\frac{1}{n^\alpha}}(x)=0$ for $x\le 0$ and $F_{\frac{1}{n^\alpha}}(x)=1$ for $x>1$. Moreover, $F_{\frac{1}{n^\alpha}}(x)=1-F_\alpha\left(\frac{\log 1/x}{\log n}+0\right)$ for $0<x\le 1$. If $x\le\frac{1}{n}$, then $\frac{\log 1/x}{\log n}+\delta>1$ for $\delta>0$. Hence $F_\alpha\left(\frac{\log 1/x}{\log n}+0\right)=1$ for $x\le\frac{1}{n}$. Consequently, we have

$$E\left(\frac{1}{n^{\alpha(\omega)}}\right)=-\int_{1/n}^1 xdF_\alpha\left(\frac{\log 1/x}{\log n}+0\right).$$

Since $\alpha(\omega)>0$ P-a.e. in Ω, so $\frac{1}{n^{\alpha(\omega)}}\to 0$ as $n\to\infty$ P-a.e. in Ω. Moreover, $\frac{1}{n^{\alpha(\omega)}}\le 1$ in Ω. By Lebesque dominated convergence theorem, there holds $E\left(\frac{1}{n^{\alpha(\omega)}}\right)\to 0$ as $n\to\infty$.

Example 2. We take $(\Omega,\boldsymbol{B},P)$ as in Example 1.

(a) Let $\alpha(\omega)=\alpha_0=\text{const.}$, where $0<\alpha_0\le 1$. Then $F_\alpha\left(\frac{\log 1/x}{\log n}+0\right)$ is equal to 1 for $x\le 1/n^{\alpha_0}$ and is equal to 0 for $x>1/n^{\alpha_0}$. Hence the right-hand side of inequality $(*)$ is equal to A/n^{α_0}.

(b) We take α as in Example 1 (b). Then

$$E\left(\frac{1}{n^{\alpha(\omega)}}\right)=\frac{K}{2n}+\frac{K}{2\log^2 n}\int_{1/n}^1 \log^2 x dx=$$

$$=\frac{K}{\log^2 n}\left(1-\frac{1}{n}\right)-\frac{K}{n\log n}\le\frac{K}{\log^2 n}.$$

Hence

$$E\left(\Lambda^l_\eta\left(f(\omega,\cdot),\frac{1}{n^{\alpha(\omega)}}\right)\right)\le\frac{AK}{\log^2 n}\quad\text{for}\quad n=1,2,\ldots$$

4. Among many applications of the Lipschitz condition, I should like to mention only that in the theory of approximation. A modular ρ on $L^0(G)$ is called *skrongly left* τ*-bounded*, if it is left τ-bounded with a function $h(t)=0$ for all $t \in G$. Let $\|\ \|_\rho$ be the norm generated in $L^0(G)$ by a convex modular ρ, i.e. $\|f\|_\rho = \inf\{u>0: \rho(u^{-1}f)\leq 1\}$ for $f \in L_\rho^0(G)$. The modular ρ is called *J-norm convex*, if $\|\int_G p(t) F(t,\cdot)dt\|_\rho \leq \int_G p(t)\|F(t,\cdot)\|_\rho\, dt$ for every integrable, nonnegetive function p on G with $\int_G p(t)dt=1$ and every function $F: G\times G \to R$, measurable in $G\times G$. If $G=\prod=\{z\in C: |z|=1\}$ in the space C of complex numbers with multiplication, or $G=R$ with addition, then $\rho(f)=\int_G |f(t)|^p\, dt$, $p\geq 1$, and $\rho(f)=\operatorname{supess}_{t\in G}|f(t)|$ are *J*-norm convex modulars. Let $E_n(f)_{\|\|_\rho}$ denote the error of best approximation in $L_\rho^0(\prod)$ of a function $f\in L_\rho^0(G)$ by trigonometric polynomials of degree $\leq n$ in the sense of the norm $\|\ \|_\rho$. It is kmown that if ρ is convex, strongly τ-bounded and *J*-norm convex in $L^0(\prod)$, then

$$E_{n-1}(f)_{\|\ \|_\rho} \leq B\Lambda_{\|\ \|_\rho}(f, \tfrac{1}{n})$$

for $m=1,2\ldots$ with some B depending on ρ (see [3], Theorem 1) – this is a generalization of the well known Jackson's direct approximation theorem. Supposing $f\in \operatorname{Lip}_{\|\|_\rho}(A,\alpha)$ with some $0<\alpha\leq 1$ we thus obtain $E_{n-1}(f)_{\|\ \|_\rho} \leq \frac{AB}{n^\alpha}$. Now, replacing the constant α by a random variable $\alpha: \Omega\to[0,1]$, the function $f: G\to \overline{R}$ by $F:\Omega\times G\to\overline{R}$; and applying Theorems 1 and 2, we obtain the following result:

Theorem 3. Let ρ be a convex, skrongly left τ-bounded and J-norm convex modular on $L^0(\prod)$ and $\eta(f)=\|f\|_\rho$ sakisty the assumptions of Theorems 1 and 2. Let $f\in \operatorname{Lip}_\eta(A,\alpha)$. Then

$$P(E_{n-1}(f(\omega,\cdot))_{\|\ \|_\rho} > \varepsilon) \leq F_\alpha\left(\frac{\log AB/\varepsilon}{\log n}\right)$$

for $0<\varepsilon<AB$, $n=1,2,\ldots$ and

$$P(E_{n-1}(f(\omega,\cdot))_{\|\ \|_\rho} > \varepsilon) \leq -AB\int_{1/n}^{1} x dF_\alpha\left(\frac{\log x}{\log n}+0\right)$$

for $n=1,2,\ldots.$

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