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EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Abstract. In this paper, we prove existence results for boundary value problems for second order differential equation with deviating arguments.

Key words: boundary value problems, Nonlinear Alternative, deviating arguments.

1. INTRODUCTION

In this paper we deal with the following differential equation with deviating arguments

(e) $$x''(t) = \mu f(t, x(t), x(\sigma(t)), x'(g(t))), \qquad \text{a.e. } t \in [0,1],$$

where $f:[0,1]\times(R^n)^4 \to R^n$ is a function satisfying Caratheodory's conditions, σ, g are continuous real valued functions defined on $[0,1]$ and μ is a real positive parameter.

We suppose that

$$-\infty < -r = \min_{t\in[0,1]}\{\sigma(t), g(t)\} < 0 \quad \text{and} \quad 1 < \max_{t\in[0,1]}\{\sigma(t), g(t)\} = d < +\infty$$

and the set $\{t\in[0,1] : g(t)=0 \text{ or } g(t)=1\}$ is finite. We also consider the following boundary conditions

(bc) $$\begin{cases} x(t) = \phi_1(t), & t\in[-r,0], \\ x(t) = \phi_2(t), & t\in[1,d], \end{cases}$$

where ϕ_1, ϕ_2 are continuously differentiable on $[-r,0]$ and $[1,d]$ respectively, R^n-valued functions.

The purpose of the present paper is to study the existence of solutions of boundary value problem (BVP) (e)-(bc). As usually, in BVP for differential equations with deviating arguments we search for solutions which satisfy the differential equation on a complact interval I and identify with an a priori given functions outside of I. For BVP concerning second order functional differential equations we refer to [5], [6], [7] and the references cited therein.

Here we will give an existence result for solutions to the BVP (e)-(bc), by using a "forbidden value" technique, introduced in [1]. (see also [2]). By this method, we do not get an a priori bound on all relevant solutions of the norm of solutions family of problems. Instead, we produce a "forbidden value" of the norm of solutions i.e a value that the norm never takes on. This suffices for the Nonlinear Alternative ([3], [4]), which requires only the existence of a suitable open set with no solutions on its boundary. In our approach we require that the supremum of a certain quantity is sufficiently large. This facet implies the existence of a "forbidden value" of the norm of the solutions, i.e. that there exists a suitable open set with no solutions on its boundary.

2. PRELIMINARIES

It is well known that the BVP (e)-(bc) can be transformed into an integral equation and consequently the problem consists of finding a fixed point for the mapping defined by this integral equation.

This mapping will include an integral whose kernel, $G(t,s)$ is the Green's function for homogeneous BVP

$$x''=0$$
$$x(0)=0, \quad x(1)=0,$$

which is explicity given by

$$G(t,s)=\begin{cases}(t-1)s & \text{if} \quad 0\le s\le t\le 1\\ (s-1)t & \text{if} \quad 0\le t\le s\le 1\end{cases}$$

We denote also by γ the solution of

$$x''=0$$
$$x(0)=\phi_1(0), \quad x(1)=\phi_2(1),$$

i.e.

$$\gamma(t)=(\phi_2(1)-\phi_1(0))t+\phi_1(0), \quad t\in[0,1].$$

If I is a compact interval of the real line R we will denote by $C(I,R^n)$ and $C^1(I,R^n)$ the usual spaces of continuous and continously differentiable, respectively, on I, R^n – valued functions. For $x\in C(I,R^n)$, we use the norm

$$\|x\|_I=\sup\{|x(t)|: t\in I\}.$$

Also, we denote by $L^1([0,1],R)$ the spase of real functions u such that $|u|$ is Lebesgue integrable on $[0,1]$, with the usual norm $\|u\|_1$.

EXISTENCE RESULTS

We shall need the following notations later:

$$K_0 = \max_{t,s\in[0,1]} |G(t,s)|, \qquad K_1 = \max_{t,s\in[0,1]} \left|\frac{\partial}{\partial t} G(t,s)\right|, \qquad K = \max\{K_0, K_1\}$$

$$\Theta_0 = \max_{t\in[0,1]} |\gamma(t)|, \quad \Theta_1 = \max_{t\in[0,1]} |\gamma(t)|, \quad \Theta = \max\{\Theta_0, \Theta_1\}$$

and

$$\Phi_0 = \max\{\|\phi_1\|_{[-r,0]}, \|\phi_2\|_{[1,d]}\}, \ \Phi_1 = \max\{\|\phi_1'\|_{[-r,0]}, \|\phi_2'\|_{[1,d]}\}, \ \Phi = \max\{\Phi_0, \Phi_1\}$$

3. MAIN RESULT

Now we can prove the following existence result:

Theorem. Let $f:[0,1]\times(R^n)^4 \to R^n$, *be a function satisfying Caratheodory's conditions,* $\phi_1 \in C^1([-r,0],R^n)$ *and* $\phi_2 \in C^1([1,d],R^n)$. *Supppose that:*

(Hf): *There exist nondecreasing functions* $\Omega_i:[0,\infty)\to[0,\infty)$ *and functions* $m_i \in L^1([0,1],R)$, $i=1,2,3,4$ *such that*

$$|f(t,u_1,u_2,u_3,u_4)| \le \sum_{i=1}^{4} m_i(t)\Omega_i(u_i)$$

for every $(t,u_1,u_2,u_3,u_4)) \in [0,1]\times R^4$.

Moreover, we suppose that there exists $\mu_0 > 0$ *such that*

$$\sup_{z\in(0,\infty)} \frac{z}{\Phi + \Theta + K\mu_0 \sum_{i=1}^{4} \|m_i\|_1 \Omega_i(z)} > 1. \tag{3.1}$$

Then for every $\mu \in (0,\mu_0]$ *the BVP* (e)-(bc) *has at least one solution.*

Proof. Consider the one parameter family of problems

$$(e_\lambda) \qquad x''(t) = \lambda\mu f(t,x(t),x(\sigma(t)),x'(t),x'(g(t))), \quad \text{a.e. } t\in[0,1]$$

$$(bc_\lambda) \qquad \begin{array}{ll} x(t) = \lambda\phi_1(t), & t\in[-r,0] \\ x(t) = \lambda\phi_2(t), & t\in[1,d], \end{array}$$

where $\lambda \in (0,1)$. Then, obviously, we have:

$$x(t) = \begin{cases} \lambda\phi_1(t), & \text{if } t\in[-r,0], \\ \lambda\gamma(t) + \lambda\int_0^1 G(t,s)\mu f(s,x(s),x(\sigma(s)),x'(s),x'(g(s)))ds, & \text{if } t\in[0,1], \\ \lambda\phi_2(t), & \text{if } t\in[1,d]. \end{cases}$$

Therefore for every $t\in[0,1]$ we have

$$|x(t)|\le\Theta_0+\int_0^1|G(t,s)||\mu f(s,x(s),x(\sigma(s)),x'(s),x'(g(s)))|ds\le$$

$$\le\Theta_0+K_0\mu\int_0^1(|m_1(s)|\Omega_1(|x(s)|))+|m_2(s)|\Omega_2(|x(\sigma(s))|)+|m_3(s)|\Omega_3(|x'(s)|)+$$

$$+|m_4(s)|\Omega_4(|x(g(s))|)ds\le$$

$$\le\Theta_0+K_0\mu(\|m_1\|_1\,\Omega_1(\|x\|_{[0,1]})+\|m_2\|_1\,\Omega_2(\|x\|_{[-r,d]})+\|m_3\|_1\,\Omega_3(\|x'\|_{[0,1]})+$$

$$+\|m_4\|_1\,\Omega_4(\max\{\Phi_1,\|x'\|_{[-r,d]}\})).$$

Now we set

$$\|x\|^*=\max\{\|x\|_{[-r,d]},\|x'\|_{[-r,0]},\|x'\|_{[0,1]},\|x'\|_{[1,d]}\}.$$

Then, since Ω_i, $i=1,\dots,4$ are nondecreasing, for every $t\in[0,1]$ we have

$$|x(t)|\le\Theta_0+K_0\mu(\|m_1\|_1\,\Omega_1(\|x\|^*)+\|m_2\|_1\,\Omega_2(\|x\|^*)+\|m_3\|_1\,\Omega_3(\|x\|^*)+$$

$$+\|m_4\|_1\,\Omega_4(\|x\|^*))$$

and hence

$$\|x\|_{[0,1]}\le\Theta_0+K_0\mu\sum_{i=l}^{4}\|m_i\|_1\,\Omega_i(\|x\|^*).$$

Since for every $t\in[-r,0]\cup[1,d]$ we have

$$|x(t)|\le\Phi_0,$$

we have finally

$$\|x\|_{[-r,d]}\le\Phi_0+\Theta_0+K_0\mu\sum_{i=l}^{4}\|m_i\|_1\,\Omega_i(\|x\|^*).\tag{3.2}$$

Also, for every $t\in[0,1]$ we have

$$|x'(t)|\le\Theta_1+\lambda\int_0^T\left|\frac{\partial}{\partial t}G(t,s)\right||\mu f(s,x(s),x(\sigma(s)),x'(s),x'(g(s)))|ds\le$$

$$\le\Theta_1+K_1\mu\int_0^1(|m_1(s)|\Omega_1(|x(s)|)+|m_2(s)|\Omega_2(|x(\sigma(s))|)+$$

$$+|m_3(s)|\Omega_3(|x'(s)|)+|m_4(s)|\Omega_4(|x'(g(s))|))ds\le$$

$$\le\Theta_1+K_1\mu(\|m_1\|_1\,\Omega_1(\|x\|_{[0,1]})+\|m_2\|_1\,\Omega_2(\|x\|_{[-r,d]})+\|m_3\|_1\,\Omega_3(\|x'\|_{[0,1]})+$$

$$+\|m_4\|_1\,\Omega_4(\max\{\Phi_1,\|x'\|_{[-r,d]}\})).$$

Hence

$$\|x'\|_{[0,1]}\le\Theta_1+K_1\mu\sum_{i=l}^{4}\|m_i\|_1\,\Omega_i(\|x\|^*).$$

Since for every $t \in [-r,0] \cup [1,d]$ we have

$$|x'(t)| \leq \Phi_1,$$

we have finally

$$\max\{\|x'\|_{[-r,0]}, \|x'\|_{[0,1]}, \|x'\|_{[1,d]}\} \leq \Phi_1 + \Theta_1 + K_1\mu \sum_{i=l}^{4} \|m_i\|_1 \Omega_i(\|x\|^*). \tag{3.3}$$

Thus by (3.2), (3.3) we obtain

$$\|x\|^* \leq \Phi + \Theta + K\mu \sum_{i=l}^{4} \|m_i\|_1 \Omega_i(\|x\|^*),$$

or

$$\frac{\|x\|^*}{\Phi + \Theta + K\mu \sum_{i=l}^{4} \|m_i\|_1 \Omega_i(\|x\|^*)} \leq 1. \tag{3.4}$$

Now fix μ, $0 \leq \mu \leq \mu_0$. Then by assumption on μ_0 there exists $M > 0$ such that

$$\frac{M}{\Phi + \Theta + K\mu \sum_{i=l}^{4} \|m_i\|_1 \Omega_i(M)} > 1. \tag{3.5}$$

Now we consider the space $D = C([-r,d], R^n) \cap C^1([-r,0], R^n) \cap C^1([0,1], R^n)$ $\cap C^1([1,d], R^n)$ endowed with the norm $\|x\|^* = \max\{\|x\|_{[-r,d]}, \|x'\|_{[-r,0]}, \|x'\|_{[0,1]},$ $\|x'\|_{[1,d]}\}$, which is a Banach space, and define the operator $N: D \to D$, by

$$Nx(t) = \begin{cases} \lambda\phi_1(t), & \text{if} \quad t \in [-r,0], \\ \lambda\gamma(t) + \lambda \int_0^1 G(t,s)\mu f(s, x(s), x(\sigma(s)), x'(t), x'(g(s)))ds, & \text{if} \quad t \in [0,1], \\ \lambda\phi_2(t), & \text{if} \quad t \in [1,d]. \end{cases}$$

We shall prove that the operator N is comletely continuous. Since f satisfies Caratheodory's conditions, N is a continuous operator. Following standard arguments we can prove that the operator $N: D \to D$, is completely continuous operator.

Moreover, it is clear that the fixed points of λN are precisely the solutions of $(e)_\lambda - (bc)_\lambda$.

Let

$$U = \{u \in D: \|u\|^* < M\}.$$

an open convex of D. Should there exist $\lambda \in (0,1)$ and $x \in \partial U$ with $x = \lambda N x$, then x is a solution of $(e)_\lambda - (bc)_\lambda$ satisfying $\|x\|^* = M$ and hence from (3.4) we have that

$$\frac{M}{\Phi + \Theta + K\mu \sum_{i=l}^{4} \|m_i\|_1 \Omega_i(M)} \leq 1,$$

which contradicts to the choice of M in (3.5). Since $0 \in U$ the Nonlinear Alternative guarantees that N has a fixed point. ■

Remark 1. It is obvious that

$$K \leq 1 \text{ and } \Theta \leq \hat{\Theta} = |\phi_2(1) - \phi_1(0)| + |\phi_1(0)|.$$

Therefore the assumption on μ_0 in Theorem can replaced by the following simpler but stronger contition:

There exists $\mu_0 < 0$ such that

$$\sup_{z \in (0,\infty)} \frac{z}{\Phi + \hat{\Theta} + \mu_0 \sum_{i=l}^{4} \| m_i \|_1 \Omega_i(z)} > 1. \tag{3.6}$$

Remark 2. It is obvious that the above result can be easily proved for the BVP (E)-(bc) where (E) stands for the equation:

$$\text{(E)} \quad x''(t) = \mu f(t, x(t), x(\sigma_1(t)), \dots, x(\sigma_k(t)), x'(t), x'(g_1(t)), \dots, x'(g_m(t))), \qquad a.e. \ t \in [0,1],$$

where σ_i, g_i, $i = 1, \dots, k$, $j = 1, \dots, m$ are continuous real valued functions defined on $[0,1]$.

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