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COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE*

ABSTRACT. We are interested in comparing the oscillatory and asymptotic properties of equations $L_n[x(t)-p(t)x(g(t))]+\delta q(t)f(x(h(t)))=0$ with those of the equations $L_n u(t)+\delta Q(t)f(u(r(t)))=0$.

KEY WORDS. Neutral differential equations, oscillatory (nonoscillatory) solution, property *A*, property *B*, quasi-derivatives.

1. INTRODUCTION

We consider neutral functional differential equations of the form

$$(\text{NE},\delta) \qquad L_n[x(t)-p(t)x(g(t))]+\delta q(t)f(x(h(t)))=0,$$

where $n\geq 2$, $\delta=+1$ or -1 and L_n is disconjugate differential operator defined recursively by

$$L_0x(t)=x(t), \qquad L_kx(t)=\frac{1}{a_k(t)}[L_{k-1}x(t)]', \qquad k=1,2,\dots,n, \ a_n=1.$$

We are interested in comparing the oscillatory and asymptotic properties of equations (NE,δ) with those of the equations

$$(\text{E},\delta) \qquad L_nu(t)+\delta Q(t)f(u(r(t)))=0.$$

We always assume that:

(*a*) $a_i\in C[[t_0,\infty),(0,\infty)]$, $t_0\geq 0$ and $\int^\infty a_i(t)dt=\infty$, $i=1,2,\dots,n-1$;

(*b*) $p\in C[[t_0,\infty),\boldsymbol{R}]$;

(*c*) $Q,q\in C[[t_0,\infty),(0,\infty)]$;

(*d*) $g,h,r\in C[[t_0,\infty),(0,\infty)]$ and g is increasing, $\lim_{t\to\infty} g(t)=\infty$, $\lim_{t\to\infty} h(t)=\infty$, $\lim_{t\to\infty} r(t)=\infty$;

(*e*) $f\in C[\boldsymbol{R},\boldsymbol{R}]$ is nondecreasing, $xf(x)>0$, for $x\neq 0$ and $xyf(xy)\geq Kxyf(x)f(y)$ $(0<K=\text{const.})$.

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By a solution of (NE,δ) we here mean a continuous function $x(t):[T_x,\infty)\to R$, $T_x\ge t_0$, such that $x(t)-p(t)\,x(g(t))$ has the continuous quasi – derivatives $L_i\,[x(t)-p(t)\,x(g(t))]$, $0\le i\le n$, and $x(t)$ satisfies (NE,δ) for all sufficiently large $t\ge T_x$. Our attention is restricted to those solutions $x(t)$ of (NE,δ) which satisfy

$$\sup\{|x(t)|:\ t\ge T\}>0,\qquad \text{for any}\quad T\ge T_x.$$

Such a solution is said to be a proper solution. We make the standing hypothesis that (NE,δ) possesses proper solutions. A proper solution of (NE,δ) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

For other related comparison results regarding the oscillatory and asymptotic behavior of differential equations with or without functional arguments the reader is referred to the paper [1-5, 8].

2. Classification of nonoscillatory solutions

We classify the possible nonoscillatory solutions of (NE,δ) and (E,δ) in a similar way as in the papers [1], [6] and [7].

Sometimes we will require the following conditions to be satisfied:

(p_1) $|p(t)|\le\lambda$ on $[t_0,\infty)$ for some constant $\lambda<1$;

(p_2) $p(t)>0$ on $[t_0,\infty)$;

(g_1) $g(t)<t$ on $[t_0,\infty)$.

Let $x(t)$ be a nonoscillatory solution of (NE,δ). From (NE,δ) and (e) it follows that the function

$$y(t)=x(t)-p(t)x(g(t)) \tag{1}$$

has to be eventually of constant sign, so that either

$$x(t)y(t)>0 \tag{2}$$

or

$$x(t)y(t)<0 \tag{3}$$

for all sufficiently large t. Assume first that (2) holds. Then the function $y(t)$ satisfies $\delta y(t)L_n y(t)<0$ eventually and from the well-known Kiguradze's lemma it follows that there is an integer $l\in\{0,1,\dots,n\}$ and a $t_1\ge t_0$ such that $(-1)^{n-l-1}\delta=1$ and for every $t\ge t_1$

$$\begin{aligned} &y(t)L_i y(t)>0, && 0\le i\le l,\\ (4)_l\qquad &(-1)^{i-l}y(t)L_i y(t)>0, && l\le i\le n \end{aligned}$$

holds.

A function $y(t)$ satisfying $(4)_l$ is said to be a nonoscillatory function of degree l. The set of all solutions $x(t)$ of (NE,δ) satisfying (2) and $(4)_l$ will be denote by N_l^+. Now assume that (3) holds. Then $y(t)$ satisfies $(-\delta)y(t)L_ny(t)<0$ for all large t and so it is a function of degree l for some $l\in\{0,1,\dots,n\}$ with $(-1)^{n-l}\delta=1$. The totality of nonoscillatory solutions $x(t)$ of (NE,δ) which satisfy (3) and $(4)_l$ will be denoted by N_l^-. The set of all possible nonoscillatory solutions of (NE,δ) we denote by N.

Assume that (p_1) and (g_1) hold. Then we have (see [6]):

$$(5_1)\qquad \begin{aligned} &N=N_1^+\cup N_3^+\cup\ldots\cup N_{n-1}^+\cup N_0^- && \text{for } \delta=1 \text{ and } n \text{ even},\\ &N=N_0^+\cup N_2^+\cup\ldots\cup N_{n-1}^+ && \text{for } \delta=1 \text{ and } n \text{ odd},\\ &N=N_0^+\cup N_2^+\cup\ldots\cup N_n^+ && \text{for } \delta=-1 \text{ and } n \text{ even},\\ &N=N_1^+\cup N_3^+\cup\ldots\cup N_n^+\cup N_0^- && \text{for } \delta=-1 \text{ and } n \text{ odd}.\end{aligned}$$

The class N_0^- must be removed from (5_1) in the case if $p(t)$ is either oscillatory or eventually negative, because in this case equation (NE,δ) cannot possess a nonoscillatory solution $x(t)$ satisfying (3).

Definition 1. Suppose that (p_1) and (g_1) hold.

(i) *Equation* $(\mathrm{NE},+1)$ *is said to have property A if for n even $N=N_0^-$ and for n odd $N=N_0^+$.*

(ii) *Equation* $(\mathrm{NE},-1)$ *is said to have property B if for n even $N=N_0^+\cup N_n^+$ and for n odd $N=N_n^+\cup N_0^-$.*

Assume that (p_2) holds. Then we have (see [7]).

$$(5_2)\qquad \begin{aligned} &N=N_1^+\cup N_3^+\cup\ldots\cup N_{n-1}^+\cup N_0^-\cup N_2^-\cup\ldots\cup N_n^- && \text{for } \delta=1 \text{ and } n \text{ even},\\ &N=N_0^+\cup N_2^+\cup\ldots\cup N_{n-1}^+\cup N_1^-\cup N_3^-\cup\ldots\cup N_n^- && \text{for } \delta=1 \text{ and } n \text{ odd},\\ &N=N_0^+\cup N_2^+\cup\ldots\cup N_n^+\cup N_1^-\cup N_3^-\cup\ldots\cup N_{n-1}^- && \text{for } \delta=-1 \text{ and } n \text{ even},\\ &N=N_1^+\cup N_3^+\cup\ldots\cup N_n^+\cup N_0^-\cup N_2^-\cup\ldots\cup N_{n-1}^- && \text{for } \delta=-1 \text{ and } n \text{ odd}.\end{aligned}$$

Definition 2. Suppose that (p_2) holds.

(i) *Equation* $(\mathrm{NE},+1)$ *is said to have property A if for n even $N=N_0^-\cup N_n^-$ and for n odd $N=N_0^+\cup N_n^-$.*

(ii) *Equation* (NE, −1) *is said to have property **B** if for n even* $N = N_0^+ \cup N_n^+$ *and for n odd* $N = N_n^+ \cup N_0^-$.

The set of all nonoscillatory solutions of degree l of (E,δ), is denote by N_l. If we denote by N the set of all nonoscillatory solutions of (E,δ), then (see[1]).

$$N = N_1 \cup N_3 \cup \ldots \cup N_{n-1} \quad \text{for } \delta = 1 \text{ and } n \text{ even},$$
$$N = N_0 \cup N_2 \cup \ldots \cup N_{n-1} \quad \text{for } \delta = 1 \text{ and } n \text{ odd},$$
$$N = N_0 \cup N_2 \cup \ldots \cup N_n \quad \text{for } \delta = -1 \text{ and } n \text{ even},$$
$$N = N_1 \cup N_3 \cup \ldots \cup N_n \quad \text{for } \delta = -1 \text{ and } n \text{ odd}.$$

Definition 3. (i) *Equation* (E, +1) *is said to have property A if for n even is oscillatory* (*i.e.* $N = \phi$) *and for n odd* $N = N_0$.

(ii) *Equation* (E, −1) *is said to have property B if for n even* $N = N_0 \cup N_n$ *and for n odd* $N = N_n$.

3. Comparison Theorems

Theorem 1. Suppose that (p_1) *and* (g_1) *hold. Moreover, assume that* $p(t)$ *is eventually negative or that* $p(t)$ *is oscillatory and satisfies*

$$p(t)\,p(g(t)) \geq 0 \quad \textit{for all large } t. \tag{6}$$

(i) *The equation* (NE, +1) *has property **A** if equation*

$$L_n u(t) + M\,q(t)\,f(u(h(t))) = 0 \quad (\textit{where } 0 < M = K f(1-\lambda)) \tag{7}$$

*has property **A**.*

(ii) *The equation* (NE, −1) *has property **B** if equation*

$$L_n u(t) - M\,q(t)\,f(u(h(t))) = 0 \tag{8}$$

*has property **B**.*

Proof. (I) Let n be even.

(i) According to (5_1), N_l^+, $l \in \{1,3,\ldots,n-1\}$ are the possible classes of nonoscillatory solutions of (NE, +1) with even n. Suppose that $N_l^+ \neq \phi$ for some $l \in \{1,3,\ldots,n-1\}$. Then for a solution $x(t)$ of (NE, +1) such that $x(t)y(t) > 0$ for large t in view of (1) we get

$$x(t) = y(t) + p(t)y(g(t)) + p(t)p(g(t))x(g(g(t))),$$

which schows that if (6) and (g_1) hold, then

(9) $$|x(t)| \geq (1-\lambda)\,|y(t)|, \quad \text{for all large } t.$$

From (9), (NE,+1) and (e) we see that $y(t)$ satisfies

(10) $$\{L_n y(t) + Mq(t) f(y(h(t)))\}\operatorname{sgn} y(t) \leq 0 \quad \text{for all large } t.$$

On the other hand, Kusano and Naito (5) showed, that (10) has a solution of degree l $(1 \leq l \leq n-1)$ if and only if equation (7) has a solution of degree l. We supposed that $1 \leq l \leq n-1$ and this contradicts the hypothesis that equation (7) has property A.

(ii) According to (5_1), N_l^+, $l \in \{0,2,...,n\}$ are the possible classes of nonoscillatory solutions of (NE,−1) with even n. Suppose that $N_l^+ \neq \phi$ for some $l \in \{2,4,...,n-2\}$. Then (9) holds and $y(t)$ satisfies

(11) $$\{L_n y(t) - Mq(t) f(y(h(t)))\}\operatorname{sgn} y(t) \geq 0 \quad \text{for all large } t.$$

Kusano and Naito [5] showed that (11) has a solution of degree l $(1 \leq l \leq n)$ if and only if equation (8) has a solution of degree l. This a contradiction.

(II) If n is odd the proof is similar to proof of case (I).

The proof is complete. ■

Example 1. We consider the equation

(12) $$\left[t^{\frac{1}{2}}\left(t^{\frac{1}{3}}[x(t)-p(t)x(g(t))]'\right)'\right]' + \frac{a}{(1-\lambda)t^{13/6}}\,x\left(\frac{t}{3}\right) = 0, \qquad t \geq 1,$$

where suppose that (p_1) and (g_1) hold. Moreover, assume that $p(t)$ is eventually negative or that $p(t)$ is oscillatory and satisfies (6).

Applying Theorem 1, case (i) ($f(x) = x$ and $K = 1$) we conclude that equation (12) has property A if equation

(13) $$\left[t^{1/2}\left(t^{1/3}u'(t)\right)'\right]' + \frac{a}{t^{13/6}}\,u\left(\frac{t}{3}\right) = 0, \qquad t \geq 1,$$

has property A.

Next, we observe that by [2, Example] equation (13) has property A if $a > \frac{49.3^{7/6}}{432\sqrt{3}}$.

Consequently, equation (12) has property A if $a > \frac{49.3^{7/6}}{432\sqrt{3}}$.

Theorem 2. Suppose that (p_1) *and* (g_1) *hold. Moreover, assume that* $p(t)$ *is eventually positive.*

(i) *The equation* (NE,+1) *has property A if equation*

(14) $$L_n u(t) + q(t) f(u(h(t))) = 0$$

has property A.

(ii) *The equation* (NE, −1) *has property **B** if equation*

(15) $$L_n u(t) - q(t) f(u(h(t))) = 0$$

*has property **B**.*

Proof. (I) Let n be even.

(i) According to (5_1), N_l^+, $l \in \{1,3,\dots,n-1\}$ and N_0^- are the possible classes of nonoscillatory solutions of (NE,+1) with even n. Suppose that $N_l^+ \neq \phi$ for some $l \in \{1,3,\dots,n-1\}$. In this case $x(t)y(t) > 0$ and

(16) $$|y(t)| \leq |x(t)| \qquad \text{for all large } t.$$

We see from (NE,+1), (16) and (*e*) that $y(t)$ is a nonoscillatory solution of differential inequality

(17) $$\{L_n \quad y(t) + q(t) f(y(h(t)))\} \quad \operatorname{sgn} y(t) \leq 0 \quad \text{for all large } t.$$

Further we proceed in the same way as in the proof of Theorem 1. ■

Example 2. We consider the equation

(18) $$[x(t) - p(t)x(g(t))]''' - \frac{a}{t^2\sqrt{t}} x(\sqrt{t}) = 0, \qquad t \geq 1, \quad a > 0,$$

where suppose that (p_1) and (g_1) hold. Moreover, assume that $p(t)$ is eventually positive.

By Theorem 2, case (ii) equation (18) has property ***B*** if equation

(19) $$u'''(t) - \frac{a}{t^2\sqrt{t}} u(\sqrt{t}) = 0, \qquad t \geq 1, \quad a > 0,$$

has property ***B***.

Next, by [3, Example 1] equation (19) has property ***B*** for any $a > 0$.

Consequently, equation (18) has property ***B***.

Theorem 3. Suppose that (p_2) *holds.*

(i) *The equation* (NE,+1) *has property **A** if equation* (14) *has property **A** and equation*

(20) $$L_n u(t) - Sq(t) f([P(g^{-1}(h(t)))]^{-1}) f(u(g^{-1}(h(t)))) = 0$$

*has property **B***

($0 < -f(-1)$. $K^2 = S$ (=const.) *and* $g^{-1}(t)$ *denotes the inverse function of* $g(t)$).

(ii) *The equation* (NE, -1) *has property* ***B*** *if equation* (15) *has property* ***B*** *and equation*

(21) $$L_n u(t) + Sq(t) f([P(g^{-1}(h(t)))]^{-1}) f(u(g^{-1}(h(t)))) = 0$$

has property ***A***.

Proof. (I) Let n be even.

(i) According to (5_2), N_l^+, $l \in \{1,3,...,n-1\}$ and N_l^-, $l \in \{0,2,...,n\}$ are the possible classes of nonoscillatory solutions of (NE, $+1$) with even n. Suppose that $N_l^+ \neq \phi$ for some $l \in \{1,3,...,n-1\}$. In this case $x(t)y(t) > 0$ and (16) holds. We see from (NE, $+1$), (16) and (e) that $y(t)$ is a nonoscillatory solution of (17). This a contradiction.

Suppose that $N_l^- \neq \phi$ for some $l \in \{2,4,...,n-2\}$. In this case $x(t)y(t) < 0$ and (see [7, Lemma 1])

(22) $$|x(t)| \geq |y(g^{-1}(t))| P(g^{-1}(t))|^{-1}, \qquad \text{for all large } t.$$

The function $y(t)$ satisfies the ineguality

(23) $$\{L_n y(t) - Sq(t) f([P(g^{-1}(h(t)))]^{-1}) f(y(g^{-1}(h(t))))\} \operatorname{sgn} y(t) \geq 0$$

for all large t. Kusano and Naito [5] showed that (23) has a solution of degree l $(1 \leq l \leq n-1)$ if and only if equation (20) has a solution of degree l. This a contradiction. The proofs of case (II) (n is odd) and case (ii) are similar to proof of Theorem 3 case (I) and will be omitted. ■

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