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OSCILLATIONS OF SOME DIFFERENCE EQUATIONS

ABSTRACT. Oscillation criteria for nonlinear difference equation of the form

$$\Delta(r_n\Delta(u_n + p_n u_{n-k})) + q_n f(n_{n-l}) = 0,\ n = 0,1,2,...,$$

are established. These criteria extend those in [13-15] and [18].

KEY WORDS: oscillations, nonoscillatory solution, difference equations.

1. INTRODUCTION

In this note we consider the nonlinear difference equation of the form

$$\Delta(r_n \Delta(u_n + p_n u_{n-k})) + q_n f(n_{n-l}) = 0,\quad n = 0,1,2,..., \tag{1}$$

where Δ denotes the forward difference operator: $\Delta v_n = v_{n+1} - v_n$ for any sequence (v_n) of real numbers, k and l are nonnegative integers, (p_n) and (q_n) are sequences of real numbers, (r_n) is a sequence of positive numbers and

$$\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty. \tag{2}$$

The function f is a real valued function satisfying $uf(u) > 0$ for $u \neq 0$. By a solution of (1) we mean a sequence (u_n) defined for $n \geq -\max\{k,l\}$, which satisfies (1) for all large n. A nontrivial solution (u_n) of (1) is said to be oscillatory if for every positive integer N there exists $n \geq N$ such that $u_n u_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

Recently, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions of „delay" and „neutral delay" type difference equations; see for example [2, 3, 6, 8-20] and Chapter 7 in the recent book by Györi and Ladas [4]. For the general theory of difference equations one can refer to [1, 5, 7].

Our purpose in this paper is to give sufficient conditions for the oscillation of solutions of (1). These criteria extend some results contained in [13-15] and [18].

2. MAIN RESULTS

The following theorem provides sufficient conditions for the oscillation of all solutions of (1).

Theorem 1. Assume that $0 \le p_n \le 1, q_n \ge 0$ *for* $n \ge n_0$ *and*

(3) $$\frac{f(u)}{u} \ge \gamma > 0 \qquad for \quad u \ne 0.$$

If there exists a sequence (h_n) *such that* $h_n > 0$ *for* $n \ge n_0$ *and*

(4) $$\sum_{n=n_0+l}^{\infty} \left[\gamma h_n q_n (1 - p_{n-l}) - \frac{r_{n-l}(\Delta h_n)^2}{4h_n} \right] = \infty,$$

then every solution of (1) *is oscillatory.*

Proof. Assume for the sake of contradiction that (1) has a nonoscillatory solution (u_n), which we may assume (and we do) to be eventually positive. Set

(5) $$z_n = u_n + p_n u_{n-k}.$$

By assumptions, there exists $n_1 \ge n_0$ such that $z_n > 0$ for $n \ge n_1$ and from (1) it follows that $\Delta(r_n \Delta z_n) \le 0$ for $n \ge n_1$. Therefore $(r_n \Delta z_n)$ is a nonincreasing sequence. We claim that

(6) $$\Delta z_n > 0 \qquad \text{for} \qquad n \ge n_1.$$

In fact, if there is an $n_2 \ge n_1$ such that $\Delta z_{n_2} \le 0$, then there is an $n_3 \ge n_2$ that $r_{n_3} \Delta z_{n_3} = c < 0$ or $r_n \Delta z_n = 0$ for all $n \ge n_2$. In the first case we have $r_n \Delta z_n \le c$ for $n \ge n_3$ and so, bo (2), we get

$$z_n \le z_{n_3} + c \sum_{i=n_3}^{n-1} \frac{1}{r_i} \to -\infty \quad \text{as} \quad n \to \infty,$$

which contradicts the fact that $z_n > 0$ for $n \ge n_1$.

In the second case, from (1) we would have $q_n \equiv 0$ eventually, which is impossible in view of (4).

Next, observe that from (1) and (5), by (3), we have

(7) $$\Delta(r_n \Delta z_n) + \gamma q_n (z_{n-l} - p_{n-l} u_{n-l-k}) \le 0.$$

Since $z_n \ge u_n$, (6) yields

$$\Delta(r_n \Delta z_n) + \gamma q_n (z_{n-l} - p_{n-l} z_{n-l-k}) \le 0,$$

which in view of (6) we get eventually, say for $n \ge n_4$

(8) $$\Delta(r_n \Delta z_n) + \gamma q_n (1 - p_{n-l}) z_{n-l} \le 0.$$

Define

$$w_n = h_n \frac{r_n \Delta z_n}{z_{n-l}} \qquad \text{for} \qquad n \ge n_4.$$

Therefore

$$\Delta w_n = \frac{h_n \Delta(r_n \Delta z_n)}{z_{n-l}} + \frac{r_{n+l} \Delta z_{n+l} \Delta h_n}{z_{n+1-l}} - \frac{r_{n+l} \Delta z_{n+l} h_n (z_{n+1-l} - z_{n-l})}{z_{n-l} z_{n+1-l}}.$$

By (8) and (6), we get

$$\Delta w_n \le -\gamma h_n q_n (1-p_{n-l}) + \frac{r_{n+1}\Delta z_{n+1}\Delta h_n}{z_{n+1-l}} - \frac{r_{n+1}\Delta z_{n+1} h_n \Delta z_{n-l}}{z_{n+1-l}^2}.$$

Using the fact that the sequence $(r_n \Delta z_n)$ is nonincreasing we have

$$\Delta w_n \le -\gamma h_n q_n (1-p_{n-l}) + \frac{r_{n+1}\Delta z_{n+1}\Delta h_n}{z_{n+1-l}} - \frac{r_{n+1}^2 (\Delta z_{n+1})^2 h_n}{r_{n-l}\, z_{n+1-l}^2}$$

and so

$$\Delta w_n \le -\gamma h_n q_n (1-p_{n-l}) + \frac{r_{n-l}(\Delta h_n)^2}{4h_n} -$$

$$-\left[r_{n+1}\left(\frac{h_n}{r_{n-l}}\right)^{1/2} \frac{\Delta z_{n+1}}{z_{n+1-l}} - \frac{\Delta h_n}{2}\left(\frac{r_{n-l}}{h_n}\right)^{1/2}\right]^2 \le$$

$$\le -\gamma h_n q_n (1-p_{n-l}) + \frac{r_{n-l}(\Delta h_n)^2}{4h_n}, \qquad n \ge n_5 = n_4 + l.$$

Summing both sides of the above inequality from n_5 to n we obtain

$$w_{n+1} \le w_{n5} - \sum_{i=n5}^{n}\left[\gamma h_i q_i (1-p_{i-l}) - \frac{r_{i-l}(\Delta h_n)^2}{4h_n}\right],$$

which in view of (4) leads to a contradiction as $n \to \infty$. Thus the proof is complete.

Remark 1. If $p_n \equiv 0$ and $f(u) = u$, then Theorem 1 and Theorem 2 of [14] are the same, and when $r_n \equiv 1$, $h_n \equiv 1$ and $f(u) = u$, the Theorem 1 gives Corollary contained in [15].

Theorem 2. Let $-1 \le p_n \le 0$, $q_n \ge 0$ for $n \ge n_0$ and f is a nondecreasing continuous function such that

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \qquad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty, \qquad \varepsilon > 0.$$

If

(9)
$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{i=n+l+1}^{\infty} q_i = \infty,$$

then every unbounded solution of (1) *is oscillatory.*

Proof. Suppose that (1) has an unbounded nonoscillatory solution (u_n) and let it is eventually positive. Then for (z_n) defined in (5) we see as before that $\Delta(r_n\Delta z_n)\le 0$ eventually, that is $(r_n\Delta z_n)$ is nonincreasing. This implies that (z_n) is eventually monotonic. Now, if (z_n) is eventually nonpositive, then, by assumption, we have $u_n \le -p_n u_{n-k} \le u_{n-k}$, which contradicts the assumption that (u_n) is unbounded. Therefore $z_n > 0$ eventually, say for $n \ge n_1 \ge n_0$. Thus as in the proof of Theorem 1 we show that (6) holds. Since $0 < z_n \le u_n$ and f is nondecreasing, we have

$$\Delta(r_n\Delta z_n) + q_n f(z_{n-l}) \le 0 \quad \text{for} \quad n \ge n_2 = n_1 + l.$$

Summing the above inequality form n to $m \ge n \ge n_2$ we get

$$r_{m+1}\Delta z_{m+1} - r_n\Delta z_n + \sum_{i=n}^{m} q_i f(z_{i-l}) \le 0.$$

After letting $m\to\infty$, we have

$$\sum_{i=n}^{\infty} q_i f(z_{i-l}) \le r_n\Delta z_n$$

and so

$$\sum_{i=n+l+1}^{\infty} q_i f(z_{i-l}) \le r_n\Delta z_n.$$

In view of monofonicity of (z_n) and f we see that

$$\frac{1}{r_n}\sum_{i=n+l+1}^{\infty} q_i \le \frac{\Delta z_n}{f(z_{n+1})} \le \int_{z_n}^{z_{n+1}} \frac{du}{f(u)} \quad \text{for} \quad n \ge n_2.$$

Summing the last inequality from n_2 to n, we obtain

$$\sum_{j=n_2}^{n} \frac{1}{r_j}\sum_{i=j+l+1}^{\infty} q_i \le \int_{z_{n_2}}^{z_{n+1}} \frac{du}{f(u)} < \int_{z_{n_2}}^{\infty} \frac{du}{f(u)} < \infty,$$

which contradicts (9). The proof is similar when (u_n) is eventually negative.

Remark 2. In the case when $r_n \equiv 1$, Theorem 2 reduces to Theorem 3 of [15]. The following criterion provides sufficient conditions for the oscillation of the difference of every solution of (1) when coefficient (q_n) is allowed to oscillate.

Theorem 3. If $p_n \equiv p \ge 0$, f is a nondecreasing function and

$$\sum_{n=0}^{\infty} q_n = \infty, \tag{10}$$

then the difference (Δu_n) of every solution (u_n) of (1) oscillates.

Proof. If not, then (1) has a solution (u_n) such that its difference (Δu_n) is nonoscillatory. Assume that the sequence (Δu_n) is eventually positive, say for $n \geq n_0$. Thus (u_n) is increasing for $n \geq n_0$, which implies that (u_n) is also nonoscillatory.

Denote

$$z_n = u_n + pu_{n-k}$$

and

(11) $$w_n = \frac{r_n \Delta z_n}{f(u_{n-l})}, \qquad n \geq n_1 = n_0 + \max\{k, l\}.$$

Then, by assumptions we have for $n \geq n_1$

$$\Delta w_n = \frac{\Delta(r_n \Delta z_n)}{f(u_{n-l})} - \frac{r_{n+1} \Delta z_{n+1} \Delta f(u_{n-l})}{f(u_{n-l}) f(u_{n+1-l})} \leq -q_n.$$

Summing the above inequality from n_1 to n, we get

$$w_{n+1} < w_{n_1} - \sum_{i=n_1}^{n} q_i$$

and, by (10), we see that $w_n \to -\infty$ as $n \to \infty$. Then (11) implies that $u_n < 0$ eventually.

Now we observe that from (10) it follows there exists $n_2 \geq n_1$ sufficiently large such that

(12) $$\sum_{i=n_2}^{n} q_i \geq 0 \qquad \text{for} \quad n \geq n_2.$$

Also we may assume that $u_{n-l} < 0$ and $\Delta u_{n-l} > 0$ for $n \geq n_2$. Summing up both sides of (1) from n_2 to n we have

$$\sum_{i=n_2}^{n} \Delta(r_n \Delta z_n) = -\sum_{i=n_2}^{n} q_i f(u_{i-l})$$

and according to summation by parts formula we may write

$$r_{n+1} \Delta z_{n+1} - r_{n_2} \Delta z_{n_2} = -f(u_{n-l}) \sum_{i=n_2}^{n} q_i + \sum_{i=n_2}^{n-1} \Delta f(u_{i-l}) \sum_{j=n_2}^{i} q_j.$$

By the assumptions, from the above equality we get

$$r_n \Delta z_{n+1} \geq r_{n_2} \Delta z_{n_2} > 0,$$

which implies

$$z_{n+2} \geq z_{n_2+1} + r_{n_2} \Delta z_{n_2} \sum_{i=n_2}^{n} \frac{1}{r_{i+1}} \to \infty \qquad \text{as} \qquad n \to \infty,$$

but this contradicts the fact that (z_n) is eventually negative. The case that (Δu_n) is eventually negative can be treated in a similar fashion and so the proof is completed.

Remark 3. Theorem 3 extends Theorem 2 of [18] and Theorem 3 of [13].

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