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ALEKSANDER WASZAK AND DARIUSZ ŻURAKOWSKI

MUSIELAK-ORLICZ SPACES OF SEQUENCES CONNECTED WITH STRONG SUMMABILITY

ABSTRACT. Let T be a space of all real sequences and let $\varphi=(\varphi_m)$, $\psi=(\psi_m)$ be any two sequences of convex φ-function. In this paper we consider two modular spaces of sequences T_φ^* and T_ψ^* connected with strong summability, which are generated by the sequences φ and ψ.

This note contains some generalization of two theorems given in [6] and [8] which determinate relationship between these spaces.

KEY WORDS: Sequence spaces, modular spaces.

1. INTRODUCTION

In papers of J. Musielak [3], W. Orlicz [8], J. Musielak and W. Orlicz [5], [6] and R. Taberski [9] there are considered and investigated some modular spaces of sequences connected with strong summability. These spaces were generated by the φ-function $\varphi(u)$ and the matrix of the first arithmetic means. Let us remark that the first method of strong summability of sequences given in [3], [8], [6] was defined for an arbitrary φ-function, while the second method considered in [8], [6] and [9] was defined only for convex φ-function which satisfies the conditions (0_1) and (∞_1).

In the general case they are not equivalent, but if $\varphi(u)=u^\alpha$, $\alpha\geq 1$, both methods coincide.

Now, we considered the same similar problem but concerning the modular spaces of strong summability of sequences with an arbitrary matrix A and with a given sequence $\varphi=(\varphi_m)$ of φ-function.

2. NOTATION AND DEFINITIONS

Let T denote a space of all real sequences. By T_f we will denote a space of „finite" real sequences (i.e. sequences with finite number of elements different from zero) and by T_b we will denote a space of bounded real sequences. Sequences which belong to T will be denoted by $x=(t_m)$, $y=(s_m)$, $|x|=(|t_m|)$, $x_k=(t_m^k)$, $\theta=(0)$, $e=(1,1,\dots)$, $x^n=(t_1,t_2,\dots,t_n,0,\dots)$, $x_p^q=(0,\dots,0,$

$t_p,\dots,t_{p+q-1},0,\dots)$. Moreover e^n, e_n and e_p^q will denote the sequence with 1 at the first n places, the sequence with 1 at the n-th place and the sequence with 1 at the p-th, $(p+1)$-st, $(p+q-1)$-st places, respectively.

By a φ-function we will undestand a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0)=0$, $\varphi(u)>0$, for $u>0$ and $\varphi(u)\to\infty$ as $u\to\infty$. Let φ be a convex φ-function satisfying conditions

$$(0_1)\quad \frac{\varphi(u)}{u}\to 0 \quad \text{as} \quad u\to 0_+, \qquad (\infty_1)\quad \frac{\varphi(u)}{u}\to\infty \quad \text{as} \quad u\to\infty.$$

Let $\varphi=(\varphi_m)$ and $\psi=(\psi_m)$ be any two sequences of convex φ-functions, which satisfy conditions (0_1) and (∞_1).

Let $A=(a_{nm})$ denote a non-negative matrix which contains no column consisting of zeros only. In order to develop a theory of sequence modular spaces, we shall need in particular theorems some additional assumptions on the matrix A:

(1) $a_{nm}\to 0$ as $n\to\infty$ for $m=1,2,\dots$,

(2) there exists a positive constant K such that $a_{n1}+a_{n2}+\dots\leq K$, for all n,

(3) $A_m=\sup\limits_n a_{nm}\to 0$ as $m\to\infty$,

(4) $\limsup\limits_{m\to\infty}\dfrac{A_m}{A_{m+1}}=a<\infty$ where a is a certain positive constant.

3. THE MODULAR SEQUENCE SPACE OF STRONGLY (A,φ)-SUMMABLE TO ZERO

We introduce the following notation

$$\sigma_n^\varphi(x)=\sum_{m=1}^{\infty}a_{nm}\varphi_m(|t_m|) \quad \text{for} \quad n=1,2,\dots \quad \text{and} \quad x\in T,$$

$$T_{\varphi 0}=\{x\in T:\ \sigma_n^\varphi(x)<\infty \ \text{for}\ n=1,2,\dots \ \text{and}\ \sigma_n^\varphi(x)\to 0 \ \text{as}\ n\to\infty\},$$

$$T_\varphi=\{x\in T:\lambda x\in T_{\varphi 0},\ \text{for and arbitrary}\ \lambda>0\}.$$

$$T_\varphi^*=\{x\in T:\lambda x\in T_{\varphi 0},\ \text{for a certain}\ \lambda>0\}.$$

Sequences x, which belong to T_φ^* are called strongly (A,φ)-summable to zero. In the space T_φ^* we can define the following functional

$$\rho_\varphi(x)=\begin{cases}\sup\limits_n \sigma_n^\varphi(x) & \text{for} \quad x\in T_{\varphi 0},\\ 0 & \text{for} \quad x\in T_\varphi^*-T_{\varphi 0}.\end{cases}$$

This functional is a modular in T_φ^* and moreover T_φ^* is a modular space in the sense of definition adopted by J. Musielak and W. Orlicz (compare [4] or [5]).

For a sequence of convex φ-function $\varphi = (\varphi_m)$ we may introduce a norm by the formula

$$\|x\|_\varphi^c = \inf\left\{\varepsilon > 0 : \rho_\varphi\left(\frac{x}{\varepsilon}\right) \le 1\right\}.$$

We say that φ-function φ satisfies the condition Δ_2 for large arguments and we write $\varphi \in \Delta_2$, if there exist constants $K > 0$, $u_0 \ge 0$ and a sequence (b_m) of elements $b_m \ge 0$ with $\sum_{m=1}^{\infty} b_m < \infty$ such that

$$\varphi_m(2|t_m|) \le K\varphi_m(|t_m|) + b_m \quad \text{for every } m \text{ and } |t_m| \ge u_0.$$

A function $\varphi = (\varphi_m)$ is called non-weaker than a function $\psi = (\psi_m)$ for large arguments and we write $\psi \overset{1}{\prec} \varphi$ if there exist positive constants K_1, K_2, nonnegative constant u_0 and a sequence b_m of elements $b_m \ge 0$ with $\sum_{m=1}^{\infty} b_m < \infty$ such that

$$\psi_m(|t_m|) \le K_1\varphi_m(K_2|t_m|) + b_m \quad \text{for every } m \text{ and } |t_m| \ge u_0.$$

If $\psi \overset{1}{\prec} \varphi$ and $\varphi \overset{1}{\prec} \psi$, then functions φ and ψ are called equivalent for large arguments and we write $\varphi \overset{1}{\sim} \psi$, (compare e.g. [4]).

The followin properties are true (for proof see [16] and [17]):

1) $T_\varphi \subset T_{\varphi 0} \subset T_\varphi^*$,
2) $T_f \subset T_\varphi$ if and only if the matrix A satisfies the condition (1),
3) if any sequence has the k-th term different from zero, then $a_{nk} \to 0$ as $n \to \infty$; if not $a_{nm} \to 0$ as $n \to \infty$ for $m = 1, 2, \ldots$, then we have $T_\varphi = T_{\varphi 0} = T_\varphi^* = \{\theta\}$,
4) T_φ^* is a linear space; T_φ is a convex set in T_φ^* closed with respect to the norm $\|\cdot\|_\varphi$,
5) φ satisfies the condition (Δ_2) for large arguments if and only if $T_\varphi^* = T_\varphi$,
6) for an arbitrary two sequences $\varphi = (\varphi_m)$ and $\psi = (\psi_m)$ of φ-functions we have

$$T_\varphi^* \cap T_b = T_\psi^* \cap T_b,$$

$$T_\varphi^* \cap T_b = T_\varphi^* \cap T_b,$$

$$T_\varphi^* \cap T_b = T_{\varphi 0}^* \cap T_b,$$

7) the spaces T_φ^* and T_φ are complete with respect to the norm $\|\cdot\|_\varphi$ (for proof see [7] and [13]),

8) let the matrix A has properties (1) ÷ (4);

if $\psi \overset{1}{\prec} \varphi$ then $T_\varphi^* \subset T_\psi^*$ and $T_\varphi \subset T_\psi$,

if $\varphi \overset{1}{\sim} \psi$ then $T_\varphi^* = T_\psi^*$ and $T_\varphi = T_\psi$.

4. MODULAR SPACE R_M

In the space T_f we may define the following funcional

$$\rho_\varphi^b(x) = \sum_{m=1}^{\infty} \varphi_m(|t_m|).$$

It is easily verified that this funcional is a modular in T_f (see [5], [9], compare also [13]). Basing on this modular, in the space T_f we can define a norm by means of the following formula

$$\|x\|_\varphi^R = \inf\left\{\varepsilon > 0: \rho_\varphi^b\left(\frac{x}{\varepsilon}\right) \le 1\right\}.$$

It is well known that this norm is homogeneous and monotonic.

Aplying this norm we define the following class of sequences, strongly $|A,\varphi|$-summable to zero. Let $T_\varphi^b \equiv R_M$ denote the class of sequences satisfying the condition

$$\|\overline{x}^n\|_\varphi^R \to 0 \quad \text{as} \quad n \to \infty$$

where

$$\overline{x}^n = \begin{cases} \varphi_m^{-1}(a_{nm})t_m & \text{if } n \ge m, \\ 0 & \text{if } n < m. \end{cases}$$

In the space T_φ^b we define a norm $\|\cdot\|_\varphi^b$ by the formula

$$\|x\|_\varphi^b = \sup_n \|\overline{x}^n\|_\varphi^R .$$

Moreover, in T_f we can define another norm

$$\|x\|_\varphi^0 = \sup_y \sum_{m=1}^{\infty} |t_m| s_m,$$

where supremum is taken over all non-negative sequence $y=(s_m)$ satisfying the inequality

$$\sum_{m=1}^{\infty} \varphi_m^*(s_m) \le 1.$$

Let φ and φ^* be two sequences of convex φ-functions, where φ^* is a complementary to φ in the sense of Young, defined by the equality

$$\varphi^*(v) = \sup_{u\ge 0}(uv - \varphi(u))$$

(see [4], [1] and [2]). Obviously φ^* is a convex φ-function satisfying conditions (0_1), (∞_1) and the Young inequality

$$uv \le \varphi(u) + \varphi^*(v)$$

(see also [1], [2], [4]).

The following properties are true:

1) for arbitrary two sequences x and y by the Young inequality we have the inequality

$$\sum_{m=p}^{p+q-1} |t_m s_m| \le \| x_p^q \|_\varphi^0 \, \| y_p^q \|_\varphi^0,$$

2) the space T_φ^b with the norm $\|\cdot\|_\varphi^b$ is a Banach space,

3) if $x \in T_\varphi^b$ then $\| x - x^k \|_\varphi^b \to 0$ as $k \to \infty$,

4) $x \in T_\varphi^b$ if and only if $\| x^k - x^j \|_\varphi^b \to 0$ as $k, j \to \infty$,

(compare [5], [9], [13]).

5. Main theorems

Let $\varphi = (\varphi_m)$ and $\psi = (\psi_m)$ be given two sequences of convex φ-functions, and let $A = (a_{nm})$ denote the matrix for which

$$a_{n1} + a_{n2} + \ldots \le K$$

and

$$\sum_{m=1}^{\infty} \psi_m^{-1}(a_{nm})[\psi_m(1)]^{-1} \le C$$

for all n, where K and C are some positive constants.

Theorem 1. If there exists a constant $\delta > 0$ satisfying the condition

(*) $$\psi(\delta uv) \le \varphi(u)\, \psi(v)$$

for all $u,v > 0$ such that $\varphi(u)\,\psi(v) < \delta$ and $\varphi(u) \ge 1$, then

$$T_{\varphi}^{*} \subset T_{\psi}^{b}.$$

Proof. Let $x=(t_m)$, $t_m \geq 0$ for all m and let $x \in T_{\varphi}^{*}$ and let λ be a given positive number satisfying the conditions

$$a_{nm}\varphi_m(\lambda t_m) \leq \delta \quad \text{and} \quad \varphi_m(\lambda t_m) \geq 1$$

for all n, m with a constant $\delta \in (0,1)$ given by $(*)$. For

$$u = \lambda t_m, \qquad v = \psi_m^{-1}(a_{nm})$$

by $(*)$ we have

(i) $$\psi_m(\delta \lambda t_m \psi_m^{-1}(a_{nm})) \leq \varphi_m(\lambda t_m) a_{nm},$$

for all m and n. For a given $\varepsilon > 0$ we may choose the positive constant λ_0 such that

$$1 < \delta \varepsilon \lambda_0$$

and that the condition $\varphi_m(\lambda_0 u) \leq 1$ implies $|u| < \varepsilon/2$. In the following let

$$x = x' + x'',$$

where $x' = (t'_m) \in T_{\varphi}^{*}$, $x'' = (t''_m) \in T_{\varphi}^{*}$ and $t''_m = t_m$ for $\varphi_m(\lambda_0 t_m) < 1$ and $t''_m = 0$ for $\varphi_m(\lambda_0 t_m) \geq 1$.

But $x' \in T_{\varphi}^{*}$, then there exists a natural number n_0 such that

(ii) $$\sum_{m=1}^{\infty} a_{nm}\varphi_m(\lambda_0 |t'_m|) \leq \delta,$$

for every $n \geq n_0$. In the following by (i) and (ii) we have

$$\rho_{\psi}^{b}\left(\frac{\overline{x}'^{n}}{\varepsilon}\right) \leq \sum_{m=1}^{\infty} \psi_m\left(\frac{t'_m \psi_m^{-1}(a_{nm})}{\varepsilon}\right) \leq \sum_{m=1}^{\infty} \psi_m(\delta \lambda_0 t'_m \psi_m^{-1}(a_{nm})) \leq$$

$$\leq \sum_{m=1}^{\infty} a_{nm}\varphi_m(\lambda_0 t'_m) \leq \delta < 1$$

for sufficiently large n and for $t'_m \neq 0$. In consequence, applying the definition of the norm $\|\cdot\|_{\psi}^{R}$, we obtain the inequality

(iii) $$\|\overline{x}'^{n}\|_{\psi}^{R} \leq \varepsilon$$

for sufficiently large n.

By the definition x'' we receive that the condition $\varphi_m(\lambda_0 t_m) < 1$ implies $t''_m < 3/2$. But the norm $\|\cdot\|_{\psi}^{R}$ is monotone, then we have

$$\left\|\overline{x}''^{n}\right\|_{\psi}^{R} \leq \left\|\left[\left(\frac{\varepsilon}{2}\right)e\right]\right\|_{\psi}^{R} \leq \frac{\varepsilon}{2}\sum_{m=1}^{n}[\psi_m(1)]\psi_m^{-1}(a_{nm}).$$

Finally we obtain

(iv) $$\left\|\overline{x}''^{n}\right\|_{\psi}^{R} \le \varepsilon_1,$$

where $2\varepsilon_1 = \varepsilon C$. But the norm $\|\cdot\|_{\psi}^{R}$ is monotone, then by (iii), (iv) we have $\left\|\overline{x}^{n}\right\|_{\psi}^{R} \le \varepsilon_2 = (\varepsilon + \varepsilon_1)$ for $n \ge n_0$, i.e. $x \in T_{\psi}^{b}$.

Theorem 2. If there exists a constant $\delta > 0$ satisfying the condition

(**) $$\varphi(u)\,\psi(v) \le \psi(\delta^{-1}uv)$$

for all $u, v > 0$ such that $uv \le \delta$ and $u \ge 1$, then $T_{\psi}^{b} \subset T_{\varphi}^{*}$.

Proof. Let us suppose that the number $\delta \in (0,1)$ be given by (**) and let $x = (t_m) \in T_{\psi}^{b}$, where $t_m \ge 0$ for all m. Now, choosing

$$\lambda t_m \psi_m^{-1}(a_{nm}) \le \delta \quad \text{and} \quad \lambda t_m \ge 1$$

for all m, we conclude by (**) that

(+) $$a_{nm}\varphi_m(\lambda t_m) \le \psi_m(\delta^{-1}\lambda t_m \psi_m^{-1}(a_{nm}))$$

where λ is a positive constant and $u = \lambda t_m$, $v = \psi_m^{-1}(a_{nm})$. In the following, we take two arbitrary positive numbers λ_0 and ε such that

$$\lambda_0 \ge \max\{1, \lambda\}$$

and

$$\varepsilon < \psi_m(1)$$

for all m and such that the inequality $\lambda_0 u \le 1$ implies the condition $\varphi_m(u) \le \varepsilon$ for every m. We write

$$x = x' + x'',$$

where $x' = (t'_m) \in T_{\psi}^{b}$, $x'' = (t''_m) \in T_{\psi}^{b}$ and $t''_m = t_m$ for $\lambda_0 \lambda t_m < 1$ and $t''_m = 0$ for $\lambda_0 \lambda t_m \ge 1$.

It is easily verified that for every m we have $\lambda_0 \lambda t''_m < 1$, whence $\varphi_m(\lambda t''_m) \le \varepsilon$ for all m. Thus there holds

$$\sum_{m=1}^{\infty} a_{nm}\varphi_m(\lambda t''_m) \le \varepsilon \sum_{m=1}^{\infty} a_{nm} \le \varepsilon K.$$

Since $x' \in T_{\psi}^{b}$, then there is an integer n_0 such that $\|\overline{x}'^{n}\|_{\psi}^{R} < \eta$ for $n \ge n_0$, where the sequence $(\overline{x}'^{n})$ is defined by the formula

$$\overline{x}'^n = \begin{cases} \psi_m^{-1}(a_{nm})t'_m & \text{for} \quad m \le n, \\ 0 & \text{for} \quad m > n. \end{cases}$$

But the norm $\|\cdot\|_\psi^R$ is monotone, then we have

$$(++) \qquad \left\| \overline{(t'_m e_m)}^n \right\|_\psi^R = t'_m [\psi_m^{-1}(1)]^{-1} \psi_m(a_{nm}) \le \left\| (\overline{x}'^n) \right\|_\psi^R < \eta.$$

Now, for

$$\eta = \inf_n \delta\, \psi_m^{-1}(\varepsilon 2^{-m})[\lambda\lambda_0\, \psi_m^{-1}(1)]^{-1}$$

by (++) we obtain the following estimates

$$t'_m[\,\psi_m^{-1}(1)]^{-1}\, \psi_m^{-1}(a_{nm}) < \delta\, \psi_m^{-1}(\varepsilon 2^{-m})[\lambda\lambda_0\, \psi_m^{-1}(1)]^{-1},$$

$$\lambda\lambda_0 t'_m\, \psi_m^{-1}(a_{nm}) < \delta\, \psi_m^{-1}(\varepsilon 2^{-m}) < \delta\, \psi_m^{-1}(\varepsilon) < \delta.$$

Finally

$$\delta^{-1}\lambda\lambda_0 t'_m\, \psi_m^{-1}(a_{nm}) < \psi_m^{-1}(\varepsilon 2^{-m})$$

and

$$(+++) \qquad \psi_m(\delta^{-1}\lambda\lambda_0 t'_m\, \psi_m^{-1}(a_{nm})) < \varepsilon 2^{-m}.$$

But, by (+) and (++) we obtain the inequalities

$$a_{nm}\varphi_m(\lambda t'_m) \le a_{nm}\varphi_m(\lambda_0 \lambda t'_m) < \varepsilon 2^{-m}$$

for sufficiently large n and $1 \le m \le n$ with $\lambda_0 \ge \max\{1, \lambda\}$. Hence

$$\sum_{m=1}^{\infty} a_{nm}\varphi_m(\lambda t'_m) < \varepsilon.$$

Finally, we have

$$\sum_{m=1}^{\infty} a_{nm}\varphi_m(\lambda t_m) < \varepsilon_2 = \varepsilon(1+K)$$

i.e. $x \in T_\varphi^*$.

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(Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Matejki 48/49, 60-769 Poznań, Poland)

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