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NONEXISTENCE CRITERIA FOR POSITIVE SOLUTIONS OF A DISCRETE ELLIPTIC EQUATION

ABSTRACT. A nonlinear elliptic type partial difference equation with a forcing term is studied in this paper. By means of an averaging technique, the problem of non-existence of positive solutions is reduced to that of forced recurrence relations. Several sample non-existence criteria are given for these recurrence relations which in turn yield non-existence criteria for the discrete elliptic equation.

KEY WORDS: elliptic type partial difference equations, positive solutions, non-existence criteria.

1. INTRODUCTION

Oscillation criteria for elliptic partial differential equations have been obtained by several authors in a number of studies. Surprisingly, only a limited number of studies [1,2] related to discrete elliptic equations are available. In this paper, we are concerned with several nonexistence criteria for „positive solutions" of a class of nonlinear elliptic type partial difference equations defined on the lattice plane.

More precisely, a lattice point $z=(i,j)$ in the plane is a point with integer coordinates. The set of all lattice points will be denoted by Z^2. We will assume that the lattice plane is endowed with the 1-norm

$$\|(i,j)\| = |i|+|j|$$

Two lattice points are said to be neighbors if their distance (induced by the norm) is one. The four neighbors of a lattice point $z=(i,j)$, namely, $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+1)$, will also denoted by z_L, z_R, z_D and z_T respectively. There are several common subsets which are useful in the sequel. First of all, the k-sphere S_k is defined by

$$S_k = \{(i,j) \in Z^2 \ \|(i,j)\| = k\}, \qquad k \geq 0.$$

Note that the size $|S_k|$ of a k-sphere is $4k$ when $k \geq 1$. Note further that the four corners of a k-sphere are $(k,0)$, $(-k,0)$, $(0,k)$ and $(0,-k)$ when $k \geq 1$. For the sake of convenience, we collect these corners into a set and denote it by C_k, where $k \geq 1$.

Consider the partial difference equation

(1.1) $$\Delta_1^2 u(i-1,j)+\Delta_2^2 u(i,j-1)+q(i,j,u(i,j))=f(i,j), \qquad (i,j)\in Z^2,$$

where $q(i,j,u)$ is a real function defined for and $u\in R$, and $f(i,j)$ is a real function defined for $(i,j)\in Z^2$. The partial differences are defined as usual, i.e.,

$$\Delta_1^2 u(i,j)=u(i+2,j)-2u(i+1,j)+u(i,j)$$

and

$$\Delta_2^2 u(i,j)=u(i,j+2)-2u(i,j+1)+u(i,j)$$

Such an equation can be regarded as a discrete analog of the partial differential equation

$$u_{xx}+u_{yy}+q(x,y,u(x,y))=f(x,y).$$

Note that by means of the notations $z=(i,j)$, z_L and z_D, we may also write (1.1) as

$$\Delta_1^2 u(z_L)+\Delta_2^2 u(z_D)+q(z,u(z))=f(z), \qquad z\in Z^2.$$

A solution of (1.1) is a double sequence $\{u(i,j)\}_{i,j=-\infty}^{\infty}$ which satisfies (1.1) for $(i,j)\in Z^2$. It is not difficult to construct solutions of (1.1.). Indeed, since (1.1) can be written in the form

$$u(i,j+1)=-u(i-1,j)+4u(i,j)-q(i,j,u(i,j))-$$
$$-u(i+1,j)-u(i,j-1)+f(i,j)$$

if we impose conditions such as

$$u(i,j)=\phi(i,j), \qquad i=0,\pm1,\pm2,...; \qquad j=-1,0,$$

then we can calculate successively

$$u(1,0);\ u(0,-2);$$
$$u(-1,1),\ u(0,2),\ u(1,1),\ u(-1,-2),\ u(0,-3),\ u(1,-2);$$
$$u(-2,1),\ u(-1,2),\ u(0,3),\ u(1,2),\ u(2,1);...,$$

from (1.1) in a unique manner. It is not clear, however, whether a positive solution i.e., $u(i,j)>0$ for all $(i,j)\in Z^2$, can exist. In this note, we will show that, if a positive solution exists, certain conditions must be imposed on the functions $q(i,j,u)$ and $f(i,j)$. For now, we will assume that $q(i,j,u)$ satisfies a commonly seen requirement, namely,

(H1) $q(i,j,u)\ge p(\|(i,j)\|)\,\psi(u)$ for all $(i,j)\in Z^2$ and $u\ge 0$, where p is a non-negative function defined on $\{0,1,...\}$ and ψ is a non-negative and convex function defined on $(0,\infty)$.

2. AVERAGING TECHNIQUE

We will develop a technique which is analogous to a well know technique in the theory of partial differential equations, namely, the technique of averaging.

Lemma 2.1. Let $\{u(i,j)\}_{i,j=-\infty}^{\infty}$ *be a real double sequence. Then for* $k \geq 1$, *we have*

$$\sum_{z\in S_{k+1}} \Delta_1^2 u(z_L) + \Delta_1^2 u(z_L) =$$

$$= 3\sum_{z\in C_k} u(z) + \sum_{z\in S_k\setminus C_k} 2u(z) + \sum_{z\in S_{k+1}} -4u(z) + \sum_{z\in C_{k+2}} u(z) + \sum_{z\in S_{k+2}\setminus C_{k+2}} 2u(z).$$

Proof. Note that the sum

$$\sum_{z\in S_{k+1}} \Delta_1^2 u(z_L) + \Delta_2^2 u(z_D)$$

is a sum of $u(z)$ where z belongs to the spheres S_k or S_{k+1} or S_{k+2} only. Furthermore, it is not difficult to see that each $u(z)$ in the sum appears twice for $z \in S_k \setminus C_k$, thrice for $z \in C_k$, -4 times for $z \in S_{k+1}$, twice for $z \in S_{k+2} \setminus C_{k+2}$ and once in C_{k+2}. ■

Let $u = \{u(i,j)\}_{i,j=-\infty}^{\infty}$ be a real double sequence. We will denote the mean of u over the sphere S_k by

$$U(k) = \frac{1}{|S_k|}\sum_{z\in S_k} u(z), \qquad k \geq 0. \tag{2.1}$$

Lemma 2.2. Let $u = \{u(i,j)\}_{i,j=-\infty}^{\infty}$ *be a solution of* (1.1). *Then its mean* U *defined by* (2.1) *over sphere* S_k *satisfies the following recurrence relations:*

$$4\{2kU(k) - 4(k+1)U(k+1) + (k+2)U(k+2)\} + \sum_{z\in S_{k+1}} q(z,u(z)) =$$

$$= \sum_{z\in S_{k+1}} f(z) - \sum_{z\in C_k} u(z) - \sum_{z\in S_{k+2}\setminus C_{k+2}} u(z), \qquad k \geq 1, \tag{2.2}$$

and

$$4\{2kU(k) - 4(k+1)U(k+1) + 2(k+2)U(k+2)\} + \sum_{z\in S_{k+1}} q(z,u(z)) =$$

$$= \sum_{z\in S_{k+1}} f(z) + \sum_{z\in C_{k+2}} u(z) - \sum_{z\in C_k} u(z), \qquad k \geq 1. \tag{2.3}$$

Proof. Summing (1.1) over S_{k+1}, we have, in view of Lemma 2.1, that

$$- \sum_{z\in S_{k+1}} q(z,u(z)) + \sum_{z\in S_{k+1}} f(z) =$$

$$= \sum_{C_k} u + 2\sum_{C_k} u + \sum_{S_k \setminus C_k} 2u + \sum_{S_{k+1}} -4u + 2\sum_{C_{k+2}} u + \sum_{S_{k+2}\setminus C_{k+2}} 2u - \sum_{C_{k+2}} u =$$

$$= \sum_{z\in S_k} 2u(z) + \sum_{z\in S_{k+1}} -4u(z) + \sum_{z\in S_{k+1}} 2u(z) + \sum_{z\in C_k} u(z) - \sum_{z\in C_{k+2}} u(z) =$$

$$= 2|S_k|U(k) - 4|S_{k+1}|U(k+1) + 2|S_{k+2}|U(k+2) + \sum_{z\in C_k} u(z) - \sum_{z\in C_{k+2}} u(z) =$$

$$= 8\{kU(k) - 2(k+1)U(k+1) + (k+2)U(k+2)\} + \sum_{z\in C_k} u(z) - \sum_{z\in C_{k+2}} u(z)$$

which is equivalent to (2.3).

Similarly,

$$- \sum_{z\in S_{k+1}} q(z,u(z)) + \sum_{z\in S_{k+1}} f(z) =$$

$$= 2\sum_{S_k\setminus C_k} u + 2\sum_{C_k} u + \sum_{C_k} u - 4\sum_{S_{k+1}} u + \sum_{C_{k+2}} u + \sum_{S_{k+2}\setminus C_{k+2}} u + \sum_{S_{k+2}\setminus C_{k+2}} u$$

$$= 2\sum_{S_k} u - 4\sum_{S_{k+1}} u + \sum_{S_{k+2}} u + \sum_{C_k} u + \sum_{S_{k+2}\setminus C_{k+2}} u$$

$$= 4\{2kU(k) - 4(k+1)U(k+1) + (k+2)U(k+2)\} + \sum_{C_k} u + \sum_{S_{k+2}\setminus C_{k+2}} u$$

which is equivalent to (2.2). ■

Let us assume that $u = \{u(i,j)\}_{i,j=-\infty}^{\infty}$ is a positive solution of (1.1), and that (H1) holds, then by means of the Jensen's inequality,

(2.4) $$\sum_{z\in S_k} q(z,u(z)) \geq \sum_{z\in S_k} p(\|(i,j)\|)\, \psi(u(i,j)) =$$

$$= p(k) \sum_{z\in S_k} \psi(u(i,j)) \geq p(k)|S_k|\, \psi(U(k)) =$$

$$= 4kp(k)\, \psi(U(k)).$$

Let us also denote the mean of the function $f(i,j)$ in (1.1) by

(2.5) $$F(k) = \frac{1}{|S_k|} \sum_{z\in S_k} f(z), \qquad k \geq 0.$$

Then, in view of (2.2) and (2.4), we have

$$4\{2(k-1)U(k-1) - 4kU(k) + (k+1)U(k+1)\} + 4kp(k)\psi(U(k)) \leq$$

$$\leq \sum_{z\in S_k} f(z) - \sum_{z\in C_{k-1}} u(z) - \sum_{z\in S_{k+1}\setminus C_{k-1}} u(z) \leq 4kF(k), \qquad k \geq 2.$$

The following result is now clear.

Theorem 2.3. Suppose (H1) *holds. If* $u=\{u(i,j)\}_{i,j=-\infty}^{\infty}$ *is a positive solution of* (1.1)*, then the mean* $U(k)$ *defined by* (2.1) *satisfies the recurrence relation*

$$2(k-1)U(k-1)-4kU(k)+(k+1)U(k+1)+ \\ +4kp(k)\,\psi(U(k))\le kF(k) \tag{2.6}$$

for $k\ge 2$.

Note that if $\Delta_1 u(i,0)\le 0$ for $i\ge 0$, $\Delta_1 u(i,0)\ge 0$ for $i\le -1$, $\Delta_2 u(0,j)\le 0$ for $j\ge 0$ and $\Delta_2 u(0,j)\ge 0$ for $j\le -1$, then

$$\sum_{z\in C_{k+1}} u(z)\le \sum_{C_{k-1}} u(z).$$

The following is now clear from (2.3) and (2.4).

Theorem 2.4. Suppose (H1) *holds. If* $u=\{u(i,j)\}_{i,j=-\infty}^{\infty}$ *is a positive solution of* (1.1) *such that* $\Delta_1 u(i,0)\le 0$ for $i\ge 0$, $\Delta_1 u(i,0)\ge 0$ for $i\le -1$, $\Delta_2 u(0,j)\le 0$ for $j\ge 0$ and $\Delta_2 u(0,j)\ge 0$ for $j\le -1$, *then the mean* $U(k)$ *defined by* (2.1) *satisfies the recurrence relations*

$$2(k-1)U(k-1)-4kU(k)+2(k+1)U(k+1)+ \\ +kp(k)\,\psi(U(k))\le kF(k) \tag{2.7}$$

for $k\ge 2$.

We remark that the recurrence relation (2.6) can also be written as

$$\Delta^2((k-1)U(k-1))-2kU(k)+(k-1)U(k-1)+kp(k)\,\psi(U(k))\le kF(k),$$

while recurrence relation (2.7) can also be written as,

$$2\Delta^2((k-1)U(k-1))+kp(k)\,\psi(U(k))\le kF(k),$$

or as

$$2\Delta(k(k-1)\Delta U(k-1))+k^2p(k)\,\psi(U(k))\le k^2F(k).$$

3. NON-EXISTENCE CRITERIA FOR RECURRENCE RELATIONS

As we have seen in the last Section, to establish non-existence criteria for positive solutions of the partial difference equation (1.1), it suffices to find non-existence criteria for the reccurence relation (2.6) or (2.7).

We first consider the following associated homogeneous recurrence relations of (2.6), i.e.

(3.1) $$\Delta^2((k-1)U(k-1)) - 2kU(k) + (k-1)U(k-1) + kp(k)\,\psi(U(k)) \le 0$$

where we recall from (H1) that

(H2) p is a non-negative function defined on $\{1,2,...\}$ and ψ is a non-negative and convex function defined on $(0,\infty)$.

Let $v(k) = kU(k)$ for $k \ge 1$. Then clearly $\{v(k)\}_{k=0}^{\infty}$ is positive if, and only if $\{U(k)\}_{k=0}^{\infty}$ is positive. Furthermore, we can rewrite (3.1) in the following form:

(3.2) $$\Delta^2 v(k-1) - 2v(k) + v(k-1) + kp(k)\psi\left(\frac{v(k)}{k}\right) \le 0, \qquad k \ge 1.$$

Theorem 3.1. Suppose (H2) *holds. Assume that there is a positive constant* M *such that* $\psi(x) \ge Mx$ *for* $x > 0$, *and* $Mp(k) \ge 2$ for $k \ge 1$. *Then* (3.2) *cannot have an eventually positive solution.*

Proof. Let $\{v(k)\}$ be an eventually positive solution of (3.2) such that $v(k) > 0$ for $k \ge N$. Note that

(3.3) $$\Delta^2 v(k-1) + (Mp(k) - 2)v(k) + v(k-1) \le$$
$$\le \Delta^2 v(k-1) - 2v(k) + v(k-1) + kp(k)\psi\left(\frac{v(k)}{k}\right) \le 0.$$

Thus $\Delta^2 v(k-1) < 0$ for all large k, which implies $\Delta v(k) > 0$ for all large k. As a consequence, we see from (3.3) that

$$\Delta^2 v(k-1) + (Mp(k) - 1)v(k-1) \le 0$$

for all large k. Let

$$w(k) = \frac{\Delta v(k)}{v(k-1)}, \qquad k \ge N+1.$$

Then it is easily verified that

$$\Delta w(k) + w(k)\frac{\Delta v(k-1)}{v(k)} + Mp(k+1) - 1 \le 0.$$

Since

(3.4) $$\frac{\Delta v(k-1)}{v(k)} \ge \frac{\Delta v(k+1)}{v(k)} = w(k+1)$$

for all large k, we see from (3.4) that

$$\Delta w(k) + 1 \le \Delta w(k) + w(k)w(k+1) + 1 \le 0$$

for all large k. By summing the above inequality from a large integer to infinity, we see that $w(k)$ tends to $-\infty$ as $k \to \infty$, which is contradiction. ■

Theorem 3.2. Assume that (H2) *holds. Suppose that there is a positive number M such that* $\psi(x) \geq Mx$ *for* $x > 0$. *Suppose further that either* (i) *there is a subsequence* $\{k_i\}$ *such that* $Mp(k_i) - 4 \geq 0$ *for all large i, or* (ii) *the sequence* $\{4 - Mp(k)\}$ *is non-negative, bounded above and*

$$\limsup_{k\to\infty} (4 - Mp(k)) = Q \in (0, \sqrt{8}).$$

Then (3.2) *cannot have an eventually positive solution.*

Proof. Let $\{v(k)\}$ be an eventually positive solution of (3.2). Then (3.3) holds. Thus

$$(Mp(k) - 4)v(k) \leq -v(k+1) - 2v(k-1) \tag{3.5}$$

for $k \geq 1$. Clearly, (i) cannot hold since the right hand side is negative.

Suppose to the contrary that (ii) holds. Let

$$r(k) = \frac{v(k)}{v(k+1)}, \qquad k \geq 0.$$

Then $r(k) > 0$ for all large k. Furthermore, if we rewrite (3.5) in the following form

$$\frac{1}{r(k)} + 2r(k-1) \leq 4 - Mp(k), \quad \text{for} \quad k \geq 1,$$

we see from (ii) that $\liminf_{k\to\infty} r(k) = \gamma > 0$. Taking limit superior on both side of the above inequality, we see that

$$\frac{1}{\gamma} + 2\gamma \leq Q,$$

or

$$2\gamma^2 - Q\gamma + 1 \leq 0.$$

However, it is easy to check that the polynomial $2\gamma^2 - Q\gamma + 1$ is positive for all γ when $Q \in (0, \sqrt{8})$. A contradiction is arrived. ∎

The number $\sqrt{8}$ in Theorem 3.2 is sharp. Indeed, consider the case $\psi(x) = x$ and $p(k) \equiv p$, then recurrence relation (3.2) has a positive solution

$$v(k) = \lambda^k, \quad k \geq 0$$

where λ is the positive root of the characteristic equation

$$\lambda^{k+1} + 2\lambda^{k-1} + (p-4)\lambda^k = 0,$$

given by

$$\lambda = \frac{4 - p - \sqrt{(4-p)^2 - 8}}{2} > 0.$$

We now turn our attention to (2.6). Again, let $v(k) = kU(k)$ for $k \geq 0$. Then we obtain from (2.6) the following recurrence relation

(3.6) $$\Delta^2 v(k-1) - 2v(k) + v(k-1) + k\psi\left(\frac{v(k)}{k}\right) \leq kF(k).$$

For the sake of convenience, in the sequel, the positive part of a function g will be denoted by g^+.

Theorem 3.3. Assume that (H2) *holds. Suppose there is a positive constant* M *such that* $\psi(z) \geq Mz$ *for* $x > 0$ *and* $Mp(k) \geq 2$ *for* $k \geq 1$. *Suppose further that there is a sequence* $\{R(k)\}$ *which has a non-positive subsequence* $\{R(k_j)\}$ *and satisfies* $\Delta^2 R(k-1) = kF(k)$ *for all large k, and that*

$$\liminf_{k\to\infty} \sum_{n=N}^{k} \sum_{j=N}^{n} \frac{\{jF(j) - (Mp(j)-2)R^+(j) - R^+(j-1)\}}{k+1-N} = -\infty,$$

where N is some large integer. Then (3.6) *cannot have an eventually positive solution.*

Proof. Suppose $\{v(k)\}$ is an eventually positive solution of (3.6). Then $w(k) = v(k) - R(k)$ satisfies

(3.7) $$\Delta^2 w(k-1) \leq -(Mp(k)-2)v(k) - v(k-1) \leq 0$$

for all large k. The sequence $\{w(k)\}$ is thus eventually concave and hence it is eventually of constant sign. The sequence $\{w(k)\}$ must be eventually positive for otherwise

$$0 < v(k_j) \leq R(k_j) \leq 0,$$

which is impossible. This implies

$$v(k) > \max\{R(k), 0\} = R^+(k)$$

for all large k. By repeated summation of (3.6), we obtain

$$v(k+1) \leq v(N) + (k+1-N)\Delta v(N-1) +$$
$$+ \sum_{n=N}^{k} \sum_{j=N}^{n} \{jF(j) - (Mp(j)-2)v(j) - v(j-1)\}.$$

Thus

$$\frac{v(k+1)}{k+1-N} \leq \frac{v(N)}{k+1-N} + \Delta v(N-1) +$$

$$+\frac{1}{k+1-N}\sum_{n=N}^{k}\sum_{j=N}^{n}\{jF(j)-(Mp(j)-2)R^+(j)-R^+(j-1)\}.$$

Taking limit inferior on both sides of the above inequality, we then arrive at a contradiction. ■

Next, we turn our attention to the recurence relation (2.7), where we recall from (H1) that p is non-negative function defined on $\{1,2,...\}$ and ψ is a non-negative and convex function defined on $(0,\infty)$. As we have mentioned before, (2.7) can be written in the form

$$\Delta^2((k-1)U(k-1)+\frac{kp(k)}{2}\ \psi(U(k))\leq\frac{k}{2}F(k),\qquad k\geq 1, \tag{3.8}$$

or

$$\Delta(k(k-1)\Delta U(k-1)+\frac{k^2p(k)}{2}\ \psi(U(k))\leq\frac{k^2}{2}F(k),\qquad k\geq 1. \tag{3.9}$$

Such recurrence relations are not uncommon. Indeed, there is quite an extensive literature on the more general equation

$$\Delta(p(k)\Delta x(k))+q(k)\ \phi(x(k))=h(k),$$

see for example Szmanda [5], He [4] or Cheng et al. [3]. Therefore, we will confine ourselves to a few sample Theorems.

First, consider the homogeneous part of (3.8), i.e.

$$\Delta^2((k-1)U(k-1)+\frac{kp(k)}{2}\ \psi(U(k))\leq 0,\qquad k\geq 1.$$

Let $v(k)=kU(k)$ for $k\geq 0$. Then we can rewrite it in the following form

$$\Delta^2 v(k-1)+\frac{kp(k)}{2}\psi\left(\frac{v(k)}{k}\right)\leq 0,\quad k\geq 1. \tag{3.10}$$

Suppose there is a positive constant M such that $\psi(x)\geq Mx$ for $x>0$. Then for any eventually positive solution $\{v(k)\}$ of (3.10), we have

$$\Delta^2 v(k-1)+\frac{Mp(k)}{2}v(k)\leq 0 \tag{3.11}$$

for all large k.

Theorem 3.4. Assume that (H2) *holds. Suppose there is a positive constant M such that $\psi(x)\geq Mx$ for $x>0$. Suppose further that*

$$\sum_{k=0}^{\infty}p(k)=\infty. \tag{3.12}$$

Then (3.10) *cannot have an eventually positive solution.*

The proof follows from the same argument used in the proof of Theorem 3.1. Or, we can employ the well known fact (see e.g. Cheng et al. [3, Theorem 4.1]) that the condition (3.12) is sufficient for the non-existence of eventually positive solutions of (3.11).

Theorem 3.5. Assume that (H2) *holds. Suppose*

$$\liminf_{k\to\infty} \frac{1}{k+1-N} \sum_{n=N}^{k} \sum_{j=N}^{n} jF(j) = -\infty.$$

Then the recurrence relation (3.8) *cannot have an eventually positive solution.*

Proof. From (3.8), we see that if $\{U(k)\}$ is an eventually positive solution, then

$$\Delta^2((k-1)U(k-1)) \le \frac{k}{2}F(k).$$

By means of repeated summation, we see that

$$(k+1)U(k+1) \le NU(N) + (k+1-N)\Delta((N-1)U(N-1)) + \\ + \sum_{n=N}^{k} \sum_{j=N}^{n} \frac{jF(j)}{2},$$

where $k > N$ and N is some large integer. Dividing the above inequality by $k+1-N$ and taking limit inferior on both sides of the resulting inequality, we see that $U(k)$ is eventually negative, which is a contradiction. ■

Finally, we state a result which is similar to Theorem 3.3, and its proof is also similar.

Theorem 3.6. Assume that (H2) *holds. Suppose* ψ *is non-decreasing on* $(0,\infty)$. *Suppose further that there is a sequence* $\{R(k)\}$ *which has a non-positive subsequence* $\{R(k_j)\}$ *and satisfies* $\Delta^2 R(k) = kF(k)/2$ *for all large k, and that*

$$\liminf_{k\to\infty} \frac{1}{2(k+1-N)} \sum_{n=N}^{k} \sum_{j=N}^{n} jF(j) - jp(j)\psi\left(\frac{R^+(j)}{j-1}\right) = -\infty.$$

Then (3.8) *cannot have an eventually positive solution.*

4. Non-existence criteria

According to the results in the previous Sections, it is now easy to obtain non-exuistence criteria for the discrete elliptic equation (1.1). For example, by means of Theorems 2.3 and 3.1, we have the following result.

Theorem 4.1. Suppose (H1) *holds. Suppose further that there is a positive constant M such that* $\psi(x) \geq Mx$ *for* $x > 0$ and $Mp(k) \geq 2$ *for* $k \geq 1$. *Then the elliptic equation*

(4.1) $$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + q(i,j,u(i,j)) = 0, \qquad (i,j) \in Z^2$$

cannot have any positive solutions.

Similarly, a combination of Theorems 2.3 and 3.3 gives rise to an alternate non-existence criteria for (4.1).

Theorem 4.2. Suppose (H1) *holds and that there is a positive constant M such that* $\psi(x) \geq Mx$ *for* $x > 0$ and $Mp(k) \geq 2$ *for* $k \geq 1$. *Suppose further that there is a sequence* $\{R(k)\}$ *which has a non-positive subsequence* $\{R(k_j)\}$ *and satisfies* $\Delta^2 R(k-1) = kF(k)$ *for all large k, where* $F(k)$ *is defined in (2.5). If*

$$\liminf_{k\to\infty} \sum_{n=N}^{k} \sum_{j=N}^{n} \frac{\{jF(j) - (Mp(j)-2)R^+(j) - R^+(j-1)\}}{k+1-N} = -\infty.$$

then (1.1) *cannot have any positive solutions.*

We will state our final result of this paper which follows from Theorems 2.4 and 3.4.

Theorem 4.3. Suppose (H1) *holds. Suppose further that there is a positive constant M such that* $\psi(x) \geq Mx$ *for* $x > 0$ *and*

$$\sum_{k=1}^{\infty} p(k) = \infty.$$

Then (4.1) *cannot have a positive solution* $\{u(i,j)\}$ *which satisfies* $\Delta_1 u(i,0) \leq 0$ *for* $i \geq 0$, $\Delta_1 u(i,0) \geq 0$ *for* $i \leq -1$, $\Delta_2 u(0,j) \leq 0$ *for* $j \geq 0$, $\Delta_2 u(0,j) \geq 0$ *for* $j \leq -1$.

There are many other non-existence criteria which can be obtained by applying our results stated in the last Section or some of the existing Theorems in the literature. However, it would be of interest to obtain necessary conditions for the non-existence of positive solutions of our discrete elliptic equation (1.1).

Finally, we remark that the concept of an eventually positive solution $\{u(i,j)\}$ for our discrete equation can be defined as one which is positive for $|i|+|j| \geq N$, where N is some positive integer. We may further define an oscillatory solution as one which is neither eventually positive nor eventually negative. In view of these definitions, our non-existence criteria will become oscillation theorems when (1.1) is a linear homogeneous equation, i.e. and

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