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ON THE BOUNDEDNESS OF SOLUTIONS FOR SYSTEM OF TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

ABSTRACT. The sufficient conditions are given for the boundedness of solutions for a system of two-dimensional linear Volterra integral equations.

KEY WORDS: system of Volterra integral equations, boundedness of solutions, two-dimensional integral inequalities.

1. INTRODUCTION

The following system of Volterra integral equations

$$u_i(x,y) = w_i(x,y) + \sum_{j=1}^{n} \int_0^x \int_0^y k_{ij}(x,y,s,t) u_j(s,t) ds dt, \quad (i = 1,2,\dots,n) \tag{1}$$

is considered, where $w_i (i=1,2,\dots,n)$ are continuous in $D\{(x,y): x,y \geq 0\}$ and k_{ij}, $(i,j = 1,2,\dots,n)$ are continuous in $\Omega = \{(x,y,t,s): 0 \leq t \leq x < \infty,$ $0 \leq s \leq y < \infty\}$.

From (1) we get

$$u(x,y) \leq w(x,y) + \int_0^x \int_0^y K(x,y,s,t) u(s,t)\, ds\, dt, \tag{2}$$

where

$$u(x,y) = \sum_{i=1}^{n} |u_i(x,y)| \geq 0,$$

$$w(x,y) = \sum_{i=1}^{n} |w_i(x,y)| \geq 0,$$

$$K(x,y,s,t) = \sum_{i=1}^{n} \max_{1 \leq j \leq n} |k_{ij}(x,y,s,t)| \geq 0.$$

If a continuous function $u(x,y)$ satisfies inequality (2), then there exists a continuous and nonnegative function $v(x,y)$, such that

(3) $$u(x,y) = w(x,y) - v(x,y) + \int_0^x \int_0^y K(x,y,s,t)u(s,t)\,ds\,dt .$$

It is well known, that

(4) $$u(x,y) = w(x,y) - v(x,y) + \int_0^x \int_0^y R(x,y,s,t)[w(s,t) - v(s,t)]\,ds\,dt ,$$

where

$$R(x,y,s,t) = \sum_{j=0}^{\infty} K_j(x,y,s,t)$$

and

$$K_j(x,y,s,t) = \int_s^x \int_t^y K(x,y,p,q)K_{j-1}(p,q,s,t)\,dp\,dq , \qquad j = 1,2,...$$

$$K_0(x,y,s,t) = K(x,y,s,t) .$$

From (4) we obtain

(5) $$u(x,y) \le w(x,y) + \int_0^x \int_0^y R(x,y,s,t)w(s,t)\,ds\,dt$$

or

(6) $$u(x,y) \le g(x,y)[1 + \int_0^x \int_0^y R(x,y,s,t)\,ds\,dt],$$

where

(7) $$g(x,y) = \sup\{w(s,t) : 0 \le s \le x,\ 0 \le t \le y\} .$$

Then it is clear.

Remark 1. If

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}}\, g(x,y) < \infty$$

and

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}} \int_0^x \int_0^y R(x,y,s,t)\,ds\,dt < \infty ,$$

then a solution $\{u_i(x,y)\}$, $i = 1,2,...,n$, of system (1) is bounded in D.

It gives the general sufficient conditions for a boundedness of solutions for a system of two-dimensional Volterra integral equations.

In fact, it is very difficult to find the resolvent kernel and we will estimate it.

2. SOME VARIANTS OF TWO-DIMENSIONAL INTEGRAL INEQUALITIES

Consider (2) with special cases of the kernel $K(x,y,s,t)$.

I. $K(x,y,s,t) = a(s,t) \geq 0$

Then we get the following iterated kernels

$$K_1(x,y,s,t) = \int_s^x \int_t^y a(p,q) a(s,t)\, dp\, dq = a(s,t) P(x,y),$$

where

$$P(x,y) = \int_s^x \int_t^y a(p,q)\, dp\, dq, \quad a(x,y) = \frac{\partial^2 P}{\partial x \partial y},$$

$$K_2(x,y,s,t) = \int_s^x \int_t^y a(p,q) K_1(p,q,s,t)\, dp\, dq =$$

$$= \int_s^x \int_t^y a(p,q) a(s,t) P(p,q)\, dp\, dq =$$

$$= a(s,t) \int_s^x \int_t^y a(p,q) P(p,q)\, dp\, dq = a(s,t) \int_s^x \int_t^y \frac{\partial^2 P}{\partial x \partial y} P(p,q)\, dp\, dq \leq$$

$$\leq a(s,t) \frac{1}{2} \int_s^x \int_t^y \frac{\partial^2 (P^2)}{\partial p \partial q}\, dp\, dq = a(s,t) \frac{[P(x,y)]^2}{2!}.$$

By the induction we obtain

$$K_1(x,y,s,t) \leq a(s,t) \frac{[P(x,y)]^i}{i!}.$$

Then

$$R(x,y,s,t) = \sum_{i=0}^{\infty} K_i(x,y,s,t) \leq a(s,t) \sum_{i=0}^{\infty} \frac{[P(x,y)]^i}{i!} =$$

$$= a(s,t) \exp\left[\int_s^x \int_t^y a(p,q)\, dp\, dq \right].$$

Hence we have an estimation for the resolvent kernel

(8) $$R(x,y,s,t) \leq a(s,t) \exp\left[\int_s^x \int_t^y a(p,q)\, dp\, dq \right].$$

In this way we proved.

Theorem 1. Let $u(x,y)$, $w(x,y)$, $a(x,y)$ be nonnegative continuous functions in D. If $u(x,y)$ satisfies the inequality

$$u(x,y) \le w(x,y) + \int_0^x \int_0^y a(s,t) u(s,t)\, ds\, dt\,, \tag{9}$$

then

$$u(x,y) \le g(x,y) \left\{ 1 + \int_0^x \int_0^y a(s,t) \exp\left[\int_s^x \int_t^y a(p,q)\, dp\, dq \right] ds\, dt \right\}, \tag{10}$$

where $g(x,y)$ is defined by formula (7).

Remark 2. If $a(s,t) = A_1(s) A_2(t) > 0$ we get better results than those obtained in [1] and [2], because we have

$$R(x,y,s,t) = A_1(s) A_2(t) \sum_{n=0}^{\infty} \frac{\left(\int_s^x \int_t^y A_1(p) A_2(q)\, dp\, dq \right)^n}{(n!)^2} <$$

$$< A_1(s) A_2(t) \exp\left[\int_s^x \int_t^y A_1(p) A_2(q)\, dp\, dq \right].$$

Lemma 1. If $a(x,y)$ is a continuous and nonnegative function in D, then

$$1 + \int_0^x \int_0^y a(s,t) \exp\left[\int_s^x \int_t^y a(p,q)\, dp\, dq \right] ds\, dt \le \exp\left[\int_0^x \int_0^y a(s,t)\, ds\, dt \right]. \tag{11}$$

Proof. To prove (11), it suffices to pose for fixed $x, y > 0$

$$F(s,t) = \exp\left[\int_s^x \int_t^y a(p,q)\, dp\, dq \right], \qquad 0 \le s \le x,\ 0 \le t \le y\,.$$

We observe that

$$F(0,0) = \exp\left[\int_0^x \int_0^y a(p,q)\, dp\, dq \right]$$

$$F(x,0) = F(0,y) = F(x,y) = 1$$

$$\frac{\partial F}{\partial t} = -F(s,t)\int_s^x a(p,t)\,dp$$

$$\frac{\partial^2 F}{\partial s \partial t} = -\frac{\partial F}{\partial t}\int_t^y a(s,q)\,dq + a(s,t)F(s,t) =$$

$$= F(s,t)\int_s^x a(p,t)\,dp\int_t^y a(s,q)\,dq + u(s,t)F(s,t) \ge a(s,t)F(s,t).$$

Integrating the above equality on the rectangle $0 \le s \le x,\ 0 \le t \le y$ we get inequality (11).

Remark 3. If $a(s,t) > 0$, then inequality (11) is strict for $x, y > 0$.

Example. If $a(s,t) = 1$, then inequality (11) becomes

$$1 + \sum_{n=1}^{\infty}\frac{(xy)^n}{n!n} < 1 + \sum_{n=1}^{\infty}\frac{(xy)^n}{n!}.$$

Corollary 1. If additionally w(x,y) in (9) *is nondecreasing respect to each variable, then* $g(x,y) = w(x,y)$ *and*

(12) $$u(x,y) \le w(x,y)\left\{1 + \int_0^x\int_0^y a(s,t)\exp\left[\int_s^x\int_t^y a(p,q)\,dp\,dq\right]ds\,dt\right\}.$$

Remark 4. The inequality (12) is stronger than

(13) $$u(x,y) \le w(x,y)\exp\left[\int_0^x\int_0^y a(s,t)\,ds\,dt\right],$$

which was obtained in [1] p. 491-492 and [2] p. 109 (see Lemma 1 and Corollary 1).

For estimate of the solution of integral inequality (2) we do not know the results in the case $K(x,y,s,t) = b(x,y)a(s,t)$, which we will consider bellow.

II. $K(x,y,s,t) = b(x,y)a(s,t)$

Theorem 2. Let $u(x,y)$, $w(x,y)$, $a(x,y)\cdot b(x,y)$ *be nonnegative and continuous functions in D* $(b(x,y) > 0)$. *If* $u(x,y)$ *satisfies the inequality*

(14) $$u(x,y) \le w(x,y) + b(x,y)\int_0^x\int_0^y a(s,t)u(s,t)\,ds\,dt\,,$$

then

(15) $$u(x,y) \le b(x,y)h(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right],$$

where

$$h(x,y) = \sup\left\{\frac{w(s,t)}{b(s,t)}:\ 0 \le s \le x,\ 0 \le t \le y\right\}.$$

Proof. From (14) we get

$$\frac{u(x,y)}{b(x,y)} \le \frac{w(x,y)}{b(x,y)} + \int_0^x\int_0^y s(s,t)u(s,t)\,ds\,dt$$

or

$$v(x,y) \le h(x,y) + \int_0^x\int_0^y a(s,t)b(s,t)v(s,t)\,ds\,dt\,,$$

where

$$v(x,y) = \frac{u(x,y)}{b(x,y)}.$$

Using inequality (13) we obtain

$$v(x,y) \le h(x,y)\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right].$$

From the above considerations we have (15).

Corollary 2. If additionally $\frac{w(x,y)}{b(x,y)}$ *is nondecreasing in D, then inequality* (14) *leads to*

(16) $$u(x,y) \le w(x,y)\int_0^x\int_0^y a(s,t)b(s,t)dsdt\,.$$

III. $K(x,y,s,t)=\sum_{p=1}^{m} A_p(x,y)B_p(s,t)$.

Theorem 3. Let $A_p(x,y)$, $B_p(x,y)$, $(p=1,2,\dots,m)$ $w(x,y)$, $u(x,y)$ *be nonnegative and continuous function in* D $(\sup\limits_{1\le p\le m} A_p(x,y)>0)$. *If* $u(x,y)$ *satisfies the inequality*

(17) $$u(x,y)\le w(x,y)+\int_0^x\int_0^y\sum_{p=1}^{m} A_p(x,y)B_p(s,t)u(s,t)\,ds\,dt,$$

then

(18) $$u(x,y)\le \inf_{1,2}\left[u_1(x,y),u_2(x,y)\right],$$

where

(19) $$u_1(x,y)=\sup_{1\le p\le m} A_p(x,y)H(x,y)\exp\left[\int_0^x\int_0^y \sup_{1\le p\le m} A_p(s,t)\sum_{p=1}^{m} B_p(s,t)\,ds\,dt\right],$$

(20) $$u_2(x,y)=\sum_{p=1}^{m} A_p(x,y)\overline{H}(x,y)\exp\left[\int_0^x\int_0^y\sum_{p=1}^{m} A_p(s,t)\sup_{1\le q\le m} B_q(s,t)\,ds\,dt\right].$$

Here

$$H(x,y)=\sup\left\{w(s,t)\Big/[\sup_{1\le p\le m} A_p(s,t)]:\ 0\le s\le x,\ 0\le t\le y\right\},$$

$$\overline{H}(x,y)=\sup\left\{w(s,t)\Big/[\sum_{p=1}^{m} B_p]:\ 0\le s\le x,\ 0\le t\le y\right\}.$$

Proof. From (17) we obtain the following inequality

$$u(x,y)\le w(x,y)+\sup_{1\le p\le m} A_p(x,y)\int_0^x\int_0^y\sum_{q=1}^{m} B_q(s,t)u(s,t)\,ds\,dt.$$

Using Theorem 2 we get

$$u(x,y)\le u_1(x,y).$$

By similar way from (17) we obtain

$$u(x,y)\le w(x,y)+\sum_{p=1}^{m} A_p(x,y)\int_0^x\int_0^y \sup_{1\le q\le m} B_q(s,t)u(s,t)\,ds\,dt.$$

Then from Theorem 2 we have

$$u(x,y) \le u_2(x,y).$$

In this manner the proof is finished.

IV. $K(x,y,s,t) \ge 0$

Theorem 4. Let $u(x,y)$, $w(x,y)$, $K(x,y,s,t)$ *be nonnegative and continuous functions in D and* Ω, *respectively and*

$$p(x,y) = \sup_{\substack{0\le s\le x\\ 0\le t\le y}} K(x,y,s,t) > 0. \tag{21}$$

If $u(x,y)$ *satisfies*

$$u(x,y) \le w(x,y) + \int_0^x\int_0^y K(x,y,s,t)u(s,t)\,ds\,dt, \tag{22}$$

then

$$u(x,y) \le p(x,y)\sup\left\{\frac{w(s,t)}{p(s,t)} : 0\le s\le x,\ 0\le t\le y\right\} \exp\left[\int_0^x\int_0^y p(s,t)\,ds\,dt\right]. \tag{23}$$

Proof. In virtue of assumptions, inequality (22) can be replaced by

$$u(x,y) \le w(x,y) + p(x,y)\int_0^x\int_0^y u(s,t)\,ds\,dt.$$

Using Theorem 2 we get inequality (23).

Remark 5. If we assume additionally in Theorem 4 that $\frac{w(x,y)}{p(x,y)}$ is nondecreasing in *D*, then

$$u(x,y) \le w(x,y)\exp\left[\int_0^x\int_0^y p(s,t)\,ds\,dt\right].$$

Remark 6. Assume additionally in Theorem 4 the function $K(x,y,s,t)$ is nondecreasing with respect to variables *s*, *t*, we get

$$p(x,y) = K(x,y,x,y)$$

and

$$u(x,y) \le K(x,y,x,y)\sup\left\{\frac{w(s,t)}{K(s,t,s,t)} : 0 \le s \le x,\, 0 \le t \le y\right\} \times$$

$$\times \exp\left[\int_0^x\int_0^y K(s,t,s,t)\,ds\,dt\right].$$

Theorem 5. Let $u(x,y)$, $w(x,y)$, $K(x,y,s,t)$ *be nonnegative and continuous functions in D and* Ω, *respectively.*

If $K(x,y,s,t)$ *is nondecreasing respect to each variable x, y and the function* $u(x,y)$ *satisfies the inequality*

$$u(x,y) \le w(x,y) + \int_0^x\int_0^y K(x,y,s,t)u(s,t)\,ds\,dt, \tag{24}$$

then

$$u(x,y) \le g(x,y)\exp\left[\int_0^x\int_0^y K(s,t,s,t)\,ds\,dt\right], \tag{25}$$

where $g(x,y)$ *is defined by formula* (7).

Proof. From inequality (24) we get

$$u(x,y) \le w(x,y) + \int_0^x\int_0^y K(s,t,s,t)u(s,t)\,ds\,dt.$$

Using inequality (13) we obtain (25).

Corollary 3. If additionally $w(x,y)$ *in (24) is nondecreasing respect to each variable, then in (25)* $g(x,y) = w(x,y)$.

3. Applications of Two-Dimensional Volterra Integral Inequalites

Theorem 6. If $w_i(x,y)$ $(i = 1,\dots,n)$ *and* $k_{ij}(x,y,s,t)$ $(i,j = 1,\dots,n)$ *are continuous functions in D and* Ω, *respectively and*

$$\sum_{i=1}^{n}\max\left\{\left|k_{ij}(x,y,s,t)\right| : 1 \le j \le n\right\} = a(s,t) \quad \text{in } \Omega,$$

then

$$(26)\qquad \sum_{i=1}^{n}|u_i(x,y)| \le \sup\left\{\sum_{i=1}^{n}|w_i(s,t)|:\ 0\le s\le x,\ 0\le t\le y\right\}\cdot \cdot\exp\left[\int_0^x\int_0^y a(s,t)\,ds\,dt\right].$$

Moreover, if

$$(27)\qquad \overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\sup\left\{\sum_{i=1}^{n}|w_i(s,t)|:\ 0\le s\le x,\ 0\le t\le y\right\}<\infty$$

and

$$(28)\qquad \overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\int_0^x\int_0^y a(s,t)\,ds\,dt<\infty,$$

then a solution $\{u_i(x,y)\}$, $i=1,\dots,n$ *of system* (1) *is bounded in D.*

Proof. The inequality (26) follows from Theorem 1 and Lemma 1. If the conditions (27) and (28) are satisfied, then the boundedness of solutions follows.

Using Theorem 2 we get.

Theorem 7. Let the assumptions of Theorem 6 *are fulfilled and*

$$\sum_{i=1}^{n}\max\left\{|k_{ij}(x,y,s,t)|:\ 1\le j\le n\right\}=b(x,y)a(s,t),$$

then

$$(29)\qquad \sum_{i=1}^{n}|u_i(x,y)|\le b(x,y)\cdot\sup\left\{\sum_{i=1}^{n}|w_i(s,t)|/b(s,t):\ 0\le s\le x,\ 0\le t\le y\right\}\cdot \cdot\exp\left[\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt\right].$$

If moreover

$$(30)\qquad \overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}b(x,y)\cdot\sup\left\{\sum_{i=1}^{n}|w_i(s,t)|/b(s,t):\ 0\le s\le x,\ 0\le t\le y\right\}<\infty$$

and

$$(31)\qquad \overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\int_0^x\int_0^y a(s,t)b(s,t)\,ds\,dt<\infty,$$

then the solutions $\{u_i(x,y)\}$, $i=1,\dots,n$, *of* (1) *are bounded in D.*

Similarly, from Theorem 3, we obtain.

Theorem 8. Let the assumptions of Theorem 6 are satisfied and

$$\sum_{i=1}^{n} \max\left\{\left|k_{ij}(x,y,s,t)\right| : 1 \le j \le n\right\} = \sum_{p=1}^{m} A_p(x,y) B_p(s,t),$$

then we obtain the inequality

$$\sum_{i=1}^{n} \left|u_i(x,y)\right| \le \inf\left\{u_1(x,y),\, u_2(x,y) : 1,2\right\}, \tag{32}$$

where

$$u_1(x,y) = \sup\left\{A_p(x,y) : 1 \le p \le m\right\} \cdot$$

$$\cdot \sup\left\{\frac{w(s,t)}{\sup\left\{A_p(s,t) : 1 \le p \le m\right\}} : 0 \le s \le x,\, 0 \le t \le y\right\} \cdot$$

$$\cdot \exp\left[\int_0^x \int_0^y \sup\left\{A_p(s,t) : 1 \le p \le m\right\} \sum_{q=1}^{m} B_q(s,t)\, ds\, dt\right]$$

and

$$u_2(x,y) = \sum_{p=1}^{m} A_p(x,y) \cdot \sup\left\{ w(s,t) \Big/ \sum_{p=1}^{m} A_p : 0 \le s \le x,\, 0 \le t \le y\right\} \cdot$$

$$\cdot \exp\left[\int_0^x \int_0^y \sum_{p=1}^{m} A_p(s,t) \sup\left\{B_q(s,t) : 1 \le q \le m\right\} ds\, dt\right].$$

Assuming additionally,

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}} \sup_{1\le p\le m} A_p(x,y) \cdot \sup_{\substack{0\le s\le x \\ 0\le t\le y}} \left\{ w(s,t) \Big/ \sup_{1\le p\le m} A_p(s,t) \right\} < \infty, \tag{33}$$

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}} \int_0^x \int_0^y \sup_{1\le p\le m} A_p(s,t) \sum_{q=1}^{m} B_q(s,t)\, ds\, dt < \infty, \tag{34}$$

or

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}} \sum_{p=1}^{m} A_p(x,y) \cdot \sup_{\substack{0\le s\le x \\ 0\le t\le y}} w(s,t) \Big/ \sum_{p=1}^{m} A_p(s,t) < \infty \tag{33'}$$

and

$$\overline{\lim_{\substack{x\to\infty \\ y\to\infty}}} \int_0^x \int_0^y \sum_{p=1}^{m} A_p(s,t) \sup_{1\le q\le m} B_q(s,t)\, ds\, dt < \infty, \tag{34'}$$

where

$$w(x,y)=\sum_{i=1}^{n}|w_i(x,y)|,$$

we get the bounded solutions $\{u_i(x,y)\}$, $i=1,\dots,n$, *of system* (1).

Theorem 9. If the assumptions of Theorem 6 *are fulfilled and*

$$\sup_{\substack{0\le s\le x\\ 0\le t\le y}}\sum_{i=1}^{n}\max_{1\le j\le n}|k_{ij}(x,y,s,t)|=p(x,y)$$

then

$$\sum_{i=1}^{n}|u_i(x,y)|\le p(x,y)\cdot\sup_{\substack{0\le s\le x\\ 0\le t\le y}}w(s,t)/p(s,t)\cdot\exp\left[\int_0^x\int_0^y p(s,t)\,ds\,dt\right] \tag{35}$$

and the solutions $u_i(x,y)$ $(i=1,\dots,n)$ *of system (1) are bounded in the infinity if*

$$\overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\,p(x,y)\cdot\sup_{\substack{0\le s\le x\\ 0\le t\le y}}w(s,t)/p(s,t)<\infty \tag{36}$$

and

$$\overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\int_0^x\int_0^y p(s,t)\,ds\,dt<\infty, \tag{37}$$

where

$$w(x,y)=\sum_{i=1}^{n}|w_i(x,y)|.$$

Proof folows from Theorem 4.

Similarly, using Theorem 5 we have.

Theorem 10. If assumptions of Theorem 6 *are satisfied and* $\sum_{i=1}^{n}\max_{1\le j\le n}|k_{ij}(x,y,s,t)|$ *is nonincreasing function respect to each variable s and t, then*

$$\sum_{i=1}^{n}|u_i(x,y)|\le g(x,y)\exp\left[\int_0^x\int_0^y\sum_{i=1}^{n}\max_{1\le j\le n}|k_{ij}(s,t,s,t)|\,ds\,dt\right], \tag{38}$$

where $g(x,y)$ *is defined by formula* (7) *and the boundedness of solutions* $\{u_i(x,y)\}$ $(i=1,\dots,n)$ *for system* (1) *follows from the following conditions:*

(39)
$$\overline{\lim_{\substack{x\to\infty\\ y\to\infty}}}\, g(x,y) < \infty$$

and

(40)
$$\overline{\lim_{\substack{x\to\infty\\ y\to\infty}}} \int_0^x \int_0^y \sum_{i=1}^{n} \max_{1\le j\le m} \left|k_{ij}(s,t,s,t)\right| ds\,dt < \infty .$$

This paper is an extension of paper [3]. Results of one are based on certain variants of two-dimensional Volterra integral equations proved here.

These results can be extended on the classes L and L^2.

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