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ON THE CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY EXPECTED VALUE OF RECORD VALUE WITH RANDOM INDEX

ABSTRACT. In this paper we give some characterizations of the exponential distribution based on the expected value of record value. These characterizations are considered for distribution functions belonging to IFRA or DFRA. The index of record value has the geometric distribution.
KEY WORDS: record value, expected value, IFRA, DFRA.

1. INTRODUCTION

Let X be a nonnegative random variable, and let $F(x)=P(X<x)$ be its distribution function. Let $\overline{F}(x)=1-F(x)$ be the survival function of X, and let $E(X)$ be the expected value of X.

We say that F has increasing failure rate average ($F\in$IFRA) if $-\frac{1}{x}\log\overline{F}(x)$ is nondecreasing in $x>0$. Similarly, F has decreasing failure rate average ($F\in$DFRA) if $-\frac{1}{x}\log\overline{F}(x)$ is nonincreasing in $x>0$.
It is known (see [1]) that $F\in$IFRA if and only if

(1) $$\overline{F}(\alpha x)\geq[\overline{F}(x)]^{\alpha} \quad \text{for all} \quad 0<\alpha<1 \quad \text{and } x>0,$$

and $F\in$DFRA if and only if

(2) $$\overline{F}(\alpha x)\leq[\overline{F}(x)]^{\alpha} \quad \text{for all} \quad 0<\alpha<1 \quad \text{and} \quad x>0.$$

We say that X is exponentially distributed if

(3) $$F(x)=1-\mathrm{e}^{-\lambda x} \qquad (x>0) \qquad \text{for some } \lambda>0.$$

We say that ν is geometrically distributed if

(4) $$P(\nu=k)=p(1-p)^{k-1}, \quad k=1,2,..., \quad \text{for some } 0<p<1.$$

Let $(X_n, n\geq 1)$ be a sequence of independent and identically distributed random variables. Define the sequence of record times $(L(n), n\geq 1)$ in the following way $L(1)=1$, $L(n)=\min\{j:X_j>X_{L(n-1)}\}$, $n\geq 2$.

Then the sequence $(R_n, n \geq 1)$, where $R_n = X_{L(n)}$, is called the sequence of record values of $(X_n, n \geq 1)$.

The following lemmas are valid:

Lemma 1. ([3]). Let $X \geq 0$ *be a random variable with distribution function F. Assume that E(X) is finite. Then*

$$E(X) = \int_0^\infty [1 - F(x)]dx .$$

Lemma 2. ([2]). Let $(X_n, n \geq 1)$ *be a sequence of identically distributed random variables with finite expected value* $E(X_1)$. *Let N be a random variable with values in* $\{1,2,...\}$. *Moreover, let N be distributed independently of the sequence* $(X_n, n \geq 1)$. *Let us suppose that the expected value* $E(N)$ *is finite. Then the expected value of the random variable*

$$Y = \sum_{k=1}^{N} X_k$$

is finite and

$$E(Y) = E(X_1)E(N).$$

2. Results

The following theorem is given in [5] (Theorem 4.5.2, p.129):

Let $(X_n, n \geq 1)$ be a sequence of independent and identically distributed positive random variables with a continuous distribution function F. Assume that the limit

$$\lim_{x \to 0^+} \frac{F(x)}{x} = \lambda, \qquad 0 < \lambda < \infty. \tag{5}$$

Moreover, assume that ν is a geometric random variable independent of the sequence $(X_n, n \geq 1)$, and condition (4) holds. The random variables X_1 and pR_ν are identically distributed if and only if F is a distribution function of the exponential law.

Theorem 1. Let $(X_n, n \geq 1)$ *be a sequence of independent and identically distributed positive random variables with a continuous distribution function F. Let condition (5) be satisfied and* $F \in \mathrm{IFRA}$. *Assume that* ν *is a geometric random variable independent of the sequence* $(X_n, n \geq 1)$, *and condition* (4) *holds. Let $E(R_\nu)<\infty$. Then X_1 has the distribution function defined in (3) if and only if*

(6) $$E(X_1)=E(pR_\nu).$$

Proof. Let $F\in$IFRA. Then $E(X_1)<\infty$. It is known ([5]) that the distribution function of R_ν is of the form

(7) $$F_{R_\nu}(x)=1-[\bar{F}(x)]^p, \qquad x>0.$$

Formula (7) gives the equality

(8) $$\bar{F}_{pR_\nu}(z)=\left[\bar{F}(z/p)\right]^p \qquad \text{for} \qquad z>0.$$

By Lemma 1 we can write condition (6) as

(9) $$\int_0^\infty \left\{\bar{F}(z)-\left[\bar{F}(z/p)\right]^p\right\}dz=0.$$

Substituting $z=py$ in (9) we get

(10) $$\int_0^\infty \{\bar{F}(py)-[\bar{F}(y)]^p\}\,p\,dy=0.$$

Since $F\in$IFRA, inequality (1) holds. Hence

$$\bar{F}(py)\ge[\bar{F}(y)]^p \qquad \text{for} \qquad y>0.$$

From (10) we obtain

(11) $\bar{F}(py)=[\bar{F}(y)]^p$ for almost all $y>0$ and a fixed $0<p<1$.

Since $\lim\limits_{x\to 0+}\frac{F(x)}{x}=\lambda$, $0<\lambda<\infty$, it follows from (11) that $\bar{F}(x)=\exp(-\lambda x)$, $x>0$, $\lambda>0$, (see [5], p.130).

Now suppose that X_1 has distribution function (3). Then from (7) we obtain that $F_{R_\nu}(y)=1-\mathrm{e}^{-\lambda py}$ for $y>0$, $\lambda>0$, $0<p<1$, and $E(pR_\nu)==\frac{1}{\lambda}=E(X_1)$.

Remark 1. Theorem 1 is also true if the condition $F\in$IFRA is replaced by the following one: $F\in$DFRA and $EX_1<\infty$. Here in the proof we use inequality (2).

We know that if there are satisfied the assumptions of Theorem 1, then the random variables

$$\sum_{i=1}^{\nu}X_i \qquad \text{and} \qquad R_\nu$$

are identically distributed if and only if F is a distribution function of the exponential law (see[4]).

Theorem 2. Assume that there are satisfied the assumptions of Theorem1. Then X_1 has the distribution function defined in (3) if and only if

$$E\left(\sum_{i=1}^{\nu} X_i\right) = E(R_\nu)\,. \tag{12}$$

Proof. If X_1 has exponential distribution function (3), then

$$E\left(\sum_{i=1}^{\nu} X_i\right) = E(R_\nu) = \frac{1}{p\lambda}\,.$$

Now let us suppose that condition (12) is satisfied. By Lemma 2 we have

$$E\left(\sum_{i=1}^{\nu} X_i\right) = E(X_1)E(\nu)\,.$$

Since $E(\nu) = \frac{1}{p}$ and Lemma 1 is valid, we conclude that

$$E\left(\sum_{i=1}^{\nu} X_i\right) = \frac{1}{p}\int_0^\infty \overline{F}(z)dz\,.$$

From (8) we obtain

$$E(R_\nu) = \frac{1}{p}\int_0^\infty [\overline{F}(z/p)]^p dz\,.$$

Hence formula (12) takes the form

$$\frac{1}{p}\int_0^\infty \{\overline{F}(z) - [\overline{F}(z/p)]^p\}dz = 0\,.$$

Next, analogously as in the proof of Theorem 1,we get that F has form (3).

Remark 2. Theorem 2 is also true if the condition $F \in$IFRA is replaced by the following one: $F \in$DFRA and $E(X_1) < \infty$.

REFERENCES

[1] R.E. Barlow, F. Proschan, *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, Inc., 1975.

[2] M. Fisz, *Rachunek prawdopodobieństwa i statystyka matematyczna,* Warszawa, 1967.

[3] J. Galambos, S. Kotz, *Characterizations of Probability Distributions,* Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[4] M. Iwińska, On the Characterization of the Exponential Distribution by Record Values with a Random Index, *Discussiones Mathematicae, Algebra and Stochastic Methods* (in print).

[5] A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, *Characterization of Distributions by the Method of Intensively Monotone Operators*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

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