FASCICULI MATHEMATICI

Nr 28 1998

DEDICATED TO PROFESSOR DOBIESŁAW BOBROWSKI
ON THE OCCASION OF HIS 70th BIRTHDAY

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ON THE STRUCTURE OF THE SET OF SOLUTIONS OF THE WEIGHTED CAUCHY PROBLEM FOR EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT. The weighted Cauchy problem

$$\frac{dx(t)}{dt} = f(x)(t), \qquad \lim_{t\to a}\frac{\|x(t)-c_0\|}{h(t)} = 0$$

is considered, where $f: C([a,b];R^n) \to L_{loc}((a,b];R^n)$ is a singular Volterra operator, $c_0 \in R^n$, $h:[a,b] \to [0,+\infty)$ is a continuous nondecreasing function, positive on $(a,b]$, and $\|\cdot\|$ is a norm in R^n. The conditions are found under which this problem possesses Kneser's property, i.e., the set of all its noncontinuable solutions is closed and connected in the topology of the space $C([a,b];R^n)$.

KEY WORDS: Evolution singular functional differential equation, weighted Cauchy problem, structure of the set of solutions, Kneser's property.

ACKNOWLEDGEMENT: This work was supported by INTAS Grant 96-1060.

1. FORMULATION OF MAIN RESULTS

The structure of the set of solutions of the Cauchy problem for systems of ordinary differential equations for the first time has been investigated by H. Kneser [14]. The theorem proved in [14] deals with differential systems with continuous right-hand sides. An analogous result for systems with right-hand sides from the Carathéodory class is contained in [2, 6]. An interesting generalization of Kneser's theorem belongs to M. Fukuhara [5] (see also [9, 10]). Kneser's type theorems are proved by V.A. Chechik [3] and I.T. Kiguradze [8, 9] for singular systems of ordinary differential equations and by T. Kaminogo [7] and C. Corduneanu [4] for evolution functional differential equations. In the present paper, these results are generalized for evolution singular functional differential equations. Sufficient conditions of local and global solvability of the weighted Cauchy problem for such equations are given in [11-13].

Throughout the paper, the use will be made of the following notation:

R is the set of real numbers, $R_+ = [0,+\infty]$;

R^n is the space of all n-dimensional column vectors $x=(x_i)_{i=1}^n$ with the elements $x_i \in R$ $(i=1,\dots,n)$ and the norm $\|x\|=\sum_{i=1}^n |x_i|$;

$R_\rho^n=\{x\in R^n : \|x\|\le \rho\}$;

if $x=(x_i)_{i=1}^n$, then $\operatorname{sgn}(x)=(\operatorname{sgn} x_i)_{i=1}^n$;

$x\cdot y$ is the scalar product of the vestors x and $y\in R^n$;

$C([a,b];R^n)$ is the space of all continuous vector functions $x:[a,b]\to R^n$ with the norm $\|x\|_C=\max\{\|x(t)\| : a\le t\le b\}$;

$C_\rho([a,b];R^n)=\{x\in C([a,b];R^n) : \|x\|_C\le\rho\}$;

$C([a,b];R_+)=\{x\in C([a,b];R) : x(t)\ge 0 \text{ for } a\le t\le b\}$;

if $x\in C([a,b];R^n)$ and $a\le s\le t\le b$, then

$$\nu(x)(s,t)=\max\{\|x(\xi)\| : s\le\xi\le t\};$$

$L([a,b];R^n)$ is the space of all summable vector functions $x:[a,b]\to R^n$ with the norm

$$\|x\|_L=\int_a^b \|x(t)\|dt;$$

$L_{loc}([a,b];R^n)$ is the space of the vector functions $x:(a,b]\to R^n$ which are summable on each subsegment of $(a,b]$, with the toplogy of convergence in the mean on each subsegment of $(a,b]$;

$L_{loc}((a,b];R_+)=\{x\in L_{loc}((a,b];R) : x(t)\ge 0 \text{ for almost all } t\in[a,b]\}$.

Definition 1.1. An operator $f:C([a,b];R^n)\to L_{loc}((a,b];R^n)$ *is called Volterra if the equality* $f(x)(t)=f(y)(t)$ *holds almost everywhere on* (a,t_0) *for any* $t_0\in(a,b]$ *and any vector functions* x *and* $y\in C([a,b];R^n)$ *satisfying* $x(t)=y(t)$ *for* $a\le t\le t_0$.

Definition 1.2. An operator $f:C([a,b];R^n)\to L_{loc}((a,b];R^n)$ *is said to satisfy the local Carathéodory conditions if it is continuous and there exists a nondecreasing with respect to the second argument function* $\gamma:(a,b]\times R_+\to R_+$ *such that* $\gamma(\cdot,\rho)\in L_{loc}((a,b];R_+)$ *for* $\rho\in R_+$ *and for any* $x\in C([a,b];R^n)$ *the inequality*

$$\|f(x)(t)\|\le\gamma(t,\|x\|_C)$$

is fulfilled almost everywhere on (a,b).

If

$$\int_a^b \gamma(t,\rho)dt < +\infty \qquad \text{for} \quad \rho \in R_+,$$

then the operator f is called *regular*; otherwise, it is called *singular*.

In this paper, we will consider the vector functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1.1}$$

with the weighted initial condition

$$\lim_{t\to a} \frac{\|x(t)-c_0\|}{h(t)} = 0. \tag{1.2}$$

It is assumed everywhere that $f: C([a,b];R^n) \to L_{loc}((a,b];R^n)$ is a Volterra, generally speaking singular, operator satisfying the local Carathéodory conditions, $c_0 \in R^n$, and $h:[a,b] \to [0,+\infty)$ is a continuous nondecreasing function, positive on $(a,b]$.

Definition 1.3. If $b_0 \in (a,b]$, *then:*

(i) *for any* $x \in C([a,b_0];R^n)$, *by* $f(x)$ *it is understood the vector function given by the equality* $f(x)(t) = f(\overline{x})(t)$ *for* $a \le t \le b_0$, *where*

$$\overline{x}(t) = \begin{cases} x(t) & \text{for} a \le t \le b_0, \\ x(b_0) & \text{for} b_0 < t \le b_0; \end{cases}$$

(ii) *a continuous vector function* $x:[a,b_0] \to R^n$ *is called a solution of the equation* (1.1) *on the segment* $[a,b_0]$ *if* x *is absolutely continuous on each segment contained in* $(a,b_0]$ *and satisfies* (1.1) *almost everywhereon* (a,b_0);

(iii) *a vector function* $x:[a,b_0) \to R^n$ *is called a solution of the equation* (1.1) *on the semi-open interval* $[a,b_0)$ *if for each* $b_1 \in (a,b_0)$ *the restriction of* x *on* $[a,b_1]$ *is a solution of this equation on* $[a,b_1]$;

(iv) *a solution* x *on the equation* (1.1) *satisfying the initial condition* (1.2) *is called a solution of the problem* (1.1), (1.2).

Definition 1.4. A solution x *of the equation* (1.1) *defined on a segment* $[a,b_0] \subset [a,b)$ (*on a semi-open interval* $[a,b_0) \subset [a,b)$) *is called continuable if for some* $b_1 \in (b_0,b]$ ($b_1 \in [b_0,b]$) *the equation* (1.1) *has on the segment* $[a,b_1]$ *a solution* y *satisfying* $x(t) = y(t)$ *for* $a \le t \le b_0$. *Otherwise, the solution* x *is called noncontinuable.*

By $I^*(f;c_0,h)$ we denote the set of those $b^* \in (a,b]$ for which the interval of definition of every noncontinuable solution of the problem (1.1), (1.2) contains the segment $[a,b^*]$.

Definition 1.5. We say the equation (1.1) *has Kneser's property if* $I^*(f;c_0,h) \neq \varnothing$ *and for every* $b^* \in I^*(f;c_0,h)$ *the set of restrictions of noncontinuable solutions on* $[a,b^*]$ *is compact and connected in the topology of the space* $C([a,b^*];R^n)$.

Let X be the set of all noncontinuable solutions of the problem (1.1), (1.2). Then the set of points of the $(n+1)$-dimensional Euclidean space

$$\{(t,x(t)): t \in [a,b],\ x \in X\} \tag{1.3}$$

is called an integral funnel of the problem (1.1), (1.2). Obviously, if the problem (1.1), (1.2) has Kneser's property, then for any $t_0 \in I^*(f;c_0,h)$ the cross-section of the funnel (1.3) by a hyperplane $t = t_0$, i.e., the set

$$\{(t_0,x(t_0)): x \in X\}$$

is closed and connected.

Theorem 1.1. Let there exist a positive number ρ *and summable functions* p *and* $q:[a,b] \to R_+$ *such that*

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_a^t p(s)ds\right)<1, \qquad \lim_{t\to a}\left(\frac{1}{h(t)}\int_a^t q(s)ds\right)=0, \tag{1.4}$$

and let for any $y \in C_\rho([a,b];R^n)$ *the inequality*

$$f(c_0+hy)(t)\operatorname{sgn}(y(t)) \leq p(t)\nu(y)(a,t)+q(t) \tag{1.5}$$

be fulfilled almost everywhere on (a,b). *Then the problem* (1.1), (1.2) *has Kneser's property.*

Corollary 1.1. Let for any $y \in C([a,b];R^n)$ *the inequality* (1.5) *be fulfilled almost everywhere on* (a,b), *where* p *and* $q:[a,b] \to R_+$ *are summable functions satisfying* (1.4). *Then each noncontinuable solution of the problem* (1.1), (1.2) *is defined on* $[a,b]$, *and the set of all such solutions is compact and connected in the topology of the space* $C([a,b];R^n)$.

An important particular case of the equation (1.1) is the daley differential equation with a fixed intial function

$$(1.6_1)\qquad \frac{dx(t)}{dt} = f_0(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))),$$

$$(1.6_2)\qquad x(t) = c(t) \quad \text{for} \quad t < a.$$

In the sequel, while dealing with the system (1.6_1), $(1,6_2)$, we will assume that $\tau_i\colon [a,b] \to R$ $(i = 1,\dots,m)$ are measurable functions satisfying

$$(1.7)\qquad \tau_i(t) \le t \quad \text{for} \quad a \le t \le b \quad (i = 1,\dots,m)$$

and $c\colon (-\infty, a) \to R^n$ is a continuous and bounded vector function. As for the vector function $f\colon (a,b] \times R^{(m+1)n} \to R^n$, it satisfies the local Carathéodory conditions, i.e., $f_0(t,\cdot,\dots,\cdot)\colon R^{(m+1)n} \to R^n$ is continuous for almost all $t \in (a,b)$, $f_0(\cdot, x_0, x_1, \dots, x_m)\colon (a,b) \to R^n$ is measurable for all x_k $(k = 0,1,\dots,m)$, and on the set $(a,b) \times R^{(m+1)n}$ the inequality

$$\|f_0(t, x_0, x_1, \dots, x_m)\| \le \gamma_0\left(t, \sum_{k=0}^{m} \|x_k\|\right)$$

is fulfilled, where $\gamma_0\colon (a,b] \times R_+ \to R_+$ is nondecreasing in the second argument and such that $\gamma_0(\cdot, \rho) \in L_{loc}((a,b]; R_+)$ for $\rho \in R_+$.

Introduce functions $\tau_{0i}\colon [a,b] \to [a,b]$ and $u_i\colon [a,b] \times R^n \to R^n$ $(i = 1,\dots,m)$ by

$$(1.8)\qquad \tau_{0i}(t) = \begin{cases} a & \text{for } \tau_i(t) < a, \\ \tau_i(t) & \text{for } \tau_i(t) \ge a, \end{cases}$$

$$(1.9)\qquad u_i(t,x) = \begin{cases} c(\tau_i(t)) & \text{for } \tau_i(t) < a, \\ x & \text{for } \tau_i(t) \ge a. \end{cases}$$

According to the well-known conception (see, e.g., [1]), the system (1.6_1), (1.6_2) is identified with the functional differential equation

$$(1.10)\qquad \frac{dx(t)}{dt} = f_0(t, x(t), u_1(t, x(\tau_{01}(t))), \dots, u_m(t, x(\tau_{0m}(t)))),$$

and under a solution of the problem (1.6_1), (1.6_2), (1.2) is meant a solution of the problem (1.10), (1.2), i.e., a solution of the problem (1.1), (1.2) in the case where the operator f is of the form

$$(1.11)\qquad f(x)(t) = f_0(t, x(t), u_1(t, x(\tau_{01}(t))), \dots, u_m(t, x(\tau_{0m}(t)))).$$

Therefore from Theorem 1.1 we have the following

Corollary 1.2. Let for some $\rho > 0$ the inequality

$$(1.12)\qquad f_0(t, c_0 + h(t)y_0, u_1(t, c_0 + h(\tau_{01}(t))y_1), \dots,$$

$$\ldots, u_m(t, c_0 + h(\tau_{0m}(t)) y_m)) \operatorname{sgn}(y_0) \le$$
$$\le \sum_{k=0}^{m} p_k(t) \| y_k \| + g(t)$$

be fulfilled on the set $(a,b] \times R_\rho^{(m+1)n}$, *where* $p_k : [a,b] \to R_+$ $(k = 0,1,\ldots,m)$ *and* $q : [a,b] \to R_+$ *are summable functions. Let, in addition, the functions* $p = \sum_{k=0}^{m} p_k$ *and* q *satisfy* (1.4). *Then the problem* (1.6_1), (1.6_2), (1.2) *has Kneser's property.*

Corollary 1.3. Let there exist $m_0 \in \{1,\ldots,m\}$ *and a continuous nondecreasing function* $\tau : [a,b] \to [a,b]$ *such that*

(1.13) $$\tau_k(t) \le \tau(t) < t \quad \text{for} \quad a < t \le b \ \ (k = m_0, \ldots, m).$$

Let, futhermore, on the set $(a,b) \times R^{(m+1)n}$ *the inequality*

(1.14) $$f_0(t, c_0 + h(t) y_0, u_1(t, c_0 + h(\tau_{01}(t)) y_1), \ldots,$$
$$u_m(t, c_0 + h(\tau_{0m}(t)) y_m)) \operatorname{sgn}(y_0) \le$$
$$\le \sum_{i=0}^{m_0-1} p_i\left(t, \sum_{k=m_0}^{m} \| y_k \|\right) \| y_i \| + q\left(t, \sum_{k=m_0}^{m} \| y_k \|\right)$$

be fulfilled, where the functions $p_i : [a,b] \times R_+ \to R_+$ $(i = 0,1,\ldots,m_0 - 1)$ *and* $q : [a,b] \times R_+ \to R_+$ *are summable in the first argument and continuous nondecreasing in the second one. Let, in addition, for some* $\rho > 0$ *the functions* $p = \sum_{i=0}^{m_0-1} p_i$ *and* q *satisfy*

(1.15) $$\limsup_{t \to a} \left(\frac{1}{h(t)} \int_a^t p(s, \rho) ds \right) < 1,$$
$$\lim_{t \to a} \left(\frac{1}{h(t)} \int_a^t q(s, \rho) ds \right) = 0.$$

Then each noncontinuable solution of the problem (1.6_1), (1.6_2), (1.2) *is defined on* $[a,b]$, *and the set of all such solutions is compact and connected in the topology of the space* $C([a,b]; R^n)$.

Remark 1.1. If $a \le \tau_i(t) \le b$ for $a \le t \le b$ $(i = 1,\ldots,m)$, then it is unnecessary to represent the initial function for the equation (1.6_1) by the equality (1.6_2). The equation (1.10) in this case coincides with the equation (1.6_1), whereas the condition (1.12) takes the form

$$f_0(t,c_0+h(t)y_0,c_0+h(\tau_1(t))y_1,\dots,c_0+h(\tau_m(t))y_m)\operatorname{sgn}(y_0)\le$$
$$\le\sum_{k=0}^{m} p_k(t)\|y_k\|+q(t).$$

Remark 1.2. In Theorem 1.1 and in its corollaries, one cannot replace the strict inequality

(1.16) $$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_a^t p(s)ds\right)<1,$$

by the nonstrict one

(1.17) $$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_a^t p(s)ds\right)\le 1.$$

Indeed, if $a=0,\ b=1,\ c_0=0,\ h(t)=t$ and

$$p(t)=1+(2-\ln t)^{-1}[\ln(2-\ln t)]^{-1},$$

then for the problem

(1.18) $$\frac{dx(t)}{dt}=\frac{p(t)}{t}x(t),\qquad \lim_{t\to 0}\frac{x(t)}{t}=0$$

all the conditions of Corollary 1.2 are fulfilled with (1.17) instead of (1.16). Nevertheless, the problem (1.18) does not have Kneser's property, since for any $\alpha\in R$ the function

$$x_\alpha(t)=\alpha t[\ln(2-\ln t)]^{-1}$$

is its solution, and hence this problem has a noncompact set of solutions.

Example 1.1. Consider the problem

$$\frac{dx(t)}{dt}=\sum_{k=1}^{m} g_k(t^{-2\mu}x(t^2),\dots,t^{-m\mu}x(t^m))t^{(1-k)\mu-1}x(t^k)+$$
$$+g_0(t,x(t),\dots,x(t^m))+t^\mu g(t),$$

(1.20) $$\lim_{t\to 0}\frac{|x(t)|}{t^\mu}=0,$$

where $m\ge 2,\ \mu>0,\ g_k:R^{m-1}\to R\ (k=1,\dots,m)$ are continuous, $g_0:(0,1]\times R^m\to R$ satisfies the local Carathéodory conditions, and $g\in L([0,1];R)$.

By Corollary 1.3, if

$$g_0(t,x_0,\dots,x_{m-1})x_0\le 0\quad\text{and}\quad \sum_{k=1}^{m}|g_k(0,\dots,0)|<\mu,$$

then every noncontinuable solution of the problem (1.19), (1.20) is defined on $[0,1]$, and the set of all such solutions is connected and compact in the topology of the space $C([0,1];R)$.

In the example under consideration one can take as g_0 the function

$$g_0(t,x_0,\dots,x_{m-1}) = -\sum_{k=0}^{m-1} r_k \exp(t^{-\alpha_k})(1+|x_k|)^{\beta_k} x_0^{2l_k-1},$$

where $r_k > 0$, $\alpha_k > 0$, $\beta_k \in R$, and l_k is a natural number.

Consequently, under the conditions of Theorem 1.1 and of its corollaries, the right-hand side of the above-considered differential equation has for $t=0$ a nonintegrable singularity of arbitrary order.

2. Auxiliary Propositions

2.1. Lemmas on the properties of the set of solutions of the functional differential equation with a parameter. In this section, we consider the functional differential equation

(2.1) $$\frac{dx(t)}{dt} = g(x;\lambda)(t)$$

with the parametr $\lambda \in \Lambda \subset R^m$ and the initial condition

(2.2) $$x(a) = c(\lambda).$$

An operator $g: C([a,b];R^n) \times D \to L([a,b];R^n)$ is said to be *continuous* if for any $x_0 \in C([a,b];R^n)$ and $\lambda_0 \in \Lambda$ we have

$$\|g(x;\lambda) - g(x_0;\lambda_0)\|_L \to 0 \quad \text{as} \quad \|x - x_0\|_C \to 0,$$

$$\lambda \in \Lambda \quad \text{and} \quad \|\lambda - \lambda_0\| \to 0.$$

We are interested in the case where

(2.3) $$c: \Lambda \to R^n \text{ is continuous}$$

and the following two conditions are fulfilled:

(C_1) $g: C([a,b];R^n) \times \Lambda \to L([a,b];R^n)$ is continuous, and $g(\cdot;\Lambda)$: $C([a,b];R^n) \to L([a,b];R^n)$ is Volterra for every $\lambda \in \Lambda$;

(C_2) for any $x \in C([a,b];R^n)$ and $\lambda \in \Lambda$ almost everywhere on $[a,b]$ the inequality

$$\|g(x;\lambda)(t)\| \le g^*(t)$$

is fulfilled, where $g^*: [a,b] \to R_+$ is a summable function not depending on x and λ.

Under a solution of the problem (2.1), (2.2) it is meant an absolutely continuous vector function $x:[a,b]\to R^n$ which almost everywhere on $[a,b]$ satisfies both the equation (2.1) and the initial condition (2.2).

We denote the set of all solutions of the problem (2.1), (2.2) by $X_{g,c}(\lambda)$ and suppose

$$X_{g,c} = \bigcup_{\lambda\in\Lambda} X_{g,c}(\lambda).$$

Lemma 2.1. Let $\Lambda\subset R^m$ *be a compact set, and let the conditions* (2.3), (C_1) *and* (C_2) *be fulfilled. Then* $X_{g,c}(\lambda)\neq\varnothing$ *for every* $\lambda\in\Lambda$, *and* $X_{g,c}$ *is compact in the topology of the space* $C([a,b];R^n)$.

Proof. Let $\lambda\in\Lambda$ be arbitrarily fixed. Then by the Schauder principle and the conditions (C_1) *and* (C_2), the integral equation

$$x(t)=c(\lambda)+\int_a^t g(x,\lambda)(s)ds$$

has at least one solution, i.e., $X_{g,c}(\lambda)\neq\varnothing$. On the other hand, every vector function $x\in X_{g,c}$ satisfies

$$\|x(t)\|\le r_0, \qquad \|x(\bar t)-x(t)\|\le\int_t^{\bar t} g^*(s)ds \quad \text{for} \quad a\le t\le \bar t\le b,$$

where

$$r_0=\max\{\|c(\lambda)\|:0\le\lambda\le 1\}+\int_a^b g^*(s)ds.$$

According to the Arzella-Ascoli lemma, this implies that $X_{g,c}$ is a precompact subset of the space $C([a,b];R^n)$. To prove the lemma, it remains to prove the closeness fo $X_{g,c}$.

Let $x_k\in X_{g,c}$ $(k=1,2,...)$ be an arbitrary uniformly convergent sequence and

$$\lim_{k\to+\infty} x_k(t)=x_0(t) \qquad \text{for} \quad a\le t\le b.$$

Then there exists a sequence $\lambda_k\in\Lambda$ $(k=1,2,...)$ such that

$$x_k(t)=c(\lambda_k)+\int_a^t g(x_k;\lambda_k)(s)ds \qquad (k=1,2,...).$$

Without loss of generality we may assume $(\lambda_k)_{k=1}^{+\infty}$ to be convergent. Suppose

$$\lambda_0 = \lim_{k\to+\infty} \lambda_k.$$

Then because of the compactness of the set Λ and due to the condition(C_1), we have $\lambda_0 \in \Lambda$ and

$$x_0(t) = c(\lambda_0) + \int_a^t g(x_0;\lambda_0)(s)ds.$$

Consequently, $x_0 \in X_{g,c}$. ■

Definition 2.1. Let $\delta \in (0, b-a)$. *An operator* $g: C([a,b];R^n) \times \Lambda \to L([a,b];R^n)$ *is said to be* δ*-Volterra, if for any* $t_0 \in [a+\delta, b]$, $\lambda \in \Lambda$ *and vector functions x and* $y \in C([a,b];R^n)$ *satisfying*

$$x(t) = y(t) \quad for \quad a \le t \le t_0 + \delta,$$

almost everywhere on $[a,t_0]$ *the equality*

$$g(x;\lambda)(t) = g(y;\lambda)(t)$$

is fulfilled.

Lemma 2.2. Let g be a δ*-Volterra operator and let the conditions* (C_1), (C_2) *and* (2.3) *be fulfilled. Then for every* $\lambda \in \Lambda$ *the problem* (2.1), (2.2) *has a unique solution* $x(\cdot;\lambda)$, *and the vector function* $x: [a,b] \times \Lambda \to R^n$ *is continuous.*

Proof. Let $\lambda \in \Lambda$ be arbitrarily fixed. Then by Lemma 2.1, the problem (2.1), (2.2) is solvable. Let us show that it has at most one solution.

Let l be an integral part of $\frac{b-a}{\delta}$, and set

$$a_i = a + i\,\delta \quad (i = 1,\dots,l), \;\; a_{l+1} = b.$$

Consider arbitrary solutions x and y of the problem (2.1), (2.2). Since the operator g is δ-Volterra and the equality $x(a) = y(a) = c(\lambda)$ holds, we have

$$x(t) = c(\lambda) + \int_a^t g(c(\lambda);\lambda)(s)ds = y(t) \quad \text{for} \quad a \le t \le a_1.$$

Assume now that for some $i \in \{1,\dots,l\}$ the equality

$$x(t) = y(t) \quad \text{for} \quad a \le t \le a_i$$

is fulfilled. Then, taking into account the fact that the operator g is δ-Volterra, we obtain

$$x(t) = c(\lambda) + \int_a^t g(x;\lambda)(s)ds =$$

$$= c(\lambda) + \int_a^t g(y;\lambda)(s)ds = y(t) \quad \text{for} \quad a \le t \le a_{i+1}.$$

If now we use the induction, then obviously we obtain $x(t) \equiv y(t)$. Thus we have proved that for every $\lambda \in \Lambda$ the problem (2.1), (2.2) has a unique solution. Denote it by $x(\cdot;\lambda)$. Then

$$x(t;\lambda) = c(\lambda) + \int_a^t g(x(\cdot;\lambda);\lambda)(s)ds \qquad \text{for} \quad a \le t \le b, \quad \lambda \in \Lambda,$$

whence by virtue of the condition (C_1) it follows that $x: [a,b] \times \Lambda \to R^n$ is continuous.

Lemma 2.3. Let $\Lambda \subset R^n$ *be a compact connected set, and let the conditions* (2.3), (C_1) *and* (C_2) *be fulfilled. Then* $X_{g,c}(\lambda) \ne \varnothing$ *for every* $\lambda \in \Lambda$ *and* $X_{g,c}$ *is compact and connected in the topology of the space* $C([a,b];R^n)$.

Proof. By Lemma 2.1, the set $X_{g,c}$ is compact. It remains to prove its connectedness. Assume to the contrary that $X_{g,c}$ is not connected. Then

$$X_{g,c} = X_{g,c}^{(1)} \cup X_{g,c}^{(2)},$$

where $X_{g,c}^{(i)}$ $(i = 1,2)$ are nonintersecting nonemplty compact sets. Therefore

(2.4) $$\rho_0 = \min\{\| x - y \|_C : x \in X_{g,c}^{(1)}, y \in X_{g,c}^{(2)}\} > 0.$$

Choose arbitrarily

(2.5) $$x^{(i)} \in X_{g,c}^{(i)} \quad (i = 1,2).$$

Then because of (2.4),

(2.6) $$\|x^{(1)} - x^{(2)}\|_C \ge \rho_0.$$

On the other hand, it is evident the existence of $\lambda^{(i)} \in \Lambda$ $(i = 1,2)$ such that

(2.7) $$x^{(i)} \in X_{g,c}(\lambda^{(i)}) \qquad (i = 1,2).$$

Let $\widetilde{\Lambda} = \Lambda \times [0,1]$. For any natural k, $(\lambda, \mu) \in \widetilde{\Lambda}$ and $x \in C([a,b];R^n)$, we set

$$\delta_k = \frac{b-a}{2k}, \quad \eta_k(x)(t) = \begin{cases} x(a) & \text{for } a \le t \le a + \delta_k, \\ x(t - \delta_k) & \text{for } a + \delta_k < t \le b, \end{cases}$$

$$g_k(x;\lambda,\mu)(t) = g(\eta_k(x);\lambda)(t) + (1-\mu)\left[g(x^{(1)};\lambda)(t) - g(\eta_k(x^{(1)});\lambda)(t)\right] + \\ + \mu\left[g(x^{(2)};\lambda)(t) - g(\eta_k(x^{(2)});\lambda)(t)\right],$$

and consider the differential equation

$$\frac{dx(t)}{dt} = g_k(x;\lambda,\mu)(t). \tag{2.9}$$

According to the conditions (C_1) and (C_2), the operator $g_k: C([a,b];R^n) \times \widetilde{\Lambda} \to L([a,b];R^n)$ is continuous and δ_k-Volterra, and for any $x \in C([a,b];R^n)$ and $(\lambda,\mu) \in \widetilde{\Lambda}$ almost everywhere on $[a,b]$ it satisfies

$$\|g_k(x;\lambda,\mu)(t)\| \le 3g^*(t). \tag{2.10}$$

By Lemma 2.2, for any natural k and $(\lambda,\mu) \in \Lambda$ the problem (2.9), (2.2) has a unique solution $x_k(\cdot;\lambda,\mu)$ which is continuous in the parameters λ and μ. Moreover, by virtue of (2.8) it is clear that

$$x_k(t;\lambda^{(1)},0) \equiv x^{(1)}(t), \qquad x_k(t;\lambda^{(2)},1) \equiv x^{(2)}(t). \tag{2.11}$$

Let

$$\varphi_k(\lambda,\mu) = \min\left\{\| x_k(\cdot;\lambda,\mu) - y\|_C : y \in X^{(1)}_{g,c}\right\} \tag{2.12}$$

By the compactness of $X^{(1)}_{g,c}$ and the continuity of $x_k:[a,b]\times\widetilde{\Lambda} \to R^n$, the function $\varphi_k : \widetilde{\Lambda} \to R_+$ is continuous. On the other hand, it follows from (2.5), (2.6) and (2.11) that

$$\varphi_k(\lambda^{(1)},0) = 0, \qquad \varphi_k(\lambda^{(2)},1) \ge \|x^{(2)} - x^{(1)}\|_C \ge \rho_0.$$

Due to the compactness and connectedness of $\widetilde{\Lambda}$, this results in the existence of $(\lambda_k,\mu_k) \in \widetilde{\Lambda}$ such that

$$\varphi_k(\lambda_k,\mu_k) = \frac{\rho_0}{2}. \tag{2.13}$$

Suppose

$$\overline{x}_k(t) = x_k(t;\lambda_k,\mu_k).$$

Because of the boundedness of the vector function $c:\Lambda \to R^n$ and the condition (2.10), the sequence $(\overline{x}_k)_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on $[a,b]$. Therefore, by the Arzella-Ascoli lemma, without loss of generality this sequence may be considered as uniformly convergent. Since Λ is compact, the sequence $(\lambda_k)_{k=1}^{+\infty}$ may also be considered as convergent. Suppose

$$x_0(t) = \lim_{k\to+\infty} \overline{x}_k(t), \qquad \lambda_0(t) = \lim_{k\to+\infty} \lambda_k. \tag{2.14}$$

Then

$$\lim_{k\to+\infty} \|\eta_k(\overline{x}_k) - x_0\|_C = 0. \tag{2.15}$$

Moreover,

(2.16) $$\lim_{k\to+\infty} \| \eta_k(x^{(i)}) - x^{(i)} \|_C = 0 \quad (i=1,2).$$

By the conditions (2.3), (C_1), (2.8) and (2.14)-(2.16), from the equality

$$\bar{x}_k(t) = c(\lambda_k) + \int_a^t g(\eta_k(\bar{x}_k);\lambda_k)(s)ds +$$
$$+(1-\mu_k)\int_a^t \left[g(x^{(1)};\lambda_k)(s) - g(\eta_k(x^{(1)});\lambda_k)(s)\right]ds +$$
$$+\mu_k\int_a^t \left[g(x^{(2)};\lambda_k)(s) - g(\eta_k(x^{(2)});\lambda_k)(s)\right]ds$$

we find that

$$x_0(t) = c(\lambda_0) + \int_a^t g(x_0;\lambda_0)(s)ds.$$

Consequently,

(2.17) $$x_0 \in X_{g,c}.$$

On the other hand, by (2.12) and (2.13) we have

$$\min\{\|\bar{x}_k - y\|_C : y \in X^{(1)}_{g,c}\} = \frac{\rho_0}{2} \quad (k=1,2,...),$$

which with regard for the compactness of $X^{(1)}_{g,c}$ and (2.14) and (2.4) implies that

$$\min\{\|x_0 - y\|_C : y \in X^{(1)}_{g,c}\} = \frac{\rho_0}{2}$$

and

$$x_0 \notin X^{(i)}_{g,c} \quad (i=1,2).$$

But this contradicts (2.17), since $X_{g,c} = X^{(1)}_{g,c} \cup X^{(2)}_{g,c}$. The obtained contradiction proves the lemma. ■

The proved lemma is a generalization of C. Corduneanu's theorem [4].

2.2. Lemma on an a priori estimate. Consider the differential inequality

(2.18) $$x'(t)\operatorname{sgn}(x(t)-c_0) \le p(t)v\left(\frac{1}{h}(x-c_0)\right)(a,t) + g(t),$$

where p and $g:[a,b]\to R_+$ are summable, and $h:(a,b]\to(0,+\infty)$ is continuous and nondecreasing.

Lemma 2.4. Let $a_0 \in [a,b),$, $b_0 \in (a,b]$, $\alpha \in (0,1)$,

$$\int_a^t p(s)ds \le \alpha h(t) \quad \textit{for} \quad a \le t \le b_0, \tag{2.19}$$

$$\varepsilon(t) = \sup\left\{\frac{1}{h(s)}\int_a^s q(\xi)d\xi : a < s \le t\right\} < +\infty \quad \textit{for} \quad a < t \le b_0, \tag{2.20}$$

and let $x: [a,b_0] \to R^n$ *be a continuous function satisfying*

$$\nu\left(\frac{1}{h}(x - c_0)\right)(a,t) \le \beta\varepsilon(t) \quad \textit{for} \quad a \le t \le a_0.\text{[1]} \tag{2.21}$$

Moreover, let x *be locally absolutely continuous on the interval* $(a_0, b_0]$ *and almost everywhere on it satisfy* (2.18). *Then*

$$\|x(t) - c_0\| \le \frac{1+\beta}{1-\alpha}\varepsilon(t)h(t) \quad \textit{for} \quad a \le t \le b_0. \tag{2.22}$$

Proof. Suppose

$$y(t) = \frac{1}{h(t)}(x(t) - c_0).$$

Then because of (2.18), almost everywhere on $[a_0, b_0]$ the inequality

$$\|x(t) - c_0\|' \le p(t)\nu(y)(a,t) + q(t)$$

holds. Hence, taking into account (2.19) and (2.20), we find that

$$\| x(s) - c_0 \| \le \| x(a_0) - c_0 \| + \int_{a_0}^t [p(\xi)\nu(y)(a,\xi) + q(\xi)]d\xi \le$$

$$\le \|x(a_0) - c_0\| + \alpha h(s)\nu(y)(a,s) + \varepsilon(s)h(s) \quad \text{for} \quad a_0 \le s \le b_0$$

and

$$\|y(s)\| \le \|y(a_0)\| + \alpha\nu(y)(a,t) + \varepsilon(t) \quad \text{for} \quad a_0 \le s \le t \le b_0,$$

since h, ε and $\nu(y)(a,\cdot)$ are nondecreasing functions.

The latter estimate and the condition (2.21) result in

$$\nu(y)(a,t) \le (1+\beta)\varepsilon(t) + \alpha\nu(y)(a,t) \quad \text{for} \quad a_0 \le t \le b_0$$

and

$$\nu(y)(a,t) \le \frac{1+\beta}{1-\alpha}\varepsilon(t) \quad \text{for} \quad a \le t \le b_0.$$

Consequently, the estimate (2.22) is valid. ■

---

[1] Under the value of the function ε at the point a is understood its right limit at that point.

2.3. Lemmas on properties of the set of solutions of the problem (1.1), (1.2).

Lemma 2.5. Let the conditions of Theorem 1.1. be fulfilled. Then $I^*(f;c_0,h)\neq\varnothing$, *and for every* $b^*\in I^*(f;c_0,h)$ *the set of restrictions on* $[a,b^*]$ *of all noncontinuable solutions of the problem* (1.1), (1.2) *is compact in the topology of the space* $C([a,b^*];R^n)$. *If, moreover,* ε *is the function given by* (2.20), *and* $b_0\in(a,b)$ *and* $\alpha\in(0,1)$ *are such that*

$$\int_a^t p(s)ds\leq\alpha h(t),\qquad \varepsilon(t)<\frac{(1-\alpha)\rho}{2}\quad \text{for}\quad a<t\leq b_0, \tag{2.23}$$

then $b_0\in I^*(f;c_0,h)$, *and every noncontinuable solution of the problem* (1.1), (1.2) *admits the estimate*

$$\|x(t)-c_0\|\leq\frac{\varepsilon(t)}{1-\alpha}h(t)<\frac{\rho}{2}h(t)\qquad \text{for}\quad a<t\leq b_0. \tag{2.24}$$

The proof of this theorem is contained in [12] (see pp. 267-269).

Lemma 2.6. Let the conditions of Theorem 1.1 be fulfilled, and let $b_0\in(a,b]$ *and* $\alpha\in(a,b)$ *be numbers appearing in Lemma 2.5. Then for every* $b^*\in I^*(f;c_0,h)$ *there exist* $\gamma^*\in L_{loc}((a,b];R_+)$ *and a continuous Volterra operator* $\tilde f: C([a,b];R^n)\to L_{loc}((a,b];R^n)$ *such that:*

(i) *for arbitrary x and* $y\in C([a,b];R^n)$ *the inequalities*

$$\|\tilde f(x)(t)\|\leq\gamma^*(t) \tag{2.25}$$

and

$$\tilde f(c_0+hy)(t)\operatorname{sgn}(y(t))\leq p(t)v(y)(a,t)+q(t) \tag{2.26}$$

are fulfilled almost everywhere on (a,b) *and* (a,b_0), *respectively.*

(ii) $b^*\in I^*(\tilde f;c_0,h)$ *and the set of restrictions on* $[a,b^*]$ *of all non-continuable solutions of the differential equation*

$$\frac{dx(t)}{dt}=\tilde f(x)(t) \tag{2.27}$$

satisfying the initial condition (1.2), *coincides with the set of restrictions of all noncontinuable solutions of the problem* (1.1), (1.2) *on* $[a,b^*]$.

Proof. Let $b^*\in I^*(f;c_0,h)$ be an arbitrarily fixed number. Without loss of generality we may assume that $b^*>b_0$. Denote by X the set of restrictions on

$[a,b^*]$ of all solutions of the problem (1.1), (1.2). By Lemma 2.5, the set X is compact in the topology of the space $C([a,b^*];R^n)$. Therefore there exists a positive number ρ^* such that

$$\|x(t)-c_0\| < \frac{\rho^*}{2}h(t) \quad \text{for} \quad x\in X, \ \ b_0\le t\le b^*.$$

Taking this condition and Lemma 2.5 into account, we have

(2.28) $$\|x(t)-c_0\| < r(t)/2 \quad \text{for} \quad x\in X, \ \ b_0<t\le b^*,$$

where

$$r(t)=\begin{cases} \rho h(t) & \text{for} \quad a\le s\le b_0, \\ \rho^* h(t) & \text{for} \quad b_0<t\le b. \end{cases}$$

Let

$$\psi(t,s)=\begin{cases} 1 & \text{for} \quad 0\le s\ge r(t)/2, \\ \dfrac{r(t)}{s}-1 & \text{for} \quad r(t)/2<s<r(t), \\ 0 & \text{for} \quad s\ge r(t). \end{cases}$$

For an arbitrary $x\in C([a,b];R^n)$, we assume thats

$$\widetilde{f}(x)(t)=\psi(t,\|x(t)-c_0\|)\,f(x)(t).$$

Obviously, $\widetilde{f}: C([a,b];R^n)\to L_{loc}((a,b];R^n)$ is a continuous Volterra operator, and

(2.29) $$\widetilde{f}(x)(t)=f(x)(t) \text{ when } \|x(t)-c_0\|\le r(t)/2.$$

On the other hand, by Definition 1.2 and the condition (1.5), almost everywhere on the intervals (a,b) and (a,b_0) the inequalities (2.25) and (2.26) are fulfilled, where $\gamma^*(t)=\gamma(t,\|c_0\|+r(t))$.

Consider the differential equation (2.27) with the initial condition (1.2). By virtue of Corollary 1 from [13] and the conditions (2.25) and (2.26), every noncontinuable solution of the problem (2.27), (1.2) is defined on the whole interval $[a,b]$. Consequently, $b^*\in I^*(\widetilde{f};c_0,h)$.

Denote by $\widetilde{X}$ the set of restrictions on $[a,b^*]$ of all noncontinuable solutions of the problem (2.27), (1.2). By (2.28) and (2.29),

$$X\subset\widetilde{X}.$$

To complete the proof of the lemma. it remains to show that the arbitrary function $\widetilde{x}$ from $\widetilde{X}$ belongs to the set X.

According to Lemma 2.5 and the condition (i),

$$\|\tilde{x}(t)-c_0\|<\frac{r(t)}{2} \quad \text{for} \quad a<t\le b_0.$$

Denote by t^* the least upper bound of the set of those $t_0\in[b_0,b^*]$ for which

$$\|\tilde{x}(t)-c_0\|<\frac{r(t)}{2} \quad \text{for} \quad a<t\le t_0.$$

Then

$$\|\tilde{x}(t)-c_0\|<\frac{r(t)}{2} \quad \text{for} \quad a<t<t^*. \tag{2.30}$$

If we assume that $t^*<b$, then we have

$$\|x(t^*)-c_0\| = \frac{r(t^*)}{2}. \tag{2.31}$$

By virtue of (2.29) and (2.30), the restriction of the function x on $[a,t^*]$ is a solution of the problem (1.1), (1.2). Therfore there exists $x\in X$ such that

$$x(t)=\tilde{x}(t) \quad \text{for} \quad a\le t\le t^*.$$

By (2.28),

$$\|x(t^*)-c_0\| = \frac{r(t^*)}{2},$$

which contradicts (2.31). The obtained contradiction proves that $t^*=b^*$. Taking into account this equality, from (2.29) and (2.30) we conclude that $\tilde{x}\in X$. ■

3. Proofs of Main Results

Proof of Theorem 1.1. By condition (1.4), there exist $b_0\in(a,b]$ and $\alpha\in(0,1)$ for which the inequalities (2.23) are fulfilled. Then by Lemma 2.5,

$$B_0\in I^*(f;c_0,h)$$

and every noncontinuable solution of the problem (1.1), (1.2) admits the estimate (2.24), where

$$\lim_{t\to a}\varepsilon(t) = 0. \tag{3.1}$$

Let b^* be an arbitrarily fixed number from the set $I^*(f;c_0,h)$ and X be the set of restrictions on restrictions on $[a,b^*]$ of all solutions of the problem (1.1), (1.2). According to Lemma 2.5, the set X is compact in the topology of the space $C([a,b^*];R^n)$. It remains to show that the set X is connected.

Without loss of generality, we will assume that $b_0<b^*$.

By Lemma 2.6, there exist $\gamma^* \in L_{loc}((a,b];R)$ and a continuous Volterra operator $\widetilde{f}: C([a,b];R^n) \to L_{loc}((a,b];R^n)$ such that for every x and $y \in C([a,b];R^n)$ the inequalities (2.25) and (2.26) are fulfilled almost everywhere on (a,b) and (a,b_0), respectively, and

$$X = \widetilde{X}, \tag{3.2}$$

where $\widetilde{X}$ is the set of restrictions on $[a,b^*]$ of all solutions of the problem (2.27), (2.1).

Suppose now that X is a nonconnected set. Then

$$X = X^{(1)} \cup X^{(2)}, \tag{3.3}$$

where $X^{(1)}$ and $X^{(2)}$ are nonintersecting compact sets, and hence

$$\rho_0 = \min\{\|x-y\|_C : x \in X^{(1)}, y \in X^{(2)}\} > 0. \tag{3.4}$$

Choose arbitrarily

$$x^{(i)} \in X^{(i)} \quad (i=1,2). \tag{3.5}$$

Then, taking into account the above arguments, we obtain

$$\|x^{(i)}(t) - c_0\| \le \frac{\varepsilon(t)}{1-\alpha} h(t) \quad \text{for} \quad a \le t \le b_0 \quad (i=1,2). \tag{3.6}$$

Let

$$a_k = a + \frac{b-a}{2k} \qquad (k=1,2,\dots).$$

For any natural k and for every $\lambda \in [0,1]$ and $x \in C([a_k,b^*];R^n)$, we assume

$$\eta_k(x;\lambda)(t) = \begin{cases} (1-\lambda)x^{(1)}(t) + \lambda x^{(2)}(t) & \text{for} \quad a \le t \le a_k \\ x(t) - x(a_k) + (1-\lambda)x^{(1)}(a_k) + \lambda x^{(2)}(a_k) & \text{for} \quad a_k < t \le b^* \end{cases} \tag{3.7}$$

and

$$g_k(x;\lambda)(t) = \widetilde{f}(\eta_k(x;\lambda))(t). \tag{3.8}$$

Then by (3.6),

$$v\left(\frac{1}{h}(\eta_k(x;\lambda) - c_0)\right)(a,t) \le \frac{1}{1-\alpha}\varepsilon(t) \quad \text{for} \quad a \le t \le a_k. \tag{3.9}$$

On the other hand, by writue of (2.25) and (2.26) the inequalities

$$\|g_k(x;\lambda)(t)\| \le \gamma^*(t) \tag{3.10}$$

and

(3.11)
$$g_k(x;\lambda)(t)\,\mathrm{sgn}(x(t)-c_0) \le$$
$$\le p(t)\nu\left(\frac{1}{h}(\eta_k(x;\lambda)-c_0)\right)(a,t)+q(t)$$

are fulfilled almost everywhere on (a_k,b^*) and (a_k,b_0), respectively. Moreover, $g_k: C([a_k,b^*];R^n)\times[0,1]\to L([a_k,b^*];R^n)$ is continuous, and $g_k(\cdot,\lambda): C([a_k,b^*];R^n)\to L([a_k,b^*];R^n)$ is Volterra for every $\lambda\in[0,1]$.

Because of the continuity of g_k and owing to the condition (3.1), for arbitrarily fixed $\lambda\in[0,1]$ and for any natural *k*, every noncontinuable solution of the initial value problem

(3.12)
$$\frac{dx(t)}{dt} = g_k(x;\lambda)(t),$$

(3.13)
$$x(a_k) = (1-\lambda)x^{(1)}(a_k)+\lambda x^{(2)}(a_k)$$

is defined on the whole interval $[a_k,b^*]$. Denote the set of all such solutions by $X_k(\lambda)$. Moreover, assume

$$X_k=\bigcup_{0\le\lambda\le1} X_k(\lambda),$$

(3.14)
$$Z_k(\lambda)=\{z=g(x;\lambda):\ x\in X_k(\lambda)\},\qquad Z_k=\bigcup_{0\le\lambda\le1} Z_k(\lambda).$$

In view of (3.13) and (3.7), for every $x\in X_k(\lambda)$ we have

$$\eta_k(x;\lambda)(t) = x(t)\qquad \text{for}\quad a_k\le t\le b^*.$$

Obviously,

(3.15) Z_k is a set of continuous on $[a,b^*]$ existensions of elements of X_k.

Let z be an arbitrary element of the set Z_k. Then there exist $\lambda\in[0,1]$ and $x\in X_k(\lambda)$ such that

$$z(t)=\eta_k(x;\lambda)(t)\quad\text{for}\quad a\le t\le b^*,\qquad z(t)=x(t)\quad\text{for}\quad a_k\le t\le b^*.$$

Therefore (3.9) and (3.11) imply that the vector function z almost everywhere on (a_k,b_0) satisfies the differential inequality

$$z'(t)\mathrm{sgn}(z(t)-c_0)\le p(t)\nu\left(\frac{1}{h}(z-c_0)\right)(a,t)+q(t)$$

and

$$\nu\left(\frac{1}{h}(z-c_0)\right)(a,t)\le\frac{1}{1-\alpha}\varepsilon(t)\qquad\text{for}\quad a\le t\le a_k.$$

Hence by Lemma 2.4 it follows that

(3.16) $$\|z(t)-c_0\| \le \beta_0\varepsilon(t)h(t) \quad \text{for} \quad a \le t \le b_0,$$

where

$$\beta_0 = \frac{2-\alpha}{(1-\alpha)^2}.$$

If along with (3.16) we take into account the condition (3.1), then it will become clear that

(3.17) $$\sup\{\|z(t)-c_0\|: z \in Z_k\} \to 0 \quad \text{as} \quad t \to a.$$

By Lemma 2.3, for an arbitrary k the set X_k is compact and connected in the topology of the space $C([a_k,b^*];R^n)$. Hence by (3.14), (3.15) and (3.17) it follows that starting from some sufficienty large k_0, the set Z_k is compact and connected in the topology of the space $C([a,b^*];R^n)$.

According to (3.7), (3.8) and (3.5)

$$x^{(i)} \in Z_k \quad (i=1,2)$$

and

(3.18) $$Z_k \cap X^{(i)} \ne \varnothing \qquad (i=1,2).$$

Because of the compactness of the set $X^{(i)}$ $(i=1,2)$ and of the connectedness and compactness of the set Z_k for $k \ge k_0$ and also according to conditions (3.4) and (3.18), for evry $k \ge k_0$ there exists $z_k \in Z_k$ such that

(3.19) $$\min\{\|z_k - y\|_C : y \in X^{(1)}\} = \frac{\rho_0}{2}.$$

On the other hand, by conditions (3.1) and (3.16), $\|z_k'\| \le \gamma^*(t)$ almost everywhere (a_k,b^*) $(k=k_0,k_0+1,\dots)$ and

(3.20) $$\|z_k(t)-c_0\| \le \beta_0\varepsilon(t)h(t) \quad \text{for} \quad a \le t \le b_0 \quad (k=k_0,k_0+1,\dots).$$

These estimates and condition (3.1) imply that the sequence $(z_k)_{k=k_0}^{+\infty}$ is uniformly bounded and equicontinuous. By Arzella-Ascoli's lemma, we can select from this sequence a uniforly converging subsequence $(z_{k_j})_{j=1}^{+\infty}$. Assume

$$x(t) = \lim_{j\to+\infty} z_{k_j}(t) \quad \text{for} \quad a \le t \le b^*.$$

Then from (3.19) and (3.20) we obtain

(3.21) $$\min\{\|x-y\| : y \in X^{(1)}\} = \frac{\rho_0}{2}$$

and

(3.22) $$\|x(t)-c_0\| \le \beta_0\varepsilon(t)h(t) \quad \text{for} \quad a \le t \le b_0.$$

By virtue of (3.2)-(3.4),

(3.23) $$x \notin \widetilde{X}.$$

Owing to (3.8),

$$z_{k_j(t)} = z_{k_j}(b_0) + \int_{b_0}^{t} \widetilde{f}(z_{k_j})(s)ds \quad \text{for} \quad a_{k_j} < t \le b^* \quad (j = 1,2,\ldots),$$

whence, with regard for the condition (2.25) and Lebesgue's theorem on the passage to limit under an integral, we find that

$$x(t) = x(b_0) + \int_{b_0}^{t} \widetilde{f}(x)(s)ds \quad \text{for} \quad a < t \le b^*.$$

On the other hand, it is clear from (3.1) and (3.22) that x satisfies the initial condition (1.2). Consequently,

$$x \in \widetilde{X},$$

which contradicts (3.23). The obtained contradiction proves the theorem. ■

To get convinced that Corollary 1.1 is valid, it is sufficient to notice that if the conditions of that corollary are fulfilled, then by Corollary 1 from [13] every noncontinuable solution of the problem (1.1), (1.2) is defined on the interval $[a,b]$, i.e., $b \in I^*(f;c_0,h)$.

Proof of Corollary 1.2. According to the conditions (1.7)-(1.9), (1.11) and (1.12), for any $y \in C_\rho([a,b];R^n)$ almost everywhere on (a,b) the inequality (1.5) is fulfilled. Hence, all the conditions of Theorem 1.1 are fulfilled, which guarantees that the problem (1.10), (1.2) has Kneser's property. ■

From the above-proven proposition, owing to Corollary 1 from [13], we obtain Corollary 1.3.

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Received on 17.01.1998 and, in revised form, on 21.05.1998.