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**ON THE EXISTENCE OF WEAK SOLUTIONS OF THE  
DARBOUX PROBLEM FOR THE HYPERBOLIC PARTIAL  
DIFFERENTIAL EQUATIONS IN BANACH SPACES**

ABSTRACT. In this paper we prove an existence theorem for the hyperbolic partial differential equation

$$z_{x,y} = f(x, y, z, z_{xy}),$$

$$z(x, 0) = 0, \quad z(0, y) = 0 \quad \text{for } x, y > 0,$$

where  $z_{x,y}$  means the second mixed derivative in the weak sense. The continuity of the function  $f$  is replaced by the weak continuity and the compactness condition is expressed in terms of the measure of weak noncompactness. This paper extends some previous results for our equation.

KEY WORDS: existence theorem, Darboux problem, hyperbolic equation.

1. Let  $J = [0, \infty)$  and  $Q = J \times J$ . Let  $(E, \|\cdot\|)$  be a weakly sequentially complete Banach space and let  $f$  be an  $E$ -valued function defined on  $\Omega = Q \times E \times E$ .

In this paper we extend existence theorem for the Darboux problem for the hyperbolic equation  $z_{xy} = f(x, y, z, z_{xy})$  on the quarterplane  $x \geq 0, y \geq 0$ , where  $f$  is a function with values in a Banach space satisfying some regularity Ambrosetti type condition expressed in terms of the measure of weak noncompactness  $\beta$  and a Lipschitz condition in the last variable,  $z_{x,y}$  means the second mixed derivative in the weak sense.

We are interested in the existence of solutions of the Darboux problem for the hyperbolic partial differential equation with implicit derivative (see [6] and [7]).

$$z_{x,y} = f(x, y, z, z_{xy})$$

$$(1) \quad z(x, 0) = 0, \quad z(0, y) = 0 \quad \text{for } x, y \geq 0.$$

We consider the problem (1) where  $z_{x,y}$  means the second mixed derivative in the weak sense (cf. [9,3]) using the method developed by Ambrosetti [1] and Goebel and Rzymowski [5].

2. Denote by  $S_\infty$  the set of nonnegative real sequences and  $\emptyset$  the zero sequence. For  $\xi = (\xi_n), \eta = (\eta_n) \in S_\infty$  we write  $\xi < \eta$  if  $\xi \neq \eta$  and  $\xi_n \leq \eta_n$ , for  $n = 1, 2, \dots$

Let  $\chi_0$  be a closed convex subset of a Hausdorff locally convex topological vector space. Let  $\Phi$  be a function which maps each nonempty subset  $Z$  of  $\chi_0$  to a sequence  $\Phi(Z) \in S_\infty$  such that

- (1)  $\Phi(\{z\} \cup Z) = \Phi(Z)$  for  $z \in \chi_0$ ,
- (2)  $\Phi(\overline{co}Z) = \Phi(Z)$  (here  $\overline{co}Z$  is the closed convex hull of  $Z$ )

and

- (3) if  $\Phi(Z) = \emptyset$  then  $\overline{Z}$  is compact.

For such  $\Phi$  we have the following theorem of Sadovskii (cf. [8], Theorem 3.4.3):

If  $T$  is continuous mapping of  $\chi_0$  into itself and  $\Phi(T(Z)) < \Phi(Z)$  for arbitrary nonempty subset  $Z$  of  $\chi_0$  with  $\Phi(Z) > \emptyset$ , then  $T$  has a fixed point in  $\chi_0$ .

3. Let  $\beta$  denote the measure of weak noncompactness introduced by De Blasi [4].

Let us recall that for a bounded subset  $A$  of  $E$ ,  $\beta(A) = \inf \{ \varepsilon > 0: \text{there exists weakly compact set } P \text{ such that } A \subset P + \varepsilon B \}$ , where  $B$  is the unit ball in  $E$ .

Moreover if  $Z$  is a set of functions on  $Q$

$$Z(x, y) = \{z(x, y) : z \in Z\}$$

and

$$\iint_{00}^{xy} Z(t, s) dt ds = \left\{ \iint_{00}^{xy} z(t, s) dt ds : z \in Z \right\}$$

for  $x, y \in J$ .

The Lemma below is an adaptation of the corresponding result of Goebel and Rzymowski ([2], [5]).

*Lemma 1. If  $W$  is bounded equicontinuous subset of usual Banach space of continuous  $E$ -valued functions defined on a compact subset  $P = [0, a] \times [0, b]$  of  $Q$ , then*

$$\beta \left( \int_0^x \int_0^y W(t,s) dt ds \right) \leq \int_0^x \int_0^y \beta(W(t,s)) dt ds$$

for  $(x,y) \in P$ .

*Theorem.* Let  $f$  be weakly continuous on bounded subset of  $\Omega$  and

$$\|f(x,y,u,v)\| \leq G(x,y,\|u\|,\|v\|) \quad \text{for } (x,y,u,v) \in \Omega.$$

Suppose that for each bounded subset  $P$  of  $Q$  there exists nonnegative constant  $k(P)$  and  $L(P) < 1$  such that

$$\beta(f(x,y,u,v)) \leq k(P)\beta(U)$$

and

$$\|f(x,y,u,v_1) - f(x,y,u,v_2)\| \leq L(P)\|v_1 - v_2\|$$

for all  $(x,y) \in P$ ,  $u,v,v_1,v_2$  in  $E$  and for any nonempty bounded subset  $U$  of  $E$ . Assume in addition that the function  $(x,y,r,s) \mapsto G(x,y,r,s)$  is monotonic nondecreasing for each  $(x,y) \in Q$  (i.e.  $0 \leq r_1 \leq r_2$  and  $0 \leq s_1 \leq s_2$  implies  $G(x,y,r_1,s_1) \leq G(x,y,r_2,s_2)$ ) and the scalar inequality

$$G \left( x,y, \int_0^x \int_0^y g(t,s) dt ds, g(x,y) \right) \leq g(x,y)$$

has a locally bounded solution  $g_0$ , existing on  $Q$ .

Under the hypothesis the problem (1) has at least one solution on  $Q$ .

*Proof.* As  $E$  is weakly sequentially complete, problem (1) is equivalent to the integral equation

$$w(x,y) = f \left( x,y, \int_0^x \int_0^y w(t,s) dt ds, w(x,y) \right),$$

whetre  $\iint$  denotes the weak Riemann integral.

Denote by  $C_w(Q,E)$  the space of all weakly continuous function from  $Q$  to  $E$  endowed with the topology of weak uniform convergence on each compact subset of  $Q$ , and by  $\chi$  the set of all  $z \in C_w(Q,E)$  with

$$\|z(x,y)\| \leq g_0(x,y) \quad \text{on } Q.$$

Let  $P$  be a compact subset of  $Q$ . From the weak continuity of  $f$  follows the existence of a function  $\delta_p : (0,\infty) \rightarrow (0,\infty)$  such that

$$\left| x^* \left[ f(x', y', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x, y)) - f(x'', y'', \int_0^{x''} \int_0^{y''} z(t, s) dt ds, z(x, y)) \right] \right| < \varepsilon$$

for any  $z \in \chi$ ,  $(x, y)$ ,  $(x', y')$  and  $(x'', y'') \in P$  with  $|x' - x''| < \delta_p(\varepsilon)$ , and  $|y' - y''| < \delta_p(\varepsilon)$ .

Consider the set  $\chi_0$  of  $z \in \chi$  possessing the following property: for each compact subset  $P$  of  $Q$ ,  $\varepsilon > 0$  and  $|x' - x''| < \delta_p(\varepsilon)$ ,  $|y' - y''| < \delta_p(\varepsilon)$  (here  $(x', y')$ ,  $(x'', y'') \in P$ ) there holds

$$\|z(x', y') - z(x'', y'')\| < (1 - L(P))^{-1} \varepsilon.$$

The set  $\chi_0$  is a closed convex and almost equicontinuous subset of  $C_w(Q, E)$ . To apply our fixed point theorem we define a continuous mapping  $T$  of  $\chi_0$  into itself by the formula

$$(Tw)(x, y) = f \left( x, y, \int_0^x \int_0^y w(t, s) dt ds, w(x, y) \right).$$

Let  $z \in \chi_0$ . Then

$$\|(Tz)(x, y)\| = G \left( x, y, \int_0^x \int_0^y \|z(t, s)\| dt ds, \|z(x, y)\| \right) \leq g_0(x, y)$$

for  $(x, y) \in Q$ . Further let  $x^* \in E^*$  and  $\|x^*\| \leq 1$ .

For  $\varepsilon > 0$  and  $(x', y')$ ,  $(x'', y'') \in P$  such that  $|x' - x''| < \delta_p(\varepsilon)$ ,  $|y' - y''| < \delta_p(\varepsilon)$  we have

$$\begin{aligned} & \left| x^* [(Tz)(x', y') - (Tz)(x'', y'')] \right| \leq \\ & \leq \left| x^* \left[ f(x', y', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x', y')) - f(x', y', \int_0^{x''} \int_0^{y''} z(t, s) dt ds, z(x'', y'')) \right] \right| + \\ & + \left| x^* \left[ f(x', y', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x'', y'')) - f(x'', y'', \int_0^{x''} \int_0^{y''} z(t, s) dt ds, z(x'', y'')) \right] \right| \end{aligned}$$



$$\begin{aligned} & \left. - f(x'', y'', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x'', y'')) \right] \leq \\ & \leq \|x^*\| \left\| f(x', y', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x', y')) - \right. \\ & \quad \left. - f\left(x', y', \int_0^{x'} \int_0^{y'} z(t, s) dt ds, z(x'', y'')\right) \right\| + \varepsilon \leq \\ & \leq L(P) \|z(x', y') - z(x'', y'')\| + \varepsilon \leq \varepsilon(1 - L(P))^{-1} \end{aligned}$$

by the Banach theorem

$$\|(Tz)(x', y') - (Tz)(x'', y'')\| \leq \varepsilon(1 - L(P))^{-1}$$

i.e.  $Tz \in \chi_0$ . Consequently  $T(\chi_0) \subset \chi_0$  and the set  $T(\chi_0)$  is strongly equicontinuous.

Applying the following Krasnoselski-type lemma:

*Lemma 2. For any  $x^* \in E^*$ ,  $\varepsilon > 0$ , and  $z \in \chi_0$  there exists a weak neighborhood  $V$  of 0 in  $E$  such that*

$$\left| x^* \left[ f(x, y, \int_0^x \int_0^y z(t, s) dt ds, z(x, y)) - \right. \right. \\ \left. \left. - f(x, y, \int_0^x \int_0^y w(t, s) dt ds, w(x, y)) \right] \right| \leq \varepsilon$$

for  $(x, y) \in P$  and  $w \in \chi_0$  such that  $w(x, y) - z(x, y) \in V$  for all  $(x, y) \in P$ .

We deduce that  $T$  is a continuous mapping.

Let  $n$  be a positive integer and let  $Z$  be a nonempty subset of  $\chi_0$ . Put  $P_n = [0, n] \times [0, n]$ ,  $k_n = K(P_n)$  and  $L_n = L(P_n)$ .

Now we shall show the basic inequality:

$$\begin{aligned} (*) \quad & \sup_{(x, y) \in P_n} \exp(-p_n(x + y)) \beta(T(Z)(x, y)) \leq \\ & \leq (p_n^{-2} k_n + L_n) \sup_{(x, y) \in P_n} \exp(-p_n(x + y)) \beta(T(Z)(x, y)), \end{aligned}$$

where  $p_n \geq 0$ .

Fix  $(x, y) \in P_n$ . By Lemma 1, we obtain

$$\begin{aligned}
\beta\left(\int_0^x \int_0^y Z(t,s) dt ds\right) &\leq \int_0^x \int_0^y \beta(Z(t,s)) dt ds \leq \\
&\leq \int_0^x \int_0^y \exp(-p_n(t+s)) \beta(Z(t,s)) \exp p_n(t+s) dt ds \leq \\
&\leq \int_0^x \int_0^y \sup_{(x,y) \in P_n} [\exp(-p_n(t+s)) \beta(Z(t,s)) \exp p_n(t+s)] dt ds \leq \\
&\leq \sup_{(x,y) \in P_n} [\exp(-p_n(t+s)) \beta(Z(t,s))] \int_0^x \int_0^y \exp p_n(t+s) dt ds \leq \\
&\leq p_n^{-2} \exp [p_n(t+s)] \sup_{(x,y) \in P_n} \exp(-p_n(t+s)) \beta(Z(t,s)).
\end{aligned}$$

Now, by the corresponding properties of  $\beta$  we have

$$\begin{aligned}
\beta(T(Z)(x,y)) &= \beta\left(f(x,y, \int_0^x \int_0^y Z(t,s) dt ds, Z(x,y)\right) \leq \\
&\leq \beta\left[f(x,y, \int_0^x \int_0^y Z(t,s) dt ds, z_1(x,y)\right] + \\
&\quad + \beta\left[f\left(x,y, \int_0^x \int_0^y Z(t,s) dt ds, Z(x,y)\right) \right] - \\
&\quad - \left[f(x,y, \int_0^x \int_0^y Z(t,s) dt ds, z_1(x,y))\right]
\end{aligned}$$

where in  $-$  we choose the same function  $z$ .

By Lipschitz condition and the properties of  $\beta$

$$\beta(T(Z)(x,y)) \leq k_n \beta\left(\int_0^x \int_0^y Z(t,s) dt ds + L_n \beta(Z(x,y))\right).$$

Therefore

$$\begin{aligned}
\exp(-p_n(x+y)) \beta(T(Z)(x,y)) &\leq \\
&\leq (p_n^{-2} k_n + L_n) \sup_{(x,y) \in P_n} \exp(-p_n(t+s)) \beta(Z(t,s))
\end{aligned}$$

and the inequality (\*) is proved.

Let  $p_n^2 > (1 - L_n)^{-1} k_n$ , ( $n = 1, 2, \dots$ ). Define

$$\Phi(Z) = \left( \sup_{(x,y) \in P_1} \exp(-p_1(x+y))\beta(Z(x,y)), \right. \\ \left. \sup_{(x,y) \in P_2} \exp(-p_2(x+y))\beta(Z(x,y)), \dots \right)$$

for any nonempty subset  $Z$  of  $\chi_0$ .

Evidently,  $\Phi(Z) \in \mathcal{S}_\infty$ . By Ascoli theorem and properties of  $\beta$  our function  $\Phi$  satisfy conditions (1)-(3) listed in section 2. From inequality (\*) it follows that  $\Phi(T(Z)) < \Phi(Z)$  whenever  $\Phi(Z) > \emptyset$ , and all assumptions of Sadowski's fixed point theorem are satisfied. Consequently,  $T$  has a fixed point in  $\chi_0$  which completes the proof.

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