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ON THE RATE OF POINTWISE CONVERGENCE OF THE KANTOROVICH-TYPE OPERATORS

ABSTRACT: For bounded or some locally bounded functions f measurable on an interval I there is estimated the rate of convergence of the Kantorovich-type operators $L_n^* f$ at those points $x \in \operatorname{Int} I$ at which the one-sided limits $f(x \pm 0)$ exist. In the main theorems the Chanturiya modulus of variation is used.

KEY WORDS: Kantorovich-type operator, rate of convergence, modulus of variation.

1. PRELIMINARIES

Let I be a finite or infinite interval and let $M(I)$ be the class of all measurable complex-valued functions bounded on I. In the case when I is an infinite interval, denote by $M_{loc}(I)$ the class of all functions measurable on I and bounded on every compact subinterval of I. Given any $n \in N := \{1,2,...\}$, let J_n be a set of indices contained in $Z := \{0,\pm 1,\pm 2,...\}$ and let I be the union of non-overlapping intervals $I_{j,n}$, $(j \in J_n)$, with increasing left [right] end points. Introduce, formally, for functions f belonging to $M(I)$ or $M_{loc}(I)$, the discrete operators L_n defined by

$$L_n f(x) := \sum_{j \in J_n} f(\xi_{j,n}) p_{j,n}(x) \qquad (x \in I,\ n \in N), \tag{1}$$

where $\xi_{j,n} \in I_{j,n}$ and $p_{j,n}$ are non-negative functions continuous on I. Denote by L_n^* the Kantorovich-type modification of operators (1), given by

$$L_n^* f(x) := \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} f(t)dt \qquad (x \in I,\ n \in N), \tag{2}$$

with $m_{j,n} = \operatorname{meas} I_{j,n}$. Assume that, for every $x \in I$,

$$\rho_n(x) := \sum_{j \in J_n} p_{j,n}(x) - 1 \to 0 \quad \text{as} \quad n \to \infty, \tag{3}$$

and that

$$\mu_{2,n}^*(x) := \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \int_{I_{j,n}} (t-x)^2 dt < \infty \qquad (n \in N). \tag{4}$$

In view of the Shisha and Mond Theorem ([4], pp. 28-29) we have

$$\lim_{n\to\infty} L_n^* f(x) = f(x)$$

at every point x of continuity of $f \in M(I)$ at which $\mu_{2,n}^*(x) \to 0$ as $n \to \infty$. Some approximation properties of certain concrete operators of the form (2) for continuous or Lebesgue-integrable functions f are investigated e.g. in [5, Chap.9], [9, Chap. II].

In this paper we present general quantitative inequalities for the rate of pointwise convergence of $L_n^* f(x)$ for functions $f \in M(I)$ (or $f \in M_{loc}(I)$) at those points $x \in \operatorname{Int} I$ at which the one-sided limits $f(x \pm 0)$ exist. In particular, inequalities of this type for the Bernstein-Kantorovich polynomials are obtained in [10]. Analogous results for operators (1) are given in [1].

For the sake of brevity we write

$$s(x) := \tfrac{1}{2}\{f(x+0) + f(x-0)\}, \qquad r(x) := \tfrac{1}{2}\{f(x+0) - f(x-0)\}.$$

Our main estimates concerning the deviation $|L_n^* f(x) - s(x)|$ are expressed in terms of the modulus of variation of the function

$$g_x(t) := \begin{cases} f(t) - f(x+0) & \text{if} \quad t > x, \\ 0 & \text{if} \quad t = x, \\ f(t) - f(x-0) & \text{if} \quad t < x, \end{cases} \qquad (t \in I).$$

Given any positive integer k, the modulus of variation $\mathrm{v}_k(g;Y)$ of a bounded function g on a finite or infinite interval Y is defined as the upper bound of the set of all numbers

$$\sum_{j=1}^{k} |g(b_j) - g(a_j)|$$

over all systems Π_k of k non-overlapping intervals (a_j, b_j) contained in Y. If $k = 0$ we take $\mathrm{v}_0(g;Y) = 0$. Clearly, $\mathrm{v}_k(g;Y)$ is a non-decreasing function of k. Some basic properties of this modulus can be found e.g. in [3].

In our considerations we use the standing notation :

$$I_x(h) := [x+h, x] \cap I \quad \text{if} \quad h < 0, \qquad I_x(h) := [x, x+h] \cap I \quad \text{if} \quad h > 0,$$

$$J_x(h) := [x-h, x+h] \cap I \quad \text{for} \quad h > 0.$$

The integral part of a real number u is denoted by $[u]$.

2. MAIN RESULTS

Let us note that under the assumption $f \in M(I)$ the operators (2) can be written in the form

(5)
$$L_n^* f(x) = \int_I f(t) H_n(x,t) dt$$

with

$$H_n(x,t) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \chi_{j,n}(t),$$

where $\chi_{j,n}$ denotes the characteristic function of the interval $I_{j,n}$. The same is also true for $f \in M_{loc}(I)$, satisfying the suitable growth condition (as in Theorem 2 below).

Consider a point $x \in \operatorname{Int} I$ at which both limits $f(x \pm 0)$ exist. It is clear that

(6)
$$L_n^* f(x) - s(x) = L_n^* g_x(x) + r(x) L_n^* \operatorname{sgn}_x(x) + s(x) \rho_n(x),$$

where

$$\operatorname{sgn}_x(t) := \begin{cases} 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x. \end{cases}$$

In order to evaluate the term $L_n^* g_x(x)$ it is convenient to use the representation (5) and write

(7)
$$L_n^* g_x(x) = \left(\int_{I_x(-a)} + \int_{I_x(b)} \right) g_x(t) H_n(x,t) dt + \vartheta_x(a,b) \int_{D_x(a,b)} g_x(t) H_n(x,t) dt$$

where $a > 0$, $b > 0$, $D_x(a,b) = I[x-a, x+b]$, $\vartheta_x(a,b) = 0$ if neither of the points $x-a$, $x+b$ belongs to Int I, and $\vartheta_x(a,b) = 1$ otherwise.

Lemma. Suppose that $x \in \operatorname{Int} I$ *and that f is bounded on an interval* $I_x(h)$, $h \neq 0$. *Choose a positive null sequence* $(d_n)_1^\infty$ *such that* $d_n \leq 1/2$ *and write* $\mu := [1/d_n]$. *Then, for every* $n \in N$,

(8)
$$\left| \int_{I_x(h)} g_x(t) H_n(x,t) dt \right| \leq$$

$$\le P_n(x,h)\left\{\sum_{i=1}^{\mu-1}\frac{1}{i^3}\mathrm{v}_i(g_x;I_x(ihd_n))+\frac{1}{\mu^2}\mathrm{v}_\mu(g_x;I_x(h))\right\},$$

where $P_n(x,h):=1+\rho_n(x)+8\mu_{2,n}^*(x)h^{-2}d_n^{-2}$.

This result follows by the same method as in [1] or [2], and we omit the details.

If the function *f* is bounded on *I* and if at least one of the points $x-a$, $x+b$ belongs to Int *I*, then the obvious inequality

$$\int_{|t-x|\ge s} H_n(x,t)\,dt \le \frac{1}{s^2}\mu_{2,n}^*(x), \qquad (x\in I,\ s>0)$$

yields the estimate

$$\left|\int_{D_x(a,b)} g_x(t)H_n(x,t)\,dt\right| \le \frac{1}{c^2}\mu_{2,n}^*(x)\mathrm{v}_1(g_x;I), \tag{9}$$

where $\mu_{2,n}^*(x)$ is defined by (4) and $c=\min\{a,b\}$.

Taking into account identities (6), (7) with $a=b=1$, inequality (9) and the Lemma (with $h=-1$ and $h=1$), we can state our main result as follows.

Theorem 1. Suppose that, for all $x\in I$ *and all* $n\in N$,

$$\sum_{j\in J_n} p_{j,n}(x)\equiv 1+\rho_n(x)\le\varphi_1(x), \tag{10}$$

$$\mu_{2,n}^*(x)\le\varphi_2(x)d_n^2, \tag{11}$$

where φ_1, φ_2 *are some positive functions continuous on I and* $(d_n)_1^\infty$ *is a positive null sequence. If* $f\in M(I)$ *and if at a point* $x\in \operatorname{Int} I$ *the one-sided limits* $f(x\pm 0)$ *exist then, for all positive integers n such that* $d_n\le 1/2$,

$$|L_n^*f(x)-s(x)|\le P(x)\left\{\sum_{i=1}^{\mu-1}\frac{1}{i^3}\mathrm{v}_i(g_x;J_x(id_n))+\frac{1}{\mu^2}\mathrm{v}_\mu(g_x;J_x(1))\right\}+$$

$$+\vartheta_x(1,1)\varphi_2(x)d_n^2\mathrm{v}_1(g_x;I)+\left|r(x)L_n^*\operatorname{sgn}_x(x)\right|+|s(x)\rho_n(x)|,$$

where $\mu=[1/d_n]$, $P(x):=2\{\varphi_1(x)+8\varphi_2(x)\}$ *and* $\rho_n(x)$ *is defined by* (3).

Following [1, Theorem 2] (or [2, Theorem 2]) one can get the result for unbounded functions f on infinite interval I.

Theorem 2. Let $I=[0,\infty)$ *or* $I=(-\infty,\infty)$ *and let conditions* (10), (11) *be fulfilled. Suppose that a function* f *of class* $M_{loc}(I)$ *satisfies the growth condition*

$$|f(x)| \le \psi(x) \qquad (x\in I) \tag{12}$$

with a positive continuous function ψ *such that for all* $n\ge n_0\in N,\ \ x\in I$

$$\sum_{j\in J_n}(m_{j,n})^{-1}p_{j,n}(x)\int\limits_{I_{j,n}}\psi^2(t)\,dt \le \varphi_3(x), \tag{13}$$

$0<\varphi_3(x)<\infty$. *If at a point* $x\in \operatorname{Int} I$ *the limits* $f(x\pm 0)$ *exist and if* A *is an arbitrary positive number for which* $|x|\le A$ *then, for every integer* $n\ge n_0$, *such that* $d_n\le 1/2$, *we have*

$$|L_n^* f(x)-s(x)|\le$$

$$\le 2P(x,A)\left\{\sum_{i=1}^{\mu-1}\frac{1}{i^3}\mathrm{v}_i\big(g_x;J_x(iAd_n)\big)+\frac{1}{\mu^2}\mathrm{v}_\mu\big(g_x;J_x(A)\big)\right\}+$$

$$+\Lambda(x,A)d_n+|r(x)L_n^*\operatorname{sgn}_x(x)|+|s(x)\rho_n(x)|,$$

where

$$P(x,A):=\varphi_1(x)+8\varphi_2(x)/A^2,$$

$$\Lambda(x,A):=A^{-1}(\varphi_2(x)\varphi_3(x))^{1/2}+\tfrac{1}{2}A^{-2}\psi(x)\varphi_2(x)$$

and the remaining quantities are of the same meaning as in Theorem 1.

Now, let us denote by $BV_p(I)$ $(1\le p<\infty)$ the class of all functions of bounded p-th power variation on I. Here, by p-th power variation of a function g on the interval $Y\subseteq I$ we will mean the upper bound of the set of non-negative numbers

$$\left\{\sum_j |g(b_j)-g(a_j)|^p\right\}^{1/p}$$

over all finite systems of non-overlapping intervals $(a_j,b_j)\subset Y$. We will denote it by $V_p(g;Y)$. Clearly, if $V_p(g;Y)<\infty$ then for every positive integer j,

$$\mathrm{v}_j(g;Y) \le j^{1-1/p} V_p(g;Y).$$

Using this inequality and proceeding similarly to [11, pp.152-153] we get from Theorem 1 the following

Corollary. Suppose that conditions (10), (11) *are satisfied. If* $f \in BV_p(I)$ *then, for all* $x \in \operatorname{Int} I$ *and* $n \in N$ *such that* $0 < d_n \le 1/2$,

$$|L_n^* f(x) - s(x)| \le Q(x) \frac{1}{\mu^{1+1/p}} \sum_{k=0}^{\mu^2-1} \frac{1}{(\sqrt{k+1})^{1-1/p}} V_p(g_x; Y_k) +$$

$$+ |r(x) L_n^* \operatorname{sgn}_x(x)| + |s(x)\rho_n(x)|,$$

where $Y_k = J_x(1/\sqrt{k})$ *if* $k = 1,2,\dots,\mu^2 - 1$, $Y_0 = I$, $Q(x) := 15\{\varphi_1(x) + 8\varphi_2(x)\}$, μ, $L_n^* \operatorname{sgn}_x(x)$, $\rho_n(x)$ *have the same meaning as in Theorem 1.*

In the case of unbounded functions f Theorem 2 leads to the analogous Corollary, too.

Remark 1. The term $L_n^* \operatorname{sgn}_x(x)$ occurring in our estimates can be written in the form

$$L_n^* \operatorname{sgn}_x(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left(\int_{t>x} \chi_{j,n}(t)dt - \int_{t<x} \chi_{j,n}(t)dt \right).$$

Suppose that x belongs to the interval $I_{l,n} = [\alpha_{l,n}, \beta_{l,n})$. Then

$$L_n^* \operatorname{sgn}_x(x) = \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left\{ \int_x^{\beta_{l,n}} \chi_{j,n}(t)dt + \sum_{k>l} \int_{I_{k,n}} \chi_{j,n}(t)dt \right\} +$$

$$- \sum_{j \in J_n} (m_{j,n})^{-1} p_{j,n}(x) \left\{ \sum_{k<l} \int_{I_{k,n}} \chi_{j,n}(t)dt + \int_{\alpha_{l,n}}^{x} \chi_{j,n}(t)dt \right\}.$$

(It is understood that the summation in the inner sums is extended over $k \in J_n$.) Further,

$$L_n^* \operatorname{sgn}_x(x) = (m_{l,n})^{-1} p_{l,n}(x)(\beta_{l,n} - x) - (m_{l,n})^{-1} p_{l,n}(x)(x - \alpha_{l,n}) +$$
$$+ \sum_{j>l} (m_{j,n})^{-1} p_{j,n}(x) m_{j,n} - \sum_{j<l} (m_{j,n})^{-1} p_{j,n}(x) m_{j,n} =$$
$$= (m_{l,n})^{-1} p_{l,n}(x)(\beta_{l,n} - 2x + \alpha_{l,n}) + \sum_{j>l} p_{j,n}(x) - \sum_{j<l} p_{j,n}(x).$$

Consequently,

$$|L_n^* \operatorname{sgn}_x(x)| \le p_{l,n}(x) + \left| \sum_{j>l} p_{j,n}(x) - \sum_{j<l} p_{j,n}(x) \right|. \tag{14}$$

The above estimate is useful in applications.

Remark 2. In view of the continuity of the function g_x at x, the right-hand side of inequality (8) converges to zero as $n \to \infty$ (see e.g. Remark 1 in [11]). Moreover, for many operators of the form (2), condition (3) is satisfied and

$$\lim_{n\to\infty} L_n^* \operatorname{sgn}_x(x) = 0 \quad \text{at every} \quad x \in \operatorname{Int} I.$$

Consequently, for these operators, the right-hand sides of the inequalities given in Theorems 1, 2 and Corollary converge to zero as n tends to infinity.

3. EXAMPLES

Let $\{\zeta_k\}_1^\infty$ be a sequence of independent and identically distributed random variables with expectations $E\zeta_k = x$ and finite variances $E(\zeta_k - E\zeta_k)^2 = \sigma^2(x)$, where x is a real parameter taking values in an interval $I \subseteq [0,\infty)$. Suppose that ζ_1 has the lattice distribution $F := \{p_{j,1}(x) : x \in I, j \in J_1\}$ concentrated on a set $J_1 \subseteq N_0 := \{0,1,2,...\}$. The operators (1) with the system $\{p_{j,n}(x) : x \in I, j \in J_n\}$ being the n-fold convolution of F with it self and $\xi_{j,n} = j/n$ are called the discrete Feller operators ([6], p. 218). Consider the corresponding Kantorovich-type operators L_n^* defined by (2) in which $\xi_{j,n} \in I_{j,n}$ and $m_{j,n} \le 1/n$ for all $j \in J_n$, $n \in N$. Suppose that $x \in I_{l,n}$ with some index $l \in J_n$ and put $\lambda = (l - nx)/\sqrt{n}\sigma(x)$. Then

$$\left| \sum_{j\le l} p_{j,n}(x) - \frac{1}{2} \right| \le \left| \sum_{j\le l} p_{j,n}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} \exp(-u^2/2)\, du \right| +$$

$$+\frac{1}{\sqrt{2\pi}}\left|\int_0^{\lambda}\exp(-u^2/2)\,du\right|.$$

If, moreover,

$$\sigma^2(x)>0 \qquad \text{and} \qquad \beta(x):=E(|\zeta_1-x|^3)<\infty$$

then, in view of the Berry-Esséen Theorem ([6], p. 515),

$$\left|\sum_{j\le l}p_{j,n}(x)-\frac{1}{2}\right|\le\frac{\tau\beta(x)}{\sqrt{n\sigma^3(x)}}+\frac{1}{\sqrt{2\pi n}\sigma(x)}$$

and

$$p_{l,n}(x)\le\frac{2\tau\beta(x)}{\sqrt{n\sigma^3(x)}}+\frac{1}{\sqrt{2\pi n}\sigma(x)},$$

where $(2\pi)^{-1/2}<\tau<0.82$ (see e.g. [2], p. 101). Applying (14) we get

$$(15)\quad \left|L_n^*\,\mathrm{sgn}_x(x)\right|\le 2\left(\left|\frac{1}{2}-\sum_{j\le l}p_{j,n}(x)\right|+p_{l,n}(x)\right)\le\frac{2}{\sqrt{n}}\left(\frac{3\tau\beta(x)}{\sigma^3(x)}+\frac{2}{\sqrt{2\pi}\sigma(x)}\right).$$

Now, we present an application of our Theorems to some concrete Feller-Kantorovich operators.

1. The Bernstein-Kantorovich polynomials $B_n^*\equiv L_n^*$ are defined by (2) with $p_{j,n}(x)=\binom{n}{j}x^j(1-x)^{n-j}$, $I_{j,n}=\left[\frac{j}{n+1},\frac{j+1}{n+1}\right]$, $x\in I=[0,1]$, $j\in J_n=\{0,1,\ldots,n\}$ and $m_{j,n}=1/(n+1)$. In this case, $\sigma^2(x)=x(1-x)$, $\beta(x)=x(1-x)\cdot$ $\cdot(2x^2-2x+1)$ ([8], p. 98). Moreover,

$$\mu_{2,n}^*(x)=\frac{3x(1-x)(n-1)+1}{3(n+1)^2}\qquad\text{for all } x\in I,\ n\in N;$$

whence $\mu_{2,n}^*(x)\le 1/4n$ for all $n\in N$ and $\mu_{2,n}^*(x)\le 3x(1-x)/n$ for $n\ge(x(1-x))^{-1}$. Consequently, Theorem 1 and Corollary apply for $x\in(0,1)$ and all $n\ge 2$, with $d_n=1/\sqrt{n}$, $\varphi_1(x)=1$, $\varphi_2(x)=1/4$, $\vartheta_x(1,1)=0$, $\rho_n(x)=0$ and, in view of (15),

$$|B_n^*\,\mathrm{sgn}_x(x)|\le 10(x^2-x+1)/\sqrt{nx(1-x)}.$$

For $n\ge(x(1-x))^{-1}$ it is convenient to choose $\varphi_2(x)=3x(1-x)$.

2. Let $S_n^* \equiv L_n^*$ be the modified Szasz-Mirakyan operators defined by (2) with $p_{j,n}(x) = (nx)^j e^{-nx}/j!$, $I_{j,n} = \left[\frac{j}{n}, \frac{j+1}{n}\right]$, $x \in I = [0,\infty)$, $j \in J_n = N_0$ and $m_{j,n} = 1/n$. In this case, $\sigma^2(x) = x$, $\beta(x) \le 8x^3 + 6x^2 + x$ ([8], p. 99), $\mu_{2,n}^*(x) = x/n + 1/3n^2$ for all $x \in I$, $n \in N$. Consequently, Theorem 1 and Corollary apply for $x > 0$ and all $n \ge 2$ with $d_n = 1/\sqrt{n}$, $\varphi_1(x) = 1$, $\varphi_2(x) = (6x+1)/12$, $\vartheta_x(1,1) = 1$, $\rho_n(x) = 0$, and

$$|S_n^* \operatorname{sgn}_x(x)| \le 10(4x^2 + 3x + 1)/\sqrt{nx},$$

by (15).

3. The Baskakov-Kantorovich operators $U_n^* \equiv L_n^*$ are defined by (2) in which

$$p_{j,n}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j}, \qquad I_{j,n} = \left[\frac{j}{n}, \frac{j+1}{n}\right], \qquad x \in I = [0,\infty),$$

$j \in J_n = N_0$, $m_{j,n} = 1/n$. Now $\sigma^2(x) = x(1+x)$, $\beta(x) \le 16x^3 + 9x^2 + x$ ([8], p. 100). It is easy to see that

$$\mu_{2,n}^*(x) = \frac{x(x+1)}{n} + \frac{1}{3n^2} \qquad \text{for all } x \in I,\ n \in N.$$

Hence, our results can be applied for $x > 0$ and all $n \ge 2$, with $d_n = 1/\sqrt{n}$, $\varphi_1(x) = 1$, $\varphi_2(x) = (1/3)(3x(x+1)+1)$, $\vartheta_x(1,1) = 1$, $\rho_n(x) = 0$. In this case inequality (15) implies

$$|U_n^* \operatorname{sgn}_x(x)| \le 10(8x^2 + 5x + 1)/\sqrt{nx(1+x)^3}.$$

Finally, let us consider the generalized Favard operators $F_n \equiv L_n$, which are not the Feller-type ones. They are defined by formula (1) in which $\xi_{j,n} = j/n$, $j \in J_n = Z$, $x \in I = (-\infty,\infty)$ and

$$p_{j,n}(x) \equiv p_{j,n}(\gamma, x) = (\sqrt{2\pi} n\gamma_n)^{-1} \exp\left(-\tfrac{1}{2}\gamma_n^{-2}\left(\frac{j}{n} - x\right)^2\right),$$

where $\gamma = (\gamma_n)_1^\infty$ is a positive null sequence such that

$$n^2\gamma_n^2 \ge \tfrac{1}{2}\pi^{-2}\log n \qquad \text{for } n \ge 2, \qquad \gamma_1^2 \ge \tfrac{1}{2}\pi^{-2}\log 2$$

(see [7]). Denote by F_n^* their Kantorovich modification of the form (2) with $I_{j,n} = [j/n, (j+1)/n]$ and $m_{j,n} = 1/n$ for all $j \in Z$ and $n \in N$. As is known ([7]), for all $x \in I$ and $n \in N$

$$|\rho_n(x)| \leq 2 \quad \text{or} \quad |\rho_n(x)| \leq 7\pi\gamma_n$$

and

$$\mu_{2,n}(x) := \sum_{j=-\infty}^{\infty} \left(\frac{j}{n} - x\right)^2 p_{j,n}(x) \leq 51\gamma_n^2.$$

An easy computation shows that

$$\mu_{2,n}^*(x) \leq \mu_{2,n}(x) + \frac{1}{n}\sqrt{\mu_{2,n}(x)}\,\sqrt{1+\rho_n(x)} + \frac{1}{3n^2}(1+\rho_n(x)) \leq 158\gamma_n^2.$$

Hence, applying Theorem 1 (or Corollary) to these operators, we can put $\varphi_1(x) = 3$, $\varphi_2(x) = 158\kappa^2$ and $d_n = \gamma_n/\kappa$, where $\kappa := \max\{1, 2\sup_{j\in N}\gamma_j\}$. In order to estimate the term $F_n^* \operatorname{sgn}_x(x)$ we use inequality (14) and, arguing similarly to [1, Sect. 4.I], we obtain

$$|F_n^* \operatorname{sgn}_x(x)| \leq 3(\sqrt{2\pi}n\gamma_n)^{-1} \leq 3\sqrt{\pi}(\log n)^{-1/2} \qquad \text{for } n \geq 2.$$

The same estimates can also be used in Theorem 2. Additionally, let us note that if unbounded function f satisfies (12) with $\psi(x) = \exp(qx^2)$, $q > 0$, then condition (13) is fulfilled for $q\gamma_n^2 \leq 3/128$ with $\varphi_3(x) = c(q)\exp(4qx^2)$, $c(q)$ being some positive constant depending on q.

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