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COMPARISON THEOREMS FOR n -th ORDER NONLINEAR DIFFERENTIAL INEQUALITIES WITH DEVIATING ARGUMENT

ABSTRACT: This paper deals with comparison theorems for oscillatoriness all solutions of the n -th order nonlinear differential inequalities with quasiderivatives and deviating argument of a form

$$y(t)[(r_{n-1}(t)(\dots r_2(t)(r_1(t)y'(t))'\dots)')' + a(t)f(y(g(t))) - b(t)] \leq 0.$$

KEY WORDS: comparison theorems, n -th order, nonlinear, differential inequality, quasiderivatives, deviating argument, oscillatory and nonoscillatory solutions.

1. INTRODUCTION

The aim of this paper is to prove some comparison theorems for n -th order nonlinear differential inequalities and equations with quasiderivatives and deviating argument of a form

$$(1) \quad y(t)[L_n y(t) + a(t)f(y(g(t))) - b(t)] \leq 0,$$

$$(2) \quad L_n y(t) + a(t)f(y(g(t))) = b(t),$$

$$(3) \quad y(t)[L_n y(t) + a(t)f(y(g(t)))] \leq 0,$$

$$(4) \quad L_n y(t) + a(t)f(y(g(t))) = 0$$

on some unbounded $< t_0, \infty$, where to so called quasiderivatives are defined by the following relations

$$(5) \quad L_0 y(t) = y(t), \quad L_i y(t) = r_i(t)[L_{i-1} y(t)]' \quad \text{for } i = 1, 2, \dots, n-1,$$

$$L_n y(t) = [L_{n-1} y(t)]'.$$

We shall consider only so called regular solutions of these inequalities and equations, i.e. solutions which are defined on some neighbourhood of infinity and which are not eventually trivial. Such a solution is said to be oscillatory if it has arbitrarily large zeroes, otherwise it is said to be nonoscillatory, i.e. it is either eventually positive one or eventually negative one.

As it will be shown in the sequel, the cases of even order and odd one are different, it will be useful to define following.

If $T \geq t_0$ and $y(t)$ is a function such that $y(t)$ and $y(g(t))$ are defined on $\langle T, \infty \rangle$, we can formally define the following integral operators.

$$\begin{aligned}
 & \text{(a) } I_{i,n}(y, T, t) = \int_t^\infty a(s) |f(y(g(s)))| ds \quad \text{for } t \geq T, \\
 (8) \quad & \text{(b) } I_{i,j}(y, T, t) = \int_t^\infty \frac{I_{i,j+1}(y, T, s)}{r_j(s)} ds \quad \text{for } t \geq T, \quad j = n-1, n-2, \dots, i, \\
 & \text{(c) } I_{i,j}(y, T, t) = \int_T^t \frac{I_{i,j+1}(y, T, s)}{r_j(s)} ds \quad \text{for } t \geq T, \quad j = i-1, i-2, \dots, 1, \\
 & \text{(d) } I_i(y, T, t) = I_{i,1}(y, T, s) \quad \text{for } t \geq T, \quad j = 1, 2, \dots, n.
 \end{aligned}$$

Remark 1. It is easy to show that from (7) we have

- a) $\lim \varphi_i(t) = 0$ for $t \rightarrow \infty$, $i = 0, 1, \dots, n-1$,
 b) $L_n \varphi_0(t) = b(t)$ for $t > t_0$.

Remark 2. Every solution of the equation (2) solves the inequality (1), too and (4) and (3) are their special cases. Therefore, everything sayed about the solutions of the inequality (1) will be valid for (3), (2) and (4), too.

In proofs the existence of nonoscillatory solutions of the given inequality or equation we shall use one very simple fixed-point theorem.

Theorem 1. Let $Y \neq 0$ be a complete lattice and $\Phi: Y \rightarrow Y$ be an isotonus operator. Then Φ has at least one fixed point in Y .

Proof. See Theorem II.3.3. in [6].

2. DEFINITIONS AND AUXILIARY ASSERTIONS

The following generalizes well known Kiguradze's lemma.

Lemma 1. Let $y(t)$ be a nonoscillatory solution of the inequality (1) on the interval $\langle t_1, \infty \rangle \subset \langle t_0, \infty \rangle$. Then there exists $T > t_1$ and $i \in \{1, 2, \dots, n\}$ such that $n-i$ is even and

$$(a) \quad y(t)y(g(t)) > 0,$$

- c) $N_0 \subset N_1$,
- d) given inequality or equation has a property (A) iff $N_2 = N_4 = \dots = N_{n-2} = N_n = 0$ for n even and $N_3 = N_5 = \dots = N_{n-2} = N_n = 0$ and $N_1 = N_0$ for n odd.

Definition 3. Let $i \in \{2, 3, \dots, n\}$. We shall say that given inequality or equation has a property (A_i) if $N_1 = 0$ and that it has a property (A_1) if $N_1 = N_0$.

Remark 4. By Lemma 1 given inequality or equation has a property (A) iff it has a property (A_i) for every $i \in \{1, 2, \dots, n\}$ such that $n - i$ is even.

Lemma 2. Let y be a nonoscillatory solution of the inequality (1) of a class N_i on $\langle t_1, \infty \rangle \subset \langle t_0, \infty \rangle$ for some $i \in \{1, 2, \dots, n\}$. Then $n - i$ is even and there exists $T > t_1$ such that (9) and

- (a) $I_{i,j}(y, T, T) < \infty$ for $j = n, n-1, \dots, i$,
- (10) (b) if $i > 1$, then $|y(t) - \varphi_0(t)| \geq |y(T) - \varphi_0(T)| + I_i(y, T, t)$,
- (c) if $i = 1$, then $|y(t) - \varphi_0(t)| \geq \lim_{s \rightarrow \infty} |y(s) - \varphi_0(s)| + I_i(y, T, t)$

for every $t \geq T$.

Proof. Without loss of generality we can assume $y(t)$ to be eventually positive. By Lemma 1 $n - i$ is even and there exists $T > t_1$ such that (9) holds for every $t \geq T$. Let us denote $z(t) = y(t) - \varphi_0(t)$. Then from the inequality (1) we have

$$[L_{n-1}z(t)]' = L_n z(t) \leq -a(t)f(y(g(t))) \quad \text{for } t \geq T.$$

Integrating this inequality from t to $s \geq t \geq T$ we get

$$L_{n-1}z(t) \geq L_{n-1}z(s) + \int_t^s a(u)f(y(g(u)))du \quad \text{for } s \geq t \geq T.$$

Using denotation (8) and the fact, that $L_{n-1}z(s) > 0$ by (9)(c), the limit $s \rightarrow \infty$ leads to

$$(11) \quad L_{n-1}z(t) \geq \int_t^\infty a(u)f(y(g(u)))du = I_{i,n}(y, T, t) \quad \text{for } t \geq T.$$

Using this inequality for $t = T$ we get the validity of (10)(a) for $j = n$.

If $i < n$, dividing inequality (11) by $r_{n-1}(t)$ and integrating it from t to $s \geq t \geq T$ we have

$$-L_{n-2}z(t) \geq -L_{n-2}z(s) + \int_t^s \frac{1}{r_{n-1}(u)} I_{i,n}(y, T, u) du \quad \text{for } s \geq t \geq T.$$

The fact that $L_{n-2}z(s) < 0$ by (9)(b) for $j = n-2$ and limit $s \rightarrow \infty$ leads to

$$-L_{n-2}z(t) \geq \int_t^\infty \frac{1}{r_{n-1}(u)} I_{i,n}(y, T, u) du = I_{i,n-1}(y, T, t) \quad \text{for } t \geq T.$$

Using this inequality for $t = T$ we get the validity of (10)(a) for $j = n-1$. Repeating this steps and using the fact, that $n-i$ is even, we finally get

$$(12) \quad L_{i-1}z(t) \geq \lim_{s \rightarrow \infty} L_{i-1}z(s) + I_{i,i}(y, T, t) \quad \text{for } t \geq T$$

and validity (10)(a) for $j = n-1, n-2, \dots, i$.

If $i = 1$, this inequality is identical with the inequality (10)(c).

If $i > 1$ then from (12) and from the fact, that $L_{i-1}z(s) > 0$ by (9)(c), we have

$$L_{i-1}z(t) \geq I_{i,i}(y, T, t) \quad \text{for } t \geq T.$$

Dividing this inequality by $r_{i-1}(t)$ and integrating it from T to $t \geq T$ and using (9)(c) and (8) we get

$$L_{i-2}z(t) \geq L_{i-2}z(T) + I_{i,i-1}(y, T, t) \geq I_{i,i-1}(y, T, t) \quad \text{for } t \geq T.$$

Repeating this step we finally get (10)(b) and this completes the proof of the lemma.

Lemma 3. Let $i \in \{2, 3, \dots, n\}$, $T \geq t_1 > t_0$ and a function $w \in C < t_1, \infty >$ are such that $n-i$ is even, $|w|$ is positive and $|w - \varphi_0|$ is positive and nondecreasing on $< t_1, \infty >$ and

- (a) $g(t) \geq t_1$,
- (13) (b) $I_{i,n}(w, T, T) < \infty$ for $j = n, n-1, \dots, i$,
- (c) $|w(t) - \varphi_0(t)| \geq |w(T) - \varphi_0(T)| + I_i(w, T, t)$

for $t \geq T$. Then the equation (2) has a nonoscillatory solution y on $< T, \infty >$ of a class N_i such that $|y(t)| \leq |w(t)|$ for $t \geq T$.

Proof. Let assume, that $w(t) > 0$ for $t \geq t_1$ (the case $w(t) < 0$ is analogous).

Let Y be a set of all functions y defined on $\langle t_1, \infty \rangle$ such that

- (14) (a) $y(t) = w(t)$ for $t \in \langle t_1, T \rangle$,
 (b) $0 < y(t) \leq w(t)$ for $t > T$,
 (c) the function $y - \varphi_0$ is nondecreasing on $\langle t_1, \infty \rangle$.

a) It is clear, that $w \in Y$ and hence $Y \neq \emptyset$.

b) Let us define point-wise ordering on Y , i.e. $y_1 \leq y_2$, if $y_1(t) \leq y_2(t)$ for every $t \geq t_1$. Let $0 \neq Y_1 \subset Y$. Then $\sup\{y, y \in Y_1\} = \sup\{y - \varphi_0, y \in Y_1\} + \varphi_0 \in Y$, because the supremum of from above bounded set of nondecreasing functions is a nondecreasing function, too and supremum holds the properties (14)(a) and (b). The same is valid for infimum. Hence the set Y is a complete lattice

c) Let us define the operator Φ on Y by the relations.

- (15) (a) $\Phi y(t) = y(t)$ for $t \in \langle t_1, T \rangle$,
 (b) $\Phi y(t) = y(T) - \varphi_0(T) + \varphi_0(t) + I_1(y, T, t)$ for $t \geq T$.

1. With regard to (13)(b), (14) and (6)(f) the definition of the operator Φ is correct.
2. With regard to (13)(c) it is obvious, that $\Phi Y \subset Y$.
3. Finally by (6)(f) $\Phi y_1 \leq \Phi y_2$ for every $y_1, y_2 \in Y$ such that $y_1 \leq y_2$.

Hence operator Φ is an isotonus one by Theorem 1 there exists $y \in Y$ such that $y = \Phi y$. Differentiating this equality and using the definition (15) we can simply show, that this fixed point y is the nonoscillatory solution of the equation (2) with wanted properties.

Lemma 4. Let n be even, $T \geq t_1 > t_0$, $D > 0$ and $w \in C \langle t_1, \infty \rangle$ we such, that $|w|$ is positive and $|w - \varphi_0|$ is positive and nonincreasing on $\langle t_1, \infty \rangle$ and

- (a) $g(t) \geq t_1$,
 (b) $I_{1,j}(w, T, T) < \infty$ for $j = n, n-1, \dots, 1$,
 (c) $|w(t) - \varphi_0(t)| \geq D + I_1(w, T, t)$

for $t \geq T$. Then the equation (2) has a nonoscillatory solution y on $\langle T, \infty \rangle$ of a class N_1 such that $|y(t)| \leq |w(t)|$ for $t \geq T$ and $\lim y(t) = D$ for $t \rightarrow \infty$, i.e. $y \in N_1 \setminus N_0$.

Proof. We can prove this lemma by the analogous way as the previous one, when we define Y as the set of all functions y defined on $\langle t_1, \infty \rangle$ such that

- (a) $y(t) = w(t)$ for $t \in \langle t_1, T \rangle$,
- (b) $0 < y(t) \leq w(t)$ for $t > T$,
- (c) function $y - \varphi_0$ is nonincreasing on $\langle t_1, \infty \rangle$,
- (d) $\lim_{t \rightarrow \infty} y(t) = D$

and the operator Φ by the relations

- (a) $\Phi y(t) = y(t)$ for $t \in \langle t_1, T \rangle$,
- (b) $\Phi y(t) = D + \varphi_0(t) + I_1(y, T, t)$ for $t \geq T$.

Lemma 5. Let $\varphi \in C \langle t_0, \infty \rangle$ have arbitrarily large zeroes and $\lim \varphi(t) = 0$ for $t \rightarrow \infty$. Then for every $t_1 > t_0$ there exists $T \geq t_1$ such that $\varphi(t) - \varphi(T) \geq 0$ for every $t \geq T$.

Proof. We can choose $T \geq t_1$ such that $\varphi(T) = \min\{\varphi(t), t \geq T\}$.

3. MAIN RESULTS

Now we can prove some comparison theorems for the investigated inequalities and equations to have a property (A).

Theorem 2. Let $i \in \{1, 2, \dots, n\}$ be such that $n - i$ is even. Then the inequality (1) has a property (A_i) if and only if the equation (2) has a property (A_i) .

Proof. Every solution of the equation (2) solves the inequality (1), too. Hence it is enough to prove one implication. Let the inequality (2) has not a property (A_i) and $i > 1$. Then the inequality (1) has some nonoscillatory solution on some $\langle t_1, \infty \rangle \subset \langle t_0, \infty \rangle$ of a class N_i . Let $w(t) = y(t)$ for $t \geq t_1$. Then by Lemma 2 the assumptions of Lemma 3 are fulfilled and by this lemma the equation (2) has some nonoscillatory solution of a class N_i , too and hence the equation (2) has

not a property (A_i) , too. The proof of the case of $i = 1$ is analogous using Lemma 2 and Lemma 4.

Corollary 1. The inequality (1) has a property (A) if and only if the equation (2) has a property (A).

Theorem 3. Let $i \in \{1, 2, \dots, n\}$ be such that $n - i$ is even. Then the equation (2) has a property (A_i) if and only if the equation (4) has a property (A_i) .

Proof. 1. Let the equation (2) have not a property (A_i) . Then there exists some nonoscillatory solution y of (2) of a class N_i on some $\langle t_1, \infty \rangle$. Without loss of generality we can assume y to be eventually positive. Let $i > 1$, then by Lemma 2 there exists $T > t_1$ such that (9) and (10) hold and by Lemma 5 we can choose T such that

$$\varphi_0(t) \geq \varphi_0(T) \quad \text{for } t \geq T.$$

From this inequality and (10)(b) we have

$$y(t) \geq y(T) + \varphi_0(t) - \varphi_0(T) + I_i(y, T, t) \geq y(T) + I_i(y, T, t) \quad \text{for } t \geq T.$$

Hence, the assumptions of Lemma 3 with $w = y$ for the equation (4) are fulfilled and by this lemma the equation (4) has not a property (A_i) , too. The case of $i = 1$ is analogous using Lemma 2 and Lemma 4.

2. The proof of the second implication is analogous.

Corollary 2. The equation (2) has a property (A) if and only if the equation (4) has a property (A).

Let us consider two inequalities of a form (1)

$$(16) \quad y(t) \left[L_n^{(1)} y(t) + a_1(t) f_1(y(g_1(t))) - b_1(t) \right] \leq 0,$$

$$(17) \quad y(t) \left[L_n^{(2)} y(t) + a_2(t) f_2(y(g_2(t))) - b_2(t) \right] \leq 0,$$

two equations of a form (2)

$$(18) \quad L_n^{(1)} y(t) + a_1(t) f_1(y(g_1(t))) = b_1(t),$$

$$(19) \quad L_n^{(2)} y(t) + a_2(t) f_2(y(g_2(t))) = b_2(t)$$

and two equations of a form (4)

$$(20) \quad L_n^{(1)}y(t) + a_1(t)f_1(y(g_1(t))) = 0,$$

$$(21) \quad L_n^{(2)}y(t) + a_2(t)f_2(y(g_2(t))) = 0,$$

where the functions a_j , b_j , g_j , $r_{j,1}$, ..., $r_{j,n-1}$ satisfy the conditions (6) for $j=1,2$ and operators $L_i^{(1)}y$ and $I_i^{(1)}(y,T,t)$, resp. $L_i^{(2)}y$ and $I_i^{(2)}(y,T,t)$, are defined by (5) and (8) for the inequality (16), resp. for the inequality (17).

Theorem 4. Let $i \in \{1,2,\dots,n\}$ be such that $n-i$ is even and

$$(a) \quad r_{1,j}(t) \geq r_{2,j}(t) \quad \text{for } j=1,2,\dots,n-1, t \geq t_0,$$

$$(22) \quad (b) \quad a_1(t) \leq a_2(t) \quad \text{for } t \geq t_0,$$

$$(c) \quad g_1(t) \leq g_2(t) \quad \text{for } t \geq t_0,$$

$$(d) \quad |f_1(y)| \leq |f_2(y)| \quad \text{for each } y \text{ real.}$$

Let the equation (20) have a property (A_i) , then the equation (21) has a property (A_i) , too.

Proof. Let the equation (21) have not a property (A_i) . Then there exists some nonoscillatory solution y of (21) of a class N_i on some $< t_1, \infty$. Without loss of generality we can assume y to be eventually positive. Then by Lemma 2 applied on (21) there exists $T > t_1$ such that (9) and (10) hold.

Let $i > 1$. By (9)(c) the function y is eventually nondecreasing and from this and from the assumption (22) we have

$$a_2(t)f_2(y(g_2(t))) \geq a_2(t)f_2(y(g_1(t))) \geq a_1(t)f_1(y(g_1(t)))$$

for $t \geq T$. By this inequality and the definition of the operators $I_i^{(1)}$ and $I_i^{(2)}$ we get

$$y(t) \geq y(T) + I_i^{(2)}(y,T,t) \geq y(T) + I_i^{(1)}(y,T,t) \quad \text{for } t \geq T.$$

Hence, the assumptions of Lemma 3 with $w = y$ for the equation (20) are fulfilled and by this lemma the equation (20) has not a property (A_i) , too.

Let $i = 1$. Then $\lim y(t) = D > 0$ for $t \rightarrow \infty$ and by (9)(b) the function y is eventually nonincreasing. Hence, there exists $S \geq T$ such that $2D \geq y(t) \geq D$ for $t \geq S$. By the assumptions (22) and by (10) we have

$$I_1^{(1)}(2D,S,S) \leq I_1^{(2)}(2D,S,S) = \frac{f_2(2D)}{f_2(D)} I_1^{(2)}(D,S,S) < \infty.$$

By this inequality we can choose $S \geq T$ such that $I_1^{(1)}(2D, S, S) \leq D$ and from this

$$2D = D + D \geq D + I_1^{(1)}(2D, S, S) \geq D + I_1^{(1)}(y, S, t)$$

for $t \geq S$. Hence, the assumptions of Lemma 4 for the equation (20) with $w \equiv 2D$ and by this lemma the equation (20) has not a property (A_1) . This completes the proof of the theorem.

Corollary 3. Let the assumptions (22) hold and equation (20) have a property (A) . Then the equation (21) has a property (A) , too.

Finally, using the theorems proved above we can simply prove the main result of this paper.

Theorem 5. Let $i \in \{1, 2, \dots, n\}$ be such that $n - i$ is even, the assumptions (22) hold and the inequality (16) has a property (A_i) . Then inequality (17) has a property (A_i) , too.

Proof. If the inequality (16) has a property (A_i) , then by Theorem 2 and Theorem 3 the equations (18) and (20) have this property. By Theorem 4 the equation (21) has this property, too. Finally, by Theorem 3 and Theorem 2 the equation (19) and the inequality (17) have a property (A_i) , too.

Theorem 6. Let the assumptions (22) hold and the inequality (16) have a property (A) . Then the inequality (17) has a property (A) , too.

Remark 5. The assumption (6)(j) cannot be omitted without the suitable substitution. For example, the equation

$$y''(t) + |y(t)|^3 \operatorname{sgn}(y(t)) = 0$$

satisfies the conditions (6) and by Atkinson [1] has a property (A) . On the other hand the equation

$$y''(t) + |y(t)|^3 \operatorname{sgn}(y(t)) = 3t^{-3}$$

does not satisfy the condition (6)(j) and it has not a property (A) , because it has a nonoscillatory solution $y(t) = 1/t$.

Theorems 2-6 and their corollaries are generalization of the analogous results given in [5] and [6] for $n = 2$ and in [3] and [2] for $b(t) = 0$.

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