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# APPROXIMATION WITH RESPECT TO A MEASURE IN A MODULAR SPACE, VI

ABSTRACT: Elements of a modular subspace of $C_R^{(n)}$ are approximated by certain singular integrals.

KEY WORDS: modular space, Lebesgue measure, $n$-th derivative, linear operator, singular intregral.

Let $C_R^{(n)}$ be the space of function $x \in R^R$ for which the derivative $x^{(l)}$, where $l = 0,1,\dots,n$, $x^{(0)}(t) \equiv x(t)$, is continuous in $R$.

Let $x_k$, $x \in C_R^{(n)}$ with $k = 1,2,\dots$. A sequence $(x_k)$ is $D^{(n)}$ – convergent in measure or $D^{(n)}$ – convergent to $x$ iff for every $\eta > 0$

$$\sum_{l=0}^{n} \mu\left(\left\{T \in R : |x_k^{(l)}(t) - x^{(l)}(t)| \geq \eta\right\}\right) \to 0$$

as $k \to \infty$, where $\mu$ is the Lebsegue measure. The limit $x$ of a sequence $(x_k)$ which is $D^{(n)}$ – convergent is well-defined.

If a sequence $(x_k)$ is $D^{(n)}$ – convergent to $x$, then $(x_k)$ tends to $x$ in measure.

*Example 1.* The sequence $(x_k)$, where

$$x_k(t) = \begin{cases} 1 + \cos(k 2^{|p|} \pi (t - 2p\pi) \\ \qquad \text{for } t \in \left\langle 2p\pi - 1/k2^{|p|}, 2p\pi + 1/k2^{|p|} \right\rangle \text{ with } p \in Z,\ k = 1,2,\dots, \\ 0 \quad \text{elsewhere in } R, \end{cases}$$

is $D^{(1)}$ – convergent in measure to $x(t) \equiv 0$.

*Example 2.* Let us denote

$$x_k(t) = \frac{1}{k^{\alpha}} \sin(k^{\beta} t) \text{ with } t \in R,\ k = 1,2,\dots,$$

where $\alpha, \beta > 0$, $x(t) \equiv 0$. Then $(x_k)$ tends to $x$ in measure. If $n\beta - \alpha < 0$, then the sequence $(x_k)$ is $D^{(n)}$ – convergent to $x$. If $n\beta - \alpha \geq 0$, then $(x_k)$ is not $D^{(n)}$ – convergent to $x$.

*Example 3.* The sequence $(x_k)$, where

$$x_k(t) = y_k(t) + z_k(t) \qquad \text{for} \qquad t \in R,$$

$y_k$ is of the form (*), $z_k(t) = (1/k)\sin(k^2 t)$, tends to $x(t) \equiv 0$ in measure and $(x_k(t))$ is not convergent to $x(t)$ uniformly with respect to $t \in R$. Moreover, the sequence $(x_k)$ is not $D^{(1)}$ – convergent to $x$.

For $x \in C_R^{(n)}$ let us write

$$\rho^{(n)}(x) = \sum_{i=1}^{\infty} \frac{\rho_i^{(n)}(x)}{2^i(1+\rho_i^{(n)}(x))}$$

where

$$\rho_i^{(n)}(x) = \sum_{l=0}^{n} \mu\left( t \in R : |x^{(l)}(t)| \geq \frac{1}{2^i} \right), \qquad i = 1,2,\ldots,$$

$\mu$ is the Lebesgue measure. $\rho^{(n)}$ is a modular in $C_R^{(n)}$ which generates modular space

$$X_{\rho(n)} = \{x \in C_R^{(n)} : \lim_{\lambda \to 0} \rho^{(n)}(\lambda x) = 0\}.$$

Since, for example, for a non-decreasing function $x \in C_R^{(n)}$ such that $x(t) \to \infty$ as $t \to \infty$ we have

$$\mu\left( t \in R : \lambda |x(t)| \geq \frac{1}{2^i} \right) = \infty$$

for every $\lambda > 0$ and for $i = 1,2,\ldots$, so $X_\rho(n) \not\subseteq C_R^{(n)}$.

It is known (see [1], [2]) that the modular $\rho$ generates the $F$-norm of the form

$$|x|_\rho = \inf\{u > 0 : \rho(x/u) \leq u\} \quad \text{for} \quad x \in X_\rho.$$

For $x_k$, $x \in X_\rho$, $k = 1,2,\ldots$, we have

$$(|x_k - x| \to 0 \text{ as } k \to \infty) \Leftrightarrow (\text{For every } \lambda > 0 \ \ \rho(\lambda(x_k - x)) \to 0 \text{ as } k \to \infty).$$

In the space $X_\rho(n)$ the convergence in the $F$-norm and the $D^{(n)}$ – convergence in measure are equivalent.

We shall approximate members of $X_\rho(n)$ by certain singular integrals of the form

(A) $$\widetilde{\rho}_m(u,x) = \int_{-\infty}^{\infty} K_m(s)\, x(u+s)ds$$

for $m = 1,2,\dots,$ $u \in R,$ where $K_m,$ $m = 1,2,\dots,$ are measurable in the sense of Lebesgue, positive almost everywhere in $R$ and such that

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} K_m(s)ds = 1.$$

We say that $(K_m)$ is a singular kernel if for every $\delta > 0$

$$\int_{-\infty}^{-\delta} K_m(s)ds \to 0, \qquad \int_{\delta}^{\infty} K_m(s)ds \to 0$$

as $m \to \infty$.

A function $x \in C_R^{(n)}$ is called $D^{(n)}$ – regular if for every $\eta > 0$ $\omega_\mu(\eta,\delta;x^{(l)}) \to 0$ as $\delta \to 0$ for $l = 0,1,\dots,n,$ where

$$\omega_\mu(\eta,\delta;x^{(l)}) = \mu\left( t \in R : \max_{|s|\le\delta} \left| \frac{\partial^l}{\partial t^l} x(s+t) - x^{(l)}(t) \right| \ge \eta \right)$$

is the modulus of continuity in measure (see [3]).

We say that $(K_m)$ is a $D^{(n)}$ – regular kernel at the point $x \in C_R^{(n)}$ if:

a) for an arbitrary $\varepsilon > 0$ there exists $\Delta > 0$ such that for $m = 1,2,\dots,$ $l = 0,1,\dots,n$ and for every $t \in R$ we have

(1) $$\mu(A) < \Delta \Rightarrow \int_A K_m(s) \left| \frac{\partial^l}{\partial t^l} x(s+t) \right| ds < \varepsilon,$$

where $A$ is an arbitrary set which is measurable in the sense of Lebesgue.

b) for $m = 1,2,\dots,$ $l = 0,1,\dots,n$ integrals

$$\int_{-\infty}^{-a} K_m(s) \frac{\partial^l}{\partial t^l} x(s+t)ds, \qquad \int_{a}^{\infty} K_m(s) \frac{\partial^l}{\partial t^l} x(s+t)ds,$$

where $a$ is an arbitrary positive real number, are convergent uniformly with respect to $t \in R$.

*Theorem. If* $x \in X_\rho(n)$ *is a* $D^{(n)}$*-regular function, the sequence* $(\widetilde{\rho}_m)$ *is of the form* (A), *where* $(K_m)$ *is a singular and* $D^{(n)}$*-regular kernel at the point x, then for every* $\lambda > 0$

$$\rho^{(n)}\{\lambda[x(\cdot) - \widetilde{\rho}_m(\cdot, x)]\} \to 0 \quad as \quad m \to \infty.$$

*Proof.* Let $x \in X_\rho(n)$. For every $u \in R$ and for $m = 1,2,\ldots$ we have $\widetilde{\rho}_m(u,1) + \varepsilon_m = 1$, where $\lim_{m\to\infty} \varepsilon_m = 0$, $|\varepsilon_m| < 1$ for $m = 1,2,\ldots$, and

$$x(u) - \widetilde{\rho}_m(u,x) = \widetilde{\rho}_m(u, x(u) - x) + \varepsilon_m x(u).$$

Hence for every $\lambda > 0$ and for $m = 1,2,\ldots$ we obtain

$$\rho^{(n)}\{\lambda[x(\cdot) - \widetilde{\rho}_m(\cdot, x)]\} \le \rho^{(n)}\{2\lambda\widetilde{\rho}_m(\cdot, x - x(\cdot))\} + \rho^{(n)}\{2\lambda\varepsilon_m x(\cdot)\}.$$

Because $x \in X_\rho(n)$, so for $m > M_1 = M_1(\varepsilon, \lambda) > 0$ we have

(2) $$\rho^{(n)}\{2\lambda\varepsilon_m x(\cdot)\} < \varepsilon/2.$$

By uniform convergence with respect to $t \in R$ of integrals

$$\int_{-\infty}^{-a} K_m(s)\frac{\partial^l}{\partial t^l}x(s+t)ds, \qquad \int_{a}^{\infty} K_m(s)\frac{\partial^l}{\partial t^l}x(s+t)ds$$

it follows that for every $l = 0,1,\ldots,n$ and for $t \in R$

$$|(2\lambda\widetilde{\rho}_m(t, x - x(t)))^{(l)}| = 2\lambda\left|\int_{-\infty}^{\infty} K_m(s)\left[\frac{\partial^l}{\partial t^l}x(s+t) - x^{(l)}(t)\right]ds\right|.$$

Therefore for $\delta > 0$ we obtain

$$\left|\int_{-\delta}^{\delta} K_m(s)\left[\frac{\partial^l}{\partial t^l}x(s+t) - x^{(l)}(t)\right]ds\right| \le (1-\varepsilon_m)\max_{|s|\le\delta}\left|\frac{\partial^l}{\partial t^l}x(s+t) - x^{(l)}(t)\right|,$$

$$\left|\int_{-\infty}^{-\delta} K_m(s)\left[\frac{\partial^l}{\partial t^l}x(s+t) - x^{(l)}(t)\right]ds\right| \le$$

$$\le \int_{-\infty}^{-\delta} K_m(s)\left|\frac{\partial^l}{\partial t^l}x(s+t)\right|ds + |x^{(l)}(t)|\int_{-\infty}^{-\delta} K_m(s)ds,$$

$$\left|\int_{\delta}^{\infty} K_m(s)\left[\frac{\partial^l}{\partial t^l}x(s+t)-x^{(l)}(t)\right]ds\right| \le$$

$$\le \int_{\delta}^{\infty} K_m(s)\left|\frac{\partial^l}{\partial t^l}x(s+t)\right|ds + |x^{(l)}(t)|\int_{\delta}^{\infty} K_m(s)ds.$$

Because $x_{t_0} \in X_\rho(n)$, where $t_0$ is an arbitrary real number, and $(K_m)$ is a $D^{(n)}$ – regular kernel, for any two positive numbers $\varepsilon, \eta$ there exists $\alpha_0' = \alpha_0'(\varepsilon,\eta) > 0$ such that for every $l = 0,1,\ldots,n$ and for every $\alpha \in (0,\alpha_0'\rangle$ we have

$$\mu(\{s \in R : \alpha\,|x_{t_0}^{(l)}(s)| \ge \eta\}) < \Delta,$$

where $\Delta = \Delta(\varepsilon) > 0$ we choose in the same way as in (1), and

$$\int_{\delta}^{\infty} K_m(s)\left|\frac{\partial^l}{\partial t^l}x(s+t_0)\right|ds < \varepsilon + \int_B K_m(s)\left|x_{t_0}^{(l)}(s)\right|ds,$$

where $B = \{s \in \langle \delta,\infty) : \alpha_0'\,|x_{t_0}^{(l)}(s) < \eta\}$. Hence we obtain

$$\int_{\delta}^{\infty} K_m(s)\left|\frac{\partial^l}{\partial t^l}x(s+t_0)\right|ds < \varepsilon + \frac{\eta}{\alpha_0'}\int_{\delta}^{\infty} K_m(s)ds$$

and similarly

$$\int_{-\infty}^{-\delta} K_m(s)\left|\frac{\partial^l}{\partial t^l}x(s+t_0)\right|ds < \varepsilon + \frac{\eta}{\alpha_0'}\int_{-\infty}^{-\delta} K_m(s)ds.$$

Therefore for an arbitrary $\eta > 0$ we have

$$\sum_{l=0}^{n} \mu(\{t \in R : |2\lambda(\widetilde{\rho}_m(t, x - x(t)))^{(l)}| \ge \eta\}) \le$$

$$\le \sum_{l=0}^{n} \mu\left(\left\{t \in R : 2\lambda(1-\varepsilon_m)\max_{|s|\le\delta}\left|\frac{\partial^l}{\partial t^l}x(s+t) - x^{(l)}(t)\right| \ge \frac{\eta}{2}\right\}\right) +$$

$$+\sum_{l=0}^{n} \mu\left(\left\{t \in R : 4\lambda\varepsilon + 2\lambda\left(|x^{(l)}(t)| + \frac{\eta}{\alpha_0'}\right)\left(\int_{-\infty}^{-\delta} K_m(s)ds + \int_{\delta}^{\infty} K_m(s)ds\right) \ge \frac{\eta}{2}\right\}\right) \le$$

$$\le \sum_{l=0}^{n} \omega_\mu\left(\frac{\eta}{8\lambda}, \delta; x^{(l)}\right) + W_1(\delta).$$

Because $x$ is a $D^{(n)}$ – regular function, so there exists $\delta_0 = \delta_0(\varepsilon, \eta, \lambda) > 0$ such that

$$\sum_{l=0}^{n} \omega_\mu \left( \frac{\eta}{8\lambda}, \delta_0; x^{(l)} \right) < \frac{\varepsilon}{2}.$$

By the singularity of $(K_m)$ it follows that for $m > M_2 = M_2(\varepsilon, \eta, \lambda) > 0$ we have

$$W_2(\delta_0) = \int_{-\infty}^{-\delta_0} K_m(s)ds + \int_{\delta_0}^{\infty} K_m(s)ds < \varepsilon',$$

where $0 < \varepsilon < \eta/(8\lambda)$, $0 < \varepsilon' < \alpha_0'(1/(4\lambda) - (2\varepsilon)/\eta)$. For $\eta' = \eta/(4\lambda) - 2\varepsilon - \varepsilon'\eta/\alpha_0'$ we obtain

$$W_1(\delta_0) \le \sum_{l=0}^{n} \mu\left( \left\{ t \in R : |x^{(l)}(t)| \left( \int_{-\infty}^{-\delta_0} K_m(s)ds + \int_{\delta_0}^{\infty} K_m(s)ds \right) \ge \eta' \right\} \right).$$

For any two positive numbers $\varepsilon$, $\eta$ there exists $\alpha_0'' = \alpha_0''(\varepsilon, \eta) > 0$ such that for every $\alpha \in (0, \alpha_0'')$ and for $l = 0, 1, \ldots, n$ we have

$$\mu(\{t \in R : \alpha |x^{(l)}(t)| \ge \eta\}) < \frac{\varepsilon}{2(n+1)}.$$

For $m > M_3 = M_3(\varepsilon, \eta, \lambda) > 0$ we obtain $W_2(\delta_0) \le \min(\varepsilon', \alpha_0'')$ and $W_1(\delta_0) < \varepsilon/2$. Therefore for $m > M_3$ we have

$$\sum_{l=0}^{n} \mu(\{t \in R : |2\lambda(\widetilde{\rho}_m(t, x - x(t)))^{(l)}| \ge \eta\}) < \varepsilon,$$

and hence for $m > M_4 = M_4(\varepsilon, \eta, \lambda) > 0$ the following inequality

(3) $$\rho^{(n)}\{2\lambda\widetilde{\rho}_m(\cdot, x - x(\cdot))\} < \varepsilon/2.$$

holds.

By (2) and (3), for each $\lambda > 0$ and $m > \max(M_1, M_4)$ we get

$$\rho^{(n)}\{\lambda[x(\cdot) - \widetilde{\rho}_m(\cdot, x)]\} < \varepsilon.$$

The proof is complete.

## References

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