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ON THE INDEFINITE INTEGRAL AND APPROXIMATION OF H-ALMOST PERIODIC FUNCTIONS

ABSTRACT. The paper gives a theorem on the indefinite integral of H-almost periodic function and a theorem on approximation of an H-almost periodic function by means Steklov functions.

KEY WORDS: almost periodic function, Hausdorff metric, indefinite integral, Steklov function.

1. PRELIMINARIES

Let F_Δ denote the class of closed and bounded with respect to the $0y$ axis point sets on the $0xy$ plane, such that their respective projections on the $0x$ axis are identical with the interval Δ (finite or infinite), and such that the intersection of every straight line $x = x_0$, $x_0 \in \Delta$ and $F \in F_\Delta$ is a closed interval. If $A, B \in F_\Delta$, then the Hausdorff distance between A and B we call the following number

$$r\{A,B\} = \max\left\{\sup_{X\in A}\inf_{Y\in B} \| X - Y \|_0,\ \sup_{X\in B}\inf_{Y\in A} \| X - Y \|_0\right\},$$

where

$$\| X - Y \|_0 = \| X(x_1, y_1) - Y(x_2, y_2) \|_0 = \max(|x_1 - x_2|, |y_1 - y_2|).$$

The following Lemma is true:

Lemma 1. Suppose that $A, B \in F_\Delta$. *In order that* $r\{A,B\} \le \delta$ *it is necessary and sufficient that*

a) *for an arbitrary* $X \in A$ *there exists* $Y \in B$ *such that* $\| X - Y \|_0 \le \delta$

and

b) *for an arbitrary* $X \in B$ *there exists* $Y \in A$ *such that* $\| X - Y \|_0 \le \delta$.

Proof of Lemma 1 can be found in [3].

By the complete graph $\overline{f}$ of the function f, defined and bounded on the interval Δ and taking real values, we call a set of points of the form

$$\overline{f} = \{(x,y) : x \in \Delta,\ I_f(x) \le y \le S_f\},$$

where I_f, S_f is the lower and upper Baire's function respectively, both with respect to the function f:

$$I_f = \lim_{\delta \to 0} \inf_{|x'-x| \le \delta} f(x'),$$

$$S_f = \lim_{\delta \to 0} \sup_{|x'-x| \le \delta} f(x').$$

The complete graph of every function, defined and bounded on the interval Δ, is an element of the set F_Δ.

The Hausdorff distance between two functions f and g, both defined and bounded on the interval Δ, is defined in the following way: it is the Hausdorff distance between their respective complete graphs, i.e. $r\{f,g\} = r\{\overline{f},\overline{g}\}$.

A bounded function $f : R \to R$ is called Hausdorff almost periodic (H-a.p.) iff for each $\varepsilon > 0$ the set

$$E_H\{\varepsilon; f\} = \{\tau \in R : r\{f, f_\tau\} \le \varepsilon\}$$

where $f_\tau(x) \equiv f(x+\tau)$, is relatively dense, i.e. there is a number $l > 0$ such that $E_H\{\varepsilon; f\} \cap (\alpha, \alpha + l) \ne \varnothing$ fir every $\alpha \in R$. Elementary properties of H-a.p. functions and connection between H-a.p. functions and almost periodic functions of other types can bo found in [5].

Let f be a bounded function defined on the interval Δ. The following equality

$$\mu_f(\delta) = \sup_{\substack{|x_1-x_2| \le \delta \\ x_1, x_2 \in \Delta}} \{ \sup_{x_1 \le x \le x_2} [| f(x_1) - f(x) | + | f(x_2) - f(x) |] - | f(x_1) - f(x_2) |\}$$

define the modulus of non-monotonicity of f.

Lemma 2. If f and g are bounded and integrable on the interval $< a,b >$ and $\mu_f(\delta) \le K\delta^\gamma$, where $K \ge 0$, $\gamma > 0$ are constants, then the following inequality

$$\int_a^b | f(x) - g(x) | dx \le (b-a) C_1 (r\{f,g\})^{\frac{\gamma}{1+\gamma}} + C_2 (r\{f,g\})^{\frac{\gamma}{1+\gamma}},$$

where C_1, C_2 *depend on* γ, K, $M = \max\{\max_{a \le x \le b} |f(x)|, \max_{a \le x \le b} |g(x)|\}$, *holds.*

The proof is in [4].

For a given positive number h and for a function $f: R \to R$ which is locally integrable put

$$S_f(x;h) := \frac{1}{2h} \int_{x-h}^{x+h} f(s)ds, \quad x \in R.$$

Then $S_f(\cdot\,;h)$ is called the Steklov function of *f*.

2. Main results

Let us write

$$H^\gamma = \{f: R \to R : f \text{ is H-a.p. such that } \mu_f(\delta) \le K\delta^\gamma, \; K \ge 0, \gamma > 0\}.$$

It is know (see[5]) that if the indefinite integral of a function $f \in H^\gamma$, which is locally integrable, is bounded, then this integral is a uniformly a.p. function (see[2]).

Now, we will prove the following theorem

Theorem 1. If $f \in H^\gamma$ *is locally integrable and the indefinite integral*

$$F(u) = \int_{u'}^{u} f(s)ds \quad \textit{for} \quad u \in R$$

is bounded, then F is V-a.p. (see[6]).

Proof. By [5] it follows that for an arbitrary $\varepsilon > 0$ and for $\tau \in E_H\{\varepsilon'; f\}$, where $\varepsilon' = (\varepsilon / L)^{(1+\gamma)/\gamma}$, $L = \max(1, l + d)$, $d = |u_1 - u_2| > 0$, $F(u_1) < \inf_{u \in R} F(u) + \varepsilon$, $F(u_2) > \sup_{u \in R} F(u) - \varepsilon$, $l = l(\varepsilon, \gamma, d)$ is the positive number which characterizes the relative density of the set of $H - (\varepsilon / (\sqrt{\varepsilon} + \varepsilon / d + d))^{(1+\gamma)/\gamma}$ – almost periods of *f*, we have

$$|F(u+\tau) - F(u)| < C_1\varepsilon + C_2\sqrt{\varepsilon} \qquad \text{for} \quad u \in R,$$

where C_1, C_2 depend on γ, K, $M = \sup_{s \in R} |f(s)|$.

Since f is bounded, so for an arbitrary $t \in R$ we have

$$V(t;F) \leq \int_{t-1}^{t+1} |f(s)|\,ds \leq 2M,$$

where $V(t;F)$ denote the Jordan variation of the function F on the interval $<t-1, t+1>$. Moreover, F is continuous.

Let us put for $\tau \in E_H\{\varepsilon'; f\}$

$$V(F - F_\tau) = \sup_{t \in R} \{|F(t) - F(t+\tau)| + |V(t; F - F_\tau)|\}.$$

Analogously to [6], we have for every $t \in R$

$$V(t; F - F_\tau) \leq \int_{t-1}^{t+1} |f(s) - f(s+\tau)|\,ds.$$

Using Lemma 2, we obtain for $t \in R$

$$\int_{t-1}^{t+1} |f(s) - f(s+\tau)|\,ds \leq C_3 (r\{f, f_\tau\})^{\frac{\gamma}{1+\gamma}},$$

where C_3 depends on γ, K, M and so $V(t; F - F_\tau) \leq C_3 \varepsilon$. Hence $V(F - F_\tau) \leq C_4 \varepsilon + C_2 \sqrt{\varepsilon}$, where $C_4 = C_1 + C_3$, i.e. F is V-a.p..

In the following we shall approximate an H-a.p. function by means of its Steklov functions.

Theorem 2. *If $f \in H^\gamma$ is locally integrable, then the Steklov function of f is uniformly a.p.* .

Proof. Using Lemma 2, we obtain for $\tau \in E_H\{\varepsilon; f\}$ and for every $x \in R$

$$\left|S_f(x;h) - S_f(x+\tau;h)\right| \leq \frac{1}{2h} \int_{x-h}^{x+h} |f(s) - f(s+\tau)|\,ds \leq$$

$$\leq C_1 \varepsilon^{\frac{\gamma}{1+\gamma}} + \frac{1}{2h} C_2 \varepsilon^{\frac{\gamma}{1+\gamma}} = \varepsilon',$$

where C_1, C_2 depend on γ, K, $M = \sup_{s \in R} |f(s)|$, $\varepsilon' \to 0$ as $\varepsilon \to 0$, i.e. τ is the ε'-almost period of $S_f(\cdot\,;h)$. Therefore the continuous Steklov function $S_f(\cdot\,;h)$ is uniformly a.p. for every $h > 0$.

Theorem 3. If the function $f: R \to R$ *is continuous and bounded, then*

$$\lim_{h \to 0} r\{f, S_f(\cdot\,;h)\} = 0.$$

Proof. Becouse f is continuous, then for every $x \in R$ we have $S_f(x;h) = f(x+\vartheta h) = g_h(x)$, where $\vartheta = \vartheta(x,h)$, $|\vartheta| \leq 1$.

For an arbitrary $X = X(x, g_h(x)) \in \overline{g}_h$ there exists $Y = Y(x+\vartheta h, g_h(x)) \in$ $\in \overline{f}$ such that $\| X - Y \|_0 \leq h$. Conversely, for an arbitrary $X = X(x, f(x)) \in \overline{f}$ there exists $Y = Y(x - \vartheta h, f(x)) \in \overline{g}_h$ such that $\| X - Y \|_0 \leq h$. By Lemma 1 it follows that $r\{f, S_f(\cdot\,;h)\} \leq h$. Hence we obtain the result.

From Theorem 2 and Theorem 3 follows the immediate corollary.

Corollary. If f *is a continuous function of class* H^γ, *then the Steklov function* $S_f(\cdot\,;h)$ *is uniformly a.p. for every* $h > 0$ *and*

$$\lim_{h \to 0} r\{f, S_f(\cdot\,;h)\} = 0.$$

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