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U-POINTS AND QUASI-U-POINTS IN ORLICZ FUNCTION SPACES

ABSTRACT: In this paper, notions of U-points and quasi-U-points are defined. There is given a characterization of U-point and quasi-U-point in Orlicz function spaces equipped with the Luxemburg norm.

KEY WORDS: Orlicz function space, U-point, quasi-U-point, locally U-space, quasi-U-space

1. INTRODUCTION

The concept of U-spaces was introduced by Gao Ji and Lau Kasing in 1991 (see [3]). U-spaces have uniformly normal structure and many other interesting properties. In this paper we introduce the notation U-points and qiasi-U-points. That kind of points are necessary to define local U-spaces and quasi-U-spaces. We give a complete characterization of U-points and quasi-U-points in Orlicz function spaces with the Luxemburg norm. Basing on that characterization, we obtain some criteria for an Orlicz function space to be the local-U-space and quasi-U-space.

Let $S(X)$ be the unit sphere of a Banach space X. A point $x \in S(X)$ is called an U-point provided for any sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$, satisfying $\| x_n + x \| \to 2$ we have $f(x_n - x) \to 0$ uniformly for all $f \in \nabla_x$, where ∇_x denotes the set of all supporting functionals of x. A point $x \in S(X)$ is called a quasi-U-point if for any $y \in S(X)$, such that $\| x + y \| = 2$, we have $f(y - x) = 0$ for all $f \in \nabla_x$. Obviously, every U-point is a quasi-U-point. A Banach space X is said to be a local (resp. quasi-) U-space if each point $x \in S(X)$ is an U (resp. a quasi-U)-point.

Denote by N and R the sets of natural and real numbers, respectively. Let (G, Σ, μ) be a measure space with a finte and atomless measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G.

A map $M : R \to [0, \infty)$ is said to be an *Orlicz function* if M is vanishing at 0, even, convex and not identically equal to 0.

Denote by $p(u)$ and $p\_(u)$ the right and left derivative of $M(u)$, respectively. An Orlicz function M is called an $N-$*function* if

$$\lim_{u\to\infty}\frac{M(u)}{u}=\infty \quad \text{and} \quad \lim_{u\to 0}\frac{M(u)}{u}=0.$$

By the *Orlicz function space* L_M we mean

$$L_M=\left\{x\in L^0:\rho_M(cx)=\int_G M(cx(t))d\mu<\infty \quad \text{for some} \quad c>0\right\}$$

equipped with so called the *Luxemburg norm*

$$\|x\|=\inf\left\{\varepsilon>0:\ \rho_M\left(\frac{x}{\varepsilon}\right)\le 1\right\}$$

or with equivalent

$$\|x\|_0=\inf_{k>0}\frac{1}{k}(1+\rho_M(kx))$$

called the *Orlicz norm*. To simplify denotations, we put $L_M=(L_M,\|\cdot\|)$ and $L_M^0=(L_M,\|\cdot\|_0)$.

For every Orlicz function M we define *the complementary function* $N:R\to[0,\infty)$ by the formula

$$N(v)=\sup_{u>0}\{u|v|-M(u)\}$$

for every $v\in R$. The complementary function N is also an Orlicz function.

We say that the Orlicz function M *satisfies the* Δ_2*-condition* if there exist a constant $k\ge 2$ and $u_0>0$ such that

$$M(2u)\le kM(u)$$

for every $|u|\ge u_0$.

We say that the Orlicz function M *satisfies the* Δ_2*-condition* if its complementary function N satisfies the Δ_2*-condition.*

We say that the Orlicz function M is *strictly convex* (write $M\in SC[0,\infty)$) if for any $u\ne v$ and $\alpha\in(0,1)$ we have

$$M(\alpha u+(1-\alpha)v)<\alpha M(u)+(1-\alpha)M(v).$$

For more details we refer to [5].

An interval $[a,b]$ is called a structural affine interval of M, or simply **SAI** of M, provided that M is affine on $[a,b]$ and it is not affine on either $[a-\varepsilon,b]$ or $[a,b+\varepsilon]$ for any $\varepsilon>0$. Let $[a_i,b_i]$ $(i=1,2,\ldots)$ be all the **SAIs** of M. Denote by

$S_M = R\setminus(\bigcup_i (a_i,b_i))$ the set of strictly convex points of *M*. Moreover, for any $x\in L_M$ we define

$$\Theta(x)=\inf\left\{c>0:\rho_M(\frac{x}{c})<\infty\right\},$$

$$d(x,E_M)=\inf\ \{\|x-y\|:y\in E_M\}$$

and for any $v\in L_N^0$,

$$K(v)=\left\{k>0:\|v\|^0=\frac{1}{k}(1+\rho_N(kv))\right\}.$$

Lemma 1. $f=v+\phi\in L_M^*$ *is a supporting functional of* $x\in S(L_M)$ *iff*

(1) $\rho_M(x)=1$,

(2) $\|\phi\|=\phi(x)$,

(3) $v(t)x(t)\geq 0$ *and* $p_-(|x(t)|)\leq k|v(t)|\leq p(|x(t)|)$, *for* $\mu-a.e.t\in G$, *where* $v\in L_N^0$, $\phi\in L_M^*$ *is a singular functional and* $k\in K(v)$.

Proof. The proof of this Lemma can be found in [10].

Lemma 2. If $x\in S(L_M)$ *and there is* $\tau>0$ *such that* $\rho_M((1+\tau)x)<\infty$, *then*

(i) $\rho_M(x)=1$,

(ii) *all supporting functionals of x belong to* L_N^0,

(iii) *for any* $k\in K(v)$ *we have* $\lim_{\mu e\to 0}\sup_{v\in\nabla_x}\rho_N(kv|_e)=0$ $(e\in\Sigma)$,

(iv) $\rho_N(p(x))<\infty$,

(v) *for any sequence* $\{x_n\}_{n=1}^{\infty}\subset S(L_M)$ *such that* $\|x_n+x\|\to 2$, *we have* $\rho_M(x_n)\to 1$ *and* $\rho_M((x_n+x)/2)\to 1$,

(vi) *for any* $y\in S(L_M)$ *with* $\|x+y\|=2$, we have $\rho_M(y)=\rho_M\left(\frac{x+y}{2}\right)=1.$

Proof. (i) is obvious.

(ii) It is easy to see that $\nabla_x\subset S(L_N^0)$ for any $x\in E_M$. Suppose that $x\in L_M\setminus E_M$ has a supporting functional of the form $f=v+\phi$, where $v\in L_N^0$ and $\phi\in L_M^*$ is a singular functional. Then, by Lemma 5.3 in [10], we get

$$\phi(x) = \|\phi\| = \sup_{z\in L_M \setminus E_M} \frac{\phi(z)}{\Theta(z)} \geq \frac{\phi(x)}{\Theta(x)} > \phi(x).$$

A contradiction.

(iii) Otherwise, there is $\varepsilon > 0$ and a sequence of measurable sets $\{e_n\}_{n=1}^{\infty} \subset G$ with $e_n \downarrow 0$ and $v_n \in \nabla_x$ such that $\rho_N(k_n v_n|_{e_n}) \geq e > 0$, where $k_n \in K(v_n)$. Since $\rho_M((1+\tau)x) < \infty$, there is $n \in N$ such that $\rho_N((1+\tau)x|_{e_n}) < \tau e/2$. Hence

$$0 = k_n \| v_n \|_M^0 - k_n \langle x, v_n \rangle = 1 + \rho_N(k_n v_n) - \langle x, k_n v_n \rangle =$$

$$= \int_G (M(x(t)) + N(k_n v_n(t)) - x(t) k_n v_n(t)) dt >$$

$$> \int_{e_n} \left(M(x(t)) + N(k_n v_n(t)) - \frac{1+\tau}{1+\tau} x(t) k_n v_n(t) \right) dt \geq$$

$$\geq \int_{e_n} \left(N(k_n v_n(t)) - \frac{1}{1+\tau} (M((1+\tau)x(t)) + N(k_n v_n(t)) \right) dt =$$

$$= \frac{\tau}{1+\tau} \rho_N(k_n v_n|_{e_n}) - \frac{1}{1+\tau} \rho_M((1+\tau)x|_{e_n}) \geq$$

$$\geq \frac{\tau\varepsilon}{1+\tau} - \frac{\tau\varepsilon}{2(1+\tau)} = \frac{\tau\varepsilon}{2(1+\tau)},$$

what is impossible.

(iv) Since

$$M((1+\tau)u) > \int_u^{(1+\tau)u} p(s) ds \geq \tau u p(u) \geq \tau N(p(u)),$$

we have

$$\rho_N(p(x)) < \frac{1}{\tau} \rho_M((1+\tau)x) < \infty.$$

(v) For any $\varepsilon > 0$, a natural number n can be found such that

$$\left\| (1+\varepsilon) \frac{x_n + x}{2} \right\| > 1.$$

Hence

$$1 < \rho_M\left((1+\varepsilon) \frac{x_n + x}{2} \right) = \rho_M\left(\frac{1+\varepsilon}{2} x_n + \frac{(1-\varepsilon)(1+\varepsilon)}{2(1-\varepsilon)} x \right) \leq$$

$$\le \frac{1+\varepsilon}{2}\rho_M(x_n)+\frac{1-\varepsilon}{2}\rho_M\left(\frac{1+\varepsilon}{1-\varepsilon}x\right)=$$

$$=\frac{1+\varepsilon}{2}\rho_M(x_n)+\frac{1-\varepsilon}{2}(\rho_M(x)+o(\varepsilon))=$$

$$=\frac{1+\varepsilon}{2}\rho_M(x_n)+\frac{1-\varepsilon}{2}(1+o(\varepsilon)).$$

Therefore, by the arbitrariness of ε, we conclude $\rho_M(x_n)\to 1$ as $n\to\infty$. Analogously, noticing that $\|1/2(((x_n+x)/2)+x)\|\to 1$ as $n\to\infty$, we can prove that $\rho_M((x_n+x)/2)\to 1$ as $n\to\infty$.

(vi) It follows immediately from (v).

Lemma 3. Suppose $M\in\nabla_2$ *and* $[a,b]$ *is a* **SAI** *of M. Then for any* $\varepsilon>0$ *there exists* $\eta>0$ *such that* $((M(u)+N(v))/2)-M((u+v)/2)<\eta$ *and* $v\in[a,b]$ *imply* $u\in[a-\varepsilon,b+\varepsilon]$.

Proof. See [2].

Theorem 1. An element $x\in S(L_M)$ *is a quasi-U-point iff the following two conditions are satisfied:*

(1) *there exists* $\tau>0$ *such that* $\rho_M((1+\tau)x)<\infty$,

(2) *if there exists a* **SAI** $[a,b]$ *of M with* $p\_(a)<p(a)$ *such that* $\mu\{t\in G:|x(t)|=a\}>0,$, *then* $\mu\{t\in G:|x(t)|\in(c,d]\}=0$ *for any* **SAI** $[c,d]$ *of M; if* $p\_(b)<p(b)$ *and* $\mu\{t\in G:|x(t)|=b\}>0$, *then* $\mu\{t\in G:|x(t)|\in[c,d)\}=0$ *for any* **SAI** $[c,d]$ *of M.*

Proof of necessity. If (1) is not true, then $\Theta(x)=1$. Denote

$$G_n=\{t\in G:n-1\le|x(t)|<n\},\ n=1,2,\dots .$$

Then $G=\bigcup_{n=1}^{\infty}G_n$. Decompose every set G_n $(n=1,2,\dots)$ into two sets G_n' and G_n'' such that $\mu(G_n')=\mu(G_n'')$. Define

$$x'(t)=\sum_{n=1}^{\infty}x(t)\,|_{G_n'},\qquad x''(t)=\sum_{n=1}^{\infty}x(t)\,|_{G_n''}.$$

Then $x=x'+x''$. Clearly,

$$\rho_M(x') < \rho_M\left(\frac{x+x'}{2}\right) < \rho_M(x) \le 1.$$

On the other hand, for any $\varepsilon \in (0,1/2)$, fix $n_0 \in N$ such that $(n_0-1)/n_0 > (1-2\varepsilon)/(1-\varepsilon)$. We have

$$\rho_M\left(\frac{1}{1-2\varepsilon}\frac{x'+x}{2}\right) > \rho_M\left(\frac{x'}{1-2\varepsilon}\right) = \sum_{n=1}^{\infty}\int_{G'_n} M\left(\frac{x(t)}{1-2\varepsilon}\right)dt \ge$$

$$\ge \sum_{n=1}^{\infty}\int_{G'_n} M\left(\frac{n-1}{1-2\varepsilon}\right)dt = \frac{1}{2}\sum_{n=1}^{\infty}\int_{G'_n} M\left(\frac{n-1}{1-2\varepsilon}\right)dt >$$

$$> \frac{1}{2}\sum_{n=1}^{\infty}\int_{G_n} M\left(\frac{n-1}{n}\frac{x(t)}{1-2\varepsilon}\right)dt > \frac{1}{2}\sum_{n>n_0}\int_{G_n} M\left(\frac{x(t)}{1-\varepsilon}\right)dt = \infty.$$

This shows that

$$\| x' \| = \left\| \frac{x'+x}{2} \right\| = 1$$

and $\Theta(x') = 1$. Similarly, we can get that

$$\| x' \| = \left\| \frac{x''+x}{2} \right\| = 1$$

and $\Theta(x'') = 1$. By $1 = \Theta(x') = d(x', E_M)$ and Hahn-Banach Theorem, there exists $\phi \in L_M^*$ such that $\phi[E_M] = 0$ and $\|\phi\| = \phi(x') = \Theta(x') = 1$, i.e. ϕ is a singular functional and supporting functional of x'.

Noticing that

$$\left\|\phi|_{\bigcup_{n=1}^{\infty} G'_n}\right\| \ge \phi|_{\bigcup_{n=1}^{\infty} G'_n}(x') = \phi\left(x'|_{\bigcup_{n=1}^{\infty} G'_n}\right) = \phi(x') = 1 = \|\phi\|$$

and

$$\|\phi\| = \left\|\phi|_{\bigcup_{n=1}^{\infty} G'_n}\right\| + \left\|\phi|_{\bigcup_{n=1}^{\infty} G''_n}\right\|,$$

we have

$$\left\|\phi|_{\bigcup_{n=1}^{\infty} G''_n}\right\| = 0.$$

Hence

$$\phi(x) = \phi(x') + \phi(x'') = \phi(x') + \phi\left(x''|_{\bigcup_{n=1}^{\infty} G''_n}\right) =$$

$$= \phi(x') + \phi|_{\bigcup_{n=1}^{\infty} G''_n}(x'') = \phi(x') = 1 = \| x \|,$$

i.e ϕ is also a supporting functional of x. But $\| x'' \| = \| (x''+x)/2 \| = 1$ and $\phi(x'') = 0$, so x is not a quasi-U-point. This contradiction proves (1).

If the first part of (2) does not hold, then there exist the **SAIs** $[a,b]$ and $[c,d]$ such that

$$p\_(a) < p(a), \qquad \mu\{t \in G : |x(t)| = a\} > 0$$

and

$$\mu\{t \in G : |x(t)| \in (c,d]\} > 0.$$

Let

$$M(u) = p(a)u + B \ \ (u \in [a,b]), \ M(u) = p(c)u + B' \ \ (u \in [c,d]).$$

Pick $\varepsilon' > 0$ satisfying

$$\mu F = \mu\{t \in G : x(t) \in [c+\varepsilon', d]\} > 0.$$

Take $\overline{E} \subset E$, $\overline{F} \subset F$ with $\mu \overline{E} = \mu \overline{F} > 0$ and choose $\varepsilon > 0$ such that $p(a)\varepsilon = p(c)\varepsilon'$ and $a + \varepsilon \le b$. Put

$$y(t) = x(t)|_{G \setminus (\overline{E} \cup \overline{F})} + (a+\varepsilon)|_{\overline{E}} + (x(t) - \varepsilon')|_{\overline{F}} .$$

Then

$$\rho_M(y) = \rho_M(x|_{G \setminus (\overline{E} \cup \overline{F})}) + M(a)\mu \overline{E} + p(a)\varepsilon \mu \overline{F} + \\ + \int_{\overline{F}} M(x(t))dt - p(c)\varepsilon' \mu \overline{F} = \rho_M(x) = 1$$

and consequently

$$\rho_M(\frac{x+y}{2}) = \rho_M(x|_{G \setminus (\overline{E} \cup \overline{F})}) + M(a)\mu\overline{E} + p(a)\frac{\varepsilon}{2}\overline{E} + \\ + \int_{\overline{F}} M(x(t))dt - p(c)\frac{\varepsilon'}{2}\mu\overline{F} = \rho_M(x) = 1.$$

Therefore $\| y \| = \| (x+y)/2 \| = 1$.

By (iv) of Lemma 2, $p\_(x(t)) \in L_N^0$. Set $v = p\_(x) / \| p\_(x) \|_N^0$. Then, by Theorem 18.5 in [5], v is a supporting functional of x, but

$$\langle x - y, v \rangle = \int_{\overline{F}} \varepsilon' p\_(x(t))dt - \int_{\overline{E}} \varepsilon p\_(x(t))dt \ge \\ \ge \varepsilon' p(c)\mu\overline{F} - \varepsilon p\_(a)\mu\overline{E} > \varepsilon' p(c)\mu\overline{F} - \varepsilon p(a)\mu\overline{E} = 0,$$

which contradics that x is a quasi-U-point.

Analogously, we can conclude that the second part of the condition (2) holds true.

Proof of sufficiency. Without loss of generality, we may assume $x(t) \geq 0$. Let $y \in S(L_M)$ and $\| x+y \| = 2$. Then, by (i) and (vi) of Lemma 2, we have

$$\rho_M(x) = \rho_M(y) = \rho_M\left(\frac{x+y}{2}\right) = 1.$$

Consequently,

$$0 = \frac{\rho_M(x)+\rho_M(y)}{2} - \rho_M\left(\frac{x+y}{2}\right) =$$

$$= \int_G \left(\frac{M(x(t))+M(y(t))}{2} - M\left(\frac{x(t)+y(t)}{2}\right)\right) dt.$$

Hence, by the convexity of *M*, we get

$$\frac{M(x(t))+M(y(t))}{2} = M\left(\frac{x(t)+y(t)}{2}\right) \text{ for } \mu-\text{a.e. } t \in G.$$

This shows that either $x(t) = y(t)$ or $x(t),\ y(t)$ belong to the same **SAI** of *M*.

Consider the following three cases:

(I) There exists a **SAI** $[a,b]$ of M with $p\_(b) < p(b)$ such that $\mu\{t \in G{:}x(t) = b\} > 0$, and for any **SAI** $[c,d]$ of *M*, $\mu\{t \in G{:}c \leq x(t) < d\} = 0$.

If $x(t)$ does not belong to any **SAI** of *M*, then $x(t) = y(t)$. If $x(t) \in [c,d]$, then $x(t) = d$. Moreover $y(t) \in [c,d]$, so $y(t) \leq x(t)$. Thus, we get $M(y(t)) \leq M(x(t))$ ($\mu-$a.e.). Combining $\rho_M(y) = 1 = \rho_M(x)$, it follows immediately that $x(t) = y(t)$ (for $\mu-$a.e. $t \in G$), i.e. $y = x$. For any $f \in L_M^*$, $f(y-x) = f(0) = 0$, i.e. *x* is a quasi-*U*-point.

(II) There exists a **SAI** $[a,b]$ of M with $p\_(a) < p(a)$ such that $\mu\{t \in G{:}x(t) = a\} > 0$, and for any **SAI** $[c,d]$ of *M*, $\mu\{t \in G{:}c \leq x(t) < d\} = 0$.

Analogously to the proof of (I), we conclude that *x* is a quasi-*U*-point.

(III) $x(t)$ belongs to a **SAI** $[a,b]$ of M and $p\_(x(t)) = p(x(t))$. By (ii) of Lemma 2, any supporting functional v of x belongs to $S(L_N^0)$. If $x(t) \in [a,b]$, then $y(t) \in [a,b]$. Set $k \in K(v)$. Then

$$kv(t) = p\_(x(t)) = p(x(t)) = p(a) = p\_(b)$$

If $y(t) \in [a,b]$, then

$$kv(t) = p(a) = p(y(t))$$

If $y(t) = b$, then

$$kv(t) = p\_(b) = p\_(y(t)).$$

By Lemma 1, v is also a supporting functional of y. Thus

$$<v, y-x> = <v, y> - <v, x> = \| y \| - \| x \| = 1 - 1 = 0,$$

i.e. x is a quasi-U-point.

Corollary 1. L_M has quasi-U-property iff

(i) $M \in \Delta_2$,

(ii) *for any* **SAI** $[a,b]$ *of M,* $p\_(a) = p(a) = p\_(b) = p(b)$.

Proof of necessity. (i) If $M \notin \Delta_2$, then we can construct $x \in S(L_M)$ with $\rho_M(x) < 1$, so for any $\varepsilon > 0$, $\rho_M((1+\varepsilon)x) = \infty$. By Theorem 1, x is not a quasi-U-point. Furthermore, L_M cannot have quasi-U-property.

(ii) If there is a **SAI** $[a,b]$ of M with $p\_(a) < p(a)$, then we can fix $E \subset G$, $F \subset G$ with $0 < \mu E < \mu G$, $0 < \mu F < \mu G$, $E \cap F = \varnothing$, a real numbers $c \in (a,b]$ and d such that

$$M(a)\mu E + M(c)\mu F + M(d)\mu(G \setminus (E \cup F)) = 1.$$

Define $x = a|_E + c|_F + d|_{G \setminus (E \cup F)}$. Then $\rho_M(x) = 1$, i.e. $\| x \| = 1$. On the other hand

$$\mu\{t \in G : x(t) = a\} = \mu E > 0$$

and

$$\mu\{t \in G : x(t) \in (a,b]\} \geq \mu F > 0.$$

Thus, by Theorem 1, x is not a quasi-U-point and consequently L_M has not quasi-U-property.

The equality $p\_(b) = p(b)$ can be proved analogously.

Proof of sufficiency. For any $x \in S(L_M)$, by $M \in \Delta_2$, we have $\rho_M((1+\tau)x) < \infty$. (2) from Theorem 1 holds true trivially. Therfore x is a quasi-U-point, i.e. L_M has quasi-U-property.

Theorem 2. An element $x \in S(L_M)$ is an U-point iff

(i) *there exists* $\tau > 0$ *such that* $\rho_M((1+\tau)x) < \infty$,

(ii) *if there is a* **SAI** $[a,b]$ *of M with* $p\_(a) < p(a)$ *such that*

$$\mu\{t \in G : |x(t)| = a\} > 0,$$

then

$$\mu\{t \in G : |x(t)| \in (c,d]\} = 0;$$

if $p\_(b) < p(b)$ *and* $\mu\{t \in G : |x(t)| = b\} > 0$, *then*

$$\mu\{t \in G : |x(t)| \in [c,d]\} = 0,$$

(iii) $M \in \nabla_2$ *or* $\mu\{t \in G : |x(t)| \in (a,b]\} = 0$ *for any* **SAI** $[a,b]$ *of* M.

Proof of necessity. Without loss of generality, we can assume $x(t) \geq 0$. Since an U-point is a quasi-U-point, by Theorem 1, (i) and (ii) are satisfied. If (iii) is not true, then there exists a **SAI** $[a,b]$ of M and $u_n \uparrow \infty$ such that

$$\mu\{t \in G : x(t) \in (a,b]\} > 0$$

and

$$M\left(\frac{u_n}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{M(u_n)}{2}.$$

Fix $\varepsilon > 0$ and $E_n \subset E$ such that

$$\mu E \overset{\Delta}{=} \mu\{t \in G : x(t) \in [a+\varepsilon, b]\} > 0$$

and

$$\int_{E_n} M(x(t)-\varepsilon)dt + M(u_n)\mu(E \setminus E_n) = \int_E M(x(t))dt.$$

Then $\mu(E \setminus E_n) \to 0$. Define

$$x_n(t) = (x(t)-\varepsilon)|_{E_n} + u_n|_{E \setminus E_n} + x(t)|_{G \setminus E}.$$

We have $\rho_M(x_n) = \rho_M(x) = 1$. Moreover,

$$\rho_M\left(\frac{x_n + x}{2}\right) = \int_{E_n} M\left(x(t) - \frac{\varepsilon}{2}\right)dt + \int_{E \setminus E_n} M\left(\frac{x(t) + u_n}{2}\right)dt + \int_{G \setminus E} M(x(t))dt \geq$$

$$\geq \frac{1}{2}\left(\int_{E_n} M(x(t))dt + \int_{E_n} M(x(t)-\varepsilon)dt\right) +$$

$$+ \int_{E \setminus E_n} M\left(\frac{u_n}{2}\right)dt + \int_{G \setminus E} M(x(t))dt \geq$$

$$\geq \frac{1}{2}\left(\int_{E_n} M(x(t))dt + \int_{E_n} M(x(t)-\varepsilon)dt \right) +$$

$$+\left(1-\frac{1}{n}\right)\frac{M(u_n)}{2}\mu(E\setminus E_n) + \int_{G\setminus E} M(x(t))dt$$

$$\geq \frac{1}{2}\left(1-\frac{1}{n}\right)(\rho_M(x_n)+\rho_M(x)) \to 1.$$

Hence

$$\| x_n + x \| \to 2.$$

Choose the supporting functional $y = \frac{1}{k}p(x)$ of x, where $k = \| p(x) \|_N^0$. Notice that

$$\int_{E_n} \varepsilon y(t)dt \geq \frac{1}{4}\varepsilon p(a)\mu E_n \to \frac{\varepsilon}{k} p(a)\mu E$$

$$\int_{E\setminus E_n} x(t)y(t)dt \to 0$$

and

$$\int_{E\setminus E_n} u_n y(t)dt \leq \| x_n|_{E\setminus E_n} \| \| y|_{E\setminus E_n} \|_N^0 \leq \| x_n \| \| y|_{E\setminus E_n} \|_N^0 \to 0.$$

Then

$$\int_G (x(t)-x_n(t))y(t)dt = \int_{E_n} \varepsilon y(t)dt + \int_{E\setminus E_n} (x(t)-u_n)y(t)dt \geq$$

$$\geq \frac{1}{k}\varepsilon p(a)\mu E_n + \int_{E\setminus E_n} x(t)y(t)dt - \int_{E\setminus E_n} u_n y(t)dt \to \frac{\varepsilon}{k} p(a)\mu E > 0,$$

which contradicts to the fact that x is an U-point.

Proof of sufficiency. Let $x_n \in S(L_M)$ and $\| x_n + x \| \to 2$. Without loss of generality assume that $x(t) \geq 0$. In the following we will prove the sufficiency consdering three cases:

(I) $\mu\{t \in G: x(t) \in (a,b]\} = 0$ for any **SAI** $[a,b]$ of M.

Denote

$$A = \left\{ t \in G : x(t) \in \bigcup_{i=1}^{\infty} \{a_i\} \right\}.$$

First we prove that

$$x_n(t) \overset{\mu}{\to} x(t) \text{ on } G \setminus A.$$

Otherwise, there exist $\varepsilon, \delta > 0$ such that

$$\mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon\} \geq \delta.$$

By Lemma 2, we have $\rho_M(x) = 1$, $\rho_M(x_n) \to 1$ and $\rho_M\left(\frac{x_n + x}{2}\right) \to 1$. Since

$$1 \leftarrow \rho_M(x_n) = \int_G M(x_n(t))dt \geq \int_{\{t \in G : |x_n(t) > D\}} M(x_n(t))dt$$

$$> M(D)\mu\{t \in G : |x_n(t)| > D\}$$

for every $D > 0$, we have

$$\mu\{t \in G : |x_n(t)| > D\} < \frac{1}{M(D)}.$$

Thus we may pick D large enough such that

$$\mu\{t \in G : |x_n(t)| > D\} < \delta/4 \quad \text{and} \quad \mu\{t \in G : |x(t)| > D\} < \delta/4.$$

Hence

$$\mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon,\ |x(t)| \leq D,\ |x_n(t)| \leq D\} > \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}.$$

If $t \in G \setminus A$, then $x(t) \neq a_i$, $i = 1, 2, \ldots$. Denote $V_\eta^i = (a_i - \eta, a_i + \eta)$. We have

$$\lim_{\eta \to 0} \mu\{t \in G \setminus A : x(t) \in V_\eta^i\} = \mu\{t \in G \setminus A : x(t) = a_i\} = 0.$$

Select η_i such that

$$\mu\{t \in G \setminus A : x(t) \in V_{\eta_i}^i\} < \frac{\delta}{4 \cdot 2^i}, \qquad i = 1, 2, \ldots.$$

Then

$$\mu\{t \in G \setminus A : x(t) \in \bigcup_{i=1}^{\infty} V_{\eta_i}^i\} < \frac{\delta}{4}.$$

Therefore

$$\mu G_n \overset{\Delta}{=} \mu\{t \in G \setminus A : |x_n(t) - x(t)| \geq \varepsilon,\ |x_n(t)| \leq D,\ |x(t)| \leq D,$$

$$x(t) \in R \setminus \bigcup_{i=1}^{\infty} V_{\eta_i}^i\} > \frac{\delta}{4}.$$

Consider the bounded closed set

$$F = \left\{ (u,v) \in R^2 : |u - v| \geq \varepsilon, |u| \leq D, |v| \leq D, v \in R \setminus \bigcup_{i=1}^{\infty} V_{\eta_i}^i \right\}.$$

It is easy to see that

$$M\left(\frac{u+v}{2}\right) < \frac{M(u)+M(v)}{2},$$

for $(u,v) \in F$. Since the set F is compact, there is $\delta' > 0$ such that

$$\max_{(u,v)\in F} \frac{2M\left(\frac{u+v}{2}\right)}{M(u)+M(v)} = 1 - \delta'$$

Hence,

$$M\left(\frac{u+v}{2}\right) \leq (1-\delta')\frac{M(u)+M(v)}{2}$$

for any $(u,v) \in F$. Consequently

$$M\left(\frac{x_n(t)+x(t)}{2}\right) \leq (1-\delta')\frac{M(x_n(t))+M(x(t))}{2}$$

for μ-a.e. $t \in G_n$. Therefore

$$\begin{aligned} 0 \leftarrow \frac{\rho_M(x_n)+\rho_M(x)}{2} - \rho_M\left(\frac{x_n+x}{2}\right) &= \\ = \int_G \left(\frac{M(x_n(t))+M(x(t))}{2} - M\left(\frac{x_n(t)+x(t)}{2}\right)\right)dt \\ \geq \int_{G_n} \left(\frac{M(x_n(t))+M(x(t))}{2} - M\left(\frac{x_n(t)+x(t)}{2}\right)\right)dt \geq \\ \geq \int_{G_n} \delta' \frac{M(x_n(t))+M(x(t))}{2} dt \geq \frac{\delta'}{2} M\left(\frac{\varepsilon}{2}\right)\frac{\delta}{4} > 0 \end{aligned}$$

and we get a contradiction. This means that $x_n(t) \xrightarrow{\mu} x(t)$ on $G \setminus A$. By the lower semicontinuity of the modular, we have

$$\liminf_n \rho_M(x_n|_{G\setminus A}) \geq \rho_M(x_n|_{G\setminus A}).$$

Since $\rho_M(x_n) \to 1 = \rho_M(x)$, the above inequality implies

$$(1) \qquad \limsup_n \rho_M(x_n|_A) \leq \rho_M(x|_A).$$

Since a is a left extreme point of **SAI** of M but it is not a right extreme point. Then, using the same argumentation as above, we have

$$\liminf_{n} x_n(t) \geq x(t)$$

for μ-a.e. $t \in A$. Combining (1), we obtain .

$$\lim_{n\to\infty} x_n(t) = x(t)$$

for μ-a.e. $t \in A$. Hence $x_n \xrightarrow{\mu} x$ on G.

Now we prove that

(2) $$\lim_{\mu e\to 0} \sup_{n} \rho_M(x_n|_e) = 0.$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \in \Sigma$ with $\mu e < \delta$ we get $\rho_M(x_n|_e) < \varepsilon$. Since $x_n \xrightarrow{\mu} x$ on G, by the Egoroff theorem, there is $e \subset G$ such that $\mu e < \delta$ and $x_n(t) \to x(t)$ uniformly on $G \setminus e$. Hence

$$\left|\rho_M(x_n|_{G\setminus e}) - \rho_M(x|_{G\setminus e})\right| < \varepsilon$$

for $n \leq n_0$. By the convergence $\rho_M(x_n) \to 1 = \rho_M(x)$, we have

$$\left|\rho_M(x_n|_e) - \rho_M(x|_e)\right| < \varepsilon$$

for n large enough. Therefore $\rho_M(x_n|_e) < 2\varepsilon$ for n large enough. Hence for any fixed n and any $e \in \Sigma$ with μe sufficiently small, we have

$$\rho_M(x_n|_e) < \varepsilon$$

unoformly, i.e (2) holds true.

By (2) and Lemma 2, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu e < \delta$ implies that

$$\rho_M(x|_e) < \varepsilon, \qquad \rho_M(x_n|_e) < \varepsilon \qquad (n = 1,2, \ldots)$$

and

$$\rho_N(y|_e) < \varepsilon \qquad \text{for} \quad y \in \nabla_x.$$

Since $x_n \xrightarrow{\mu} x$, applying again the Egoroff theorem, a set $e \in \Sigma$ can be found such that $\mu e < \delta$ and $x_n \to x$ uniformly on $G \setminus e$. Hence,

$$|x_n(t) - x(t)| < \varepsilon$$

for $t \in G \setminus e$ and n large enough. Thus

$$< x_n - x, y > = \int_{G\setminus e} (x_n(t) - x(t))y(t)dt + \int_{e} (x_n(t) - x(t))y(t)dt \leq$$

$$\leq \varepsilon \| \chi_G \| + \rho_M(x_n|_e) + \rho_M(x|_e) + 2\rho_N(y|_e) < \varepsilon \| \chi_G \| + 4\varepsilon$$

uniformly with respect to $y \in \nabla_x$, i.e. x is an *U*-point.

(II) $M \in \nabla_2$ and $\mu\{t \in G : x(t) \in (a,b]\} = 0$ for any **SAI** $[a,b]$ of *M*. Denote

$$B = \left\{ t \in G : x(t) \in \bigcup_{i=1}^{\infty} \{b_i\} \right\}$$

First, we will prove that

(3) $$\lim_{\mu e \to 0} \sup_n \rho_M(x_n|_e) = 0.$$

Otherwise, there exist $e_n \subset G$ with $\mu e_n \downarrow 0$ such that $\rho_M(x_n|_e) \geq \varepsilon$. Without loss of generality we can assume, $x_n(t) \geq u_0$ for $t \in e_n$. Since $M \in \nabla_2$, there exists $\delta \in (0,1)$ such that for $u > u_0$

$$M\left(\frac{u}{1+\tau'}\right) \leq \frac{1}{1+\tau'}(1-\delta)M(u),$$

where $\frac{1}{1-\tau'} = 1+\tau$. Hence

$$1 \leftarrow \rho_M\left(\frac{x_n(t)+x(t)}{2}\right) =$$

$$= \int_{G\setminus e_n} M\left(\frac{x_n(t)+x(t)}{2}\right)dt + \int_{e_n} M\left(\frac{1+\tau'}{2}\cdot\frac{x_n(t)}{1+\tau'} + \frac{1-\tau'}{2}\cdot\frac{x(t)}{1-\tau'}\right)dt \leq$$

$$\leq \int_{G\setminus e_n} M\left(\frac{x_n(t)+x(t)}{2}\right)dt + \frac{1+\tau'}{2}\int_{e_n} M\left(\frac{x_n(t)}{1+\tau'}\right)dt + + \frac{1-\tau'}{2}\int_{e_n} M((1+\tau)x(t))dt \leq$$

$$\leq \frac{1}{2}\int_{G\setminus e_n} M(x_n(t))dt + \frac{1}{2}\int_{G\setminus e_n} M(x(t))dt + \frac{1-\delta}{1+\tau'}\cdot\frac{1+\tau'}{2}\int_{e_n} M(x_n(t))dt +$$

$$+ \rho_M((1+\tau)x|_{e_n}) \leq$$

$$\leq \frac{\rho_M(x_n) + \rho_M(x)}{2} - \frac{\delta\varepsilon}{2} + \rho_M((1+\tau)x|_{e_n}) \to 1 - \frac{\delta\varepsilon}{2}.$$

This contradiction proves (3).

Analogously as in the proof of (1), we can deduce that $x_n \xrightarrow{\mu} x$ on *G\B*. Using (3), we obtain

$$\lim_{n\to\infty} \rho_M(x_n|_{G\setminus B}) = \rho_M(x|_{G\setminus B})$$

and consequently

$$\lim_{n\to\infty} \rho_M(x_n|_B) = \rho_M(x|_B).$$

Since b is a right extreme point of **SAI** of M and not a left extreme point, repeating the same argumentation as above, we get

$$\lim_{n\to\infty} x_n(t) = x(t), \quad \mu-\text{a.e. on } B.$$

Thus, we obtain that $x_n(t) \xrightarrow{\mu} x(t)$ on G. Similarly as in the proof of (I), we can verify that x is an U-point.

(III) $M \in \nabla_2$ and $x(t)$ does not belongs to any **SAI** of M or $x(t)$ belongs to some **SAI** and $p\_(x(t)) = p(x(t))$.
For every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \in \Sigma$ with $\mu e < \delta$ we have

$$\rho_M(x|_e) < \varepsilon, \quad \rho_M(x_n|_e) < \varepsilon \quad \text{and} \quad \rho_N(kv|_e) < \varepsilon$$

for each $v \in \nabla_x$, and $k \in K(v)$. Denote

$$E_i = \{t \in G : x(t) \in [a_i, b_i]\} \qquad (i=1,2,...)$$

and

$$E_0 = G \setminus \bigcup_{i=1}^{\infty} E_i.$$

Analogously as in the proof of (I), we can prove $x_n \xrightarrow{\mu} x$ on E_0. Choose $F_0 \subset E_0$ such that $\mu F_0 < \delta$ implies $x_n(t) \to x(t)$ uniformly on $E_0 \setminus F_0$, i.e. there exists $n_0 \in N$ such that $n > n_0$ implies

$$|x_n(t) - x(t)| < \varepsilon$$

and

$$|M(x_n(t)) - M(x(t))| < \varepsilon$$

for $t \in E_0 \setminus F_0$. Hence, for $n > n_0$, we have

$$x_n(t)kv(t) > (x(t) - \varepsilon)kv(t) = M(x(t)) + N(kv(t)) - \varepsilon kv(t) >$$
$$> M(x_n(t)) - \varepsilon + N(kv(t)) - \varepsilon kv(t)$$

for all $t \in E_0 \setminus F_0$. Since $\bigcup_{i=1}^{\infty} \mu E_i \le \mu G$, there is $m \in N$ such that $\mu(\bigcup_{i>m} E_i) < \delta$.

We have

$$up(a_i) = M(u) + N(p(a_i))$$

for $u \in [a_i, b_i]$. So there exists $\beta > 0$ such that

$$up(a_i) > M(u) + N(p(a_i)) - \varepsilon$$

whenever $u \in [a_i - \beta, b_i + \beta]$ (i=1,2, ... ,m). Fixing $\beta > 0$, by Lemma 2, we can find $\eta > 0$ such that

$$\frac{M(u)+M(v)}{2} - M\left(\frac{u+v}{2}\right) < \eta$$

and $v \in [a_i, b_i]$, imply $u \in [a_i - \beta, b_i + \beta]$ (i=1,2,...,m). Since

$$f_n(t) = \frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t)+x(t)}{2}\right) \xrightarrow{\mu} 0.$$

Then

$$\mu F_n = \mu\{t \in G : f_n(t) \geq \eta\} < \delta$$

for $n > n_0' > n_0$. Hence, if $t \in \bigcup_{i=1}^{\infty} E_i \setminus F_n$, then $x(t) \in [a_i, b_i]$ and $f_n(t) < \eta$. So $x_n(t) \in [a_i - \beta, b_i + \beta]$ (i=1,2,...,m). Furthermore,

$$x_n(t)p(a_i) > M(x_n(t)) + N(p(a_i)) - \varepsilon \qquad (i = 1,2,...,m).$$

By the definition of E_i, we conclude $x(t) \in [a_i, b_i]$. Hence for $v \in \nabla_x$,

$$kv(t) = p(x(t)) = p\_(x(t)) = p(a_i),$$

i.e.

$$x_n(t)kv(t) > M(x_n(t)) + N(kv(t)) - \varepsilon$$

for $t \in \bigcup_{i=1}^{m} E_i \setminus F_n$. Thus, we conclude that

$$< x_n, kv > = \int_{(E_0 \setminus F_0) \cup \left(\bigcup_{i=1}^{\infty} E_i \setminus F_n\right)} x_n(t)kv(t)dt + \int_{F_0 \cup \left(\bigcup_{i=1}^{m} E_i \cap F_n\right) \cup \left(\bigcup_{i>m} E_i\right)} x_n(t)kv(t)dt \geq$$

$$\geq \int_{(E_0 \setminus F_0) \cup \left(\bigcup_{i=1}^{\infty} E_i \setminus F_n\right)} (M(x_n(t)) + N(kv(t)))dt - o(\varepsilon) =$$

$$= \int_G (M(x_n(t)) + N(kv(t)))dt - o(\varepsilon) = 1 + \rho_N(kv) - o(\varepsilon) = k - o(\varepsilon).$$

This implies $(x_n, v) > 1 - o(\varepsilon)$, i.e. $(x_n, v) \to 1$. Hence x is an U-point.

Corollary 2. L_M has local U-property iff

(i) $M \in \Delta_2$

(ii) *for any* **SAI** $[a,b]$ of *M*, $p\_(a) = p(a) = p\_(b) = p(b)$,

(iii) $M \in SC[0,\infty)$ *or* $M \in \nabla_2$.

Proof of necessity. If L_M has local *U*-property, then it has quasi-*U*-property. So (i) and (ii) are satisfied. If (iii) is not true, then $M \notin \nabla_2$ and there exists a **SAI** $[a,b]$ of *M*. Pick $E \subset G$, $F \subset G \setminus E$ and *c* large enough such that $0 < \mu E < \mu G$ and $M(b)\mu E + M(c)\mu F = 1$. Set $x = b|_E + c|_F$. Then $\rho_M(x) = 1$ and $\mu\{t \in G : x(t) \in (a,b]\} \geq \mu E > 0$.

Hence, $x \in S(L_M)$ is not *U*-point. Therefore L_M cannot have local *U*-property. This contradiction proves (iii).

Proof of sufficiency. Using the same argumentation as in the proof of Corollary 1, conditions (i) and (ii) of Theorem 2 hold true. If $M \in \nabla_2$ or $M \in SC[0,\infty)$, then $\mu\{t \in G : |x(t)| \in (a,b]\} = \mu\varnothing = 0$. Hence *x* is an *U*-point and consequently L_M has local *U*-property.

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