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ON THE OSCILLATION OF ODD ORDER NEUTRAL DIFFERENTIAL EQUATIONS*

ABSTRACT: Consider the odd order neutral differential equation

$$\frac{d^n}{dt^n}(y(t)-py(t-\tau))+\sum_{i=1}^{m}q_i(t)y(t-\sigma_i)=0,$$

where $p\geq 0$, $n\geq 1$ is an odd integer, $\tau>0$, $\sigma_i\geq 0$, $q_i\in C([t_0,\infty),R_+)$, $i=1,2,\ldots,m$. Some new sufficient conditions for oscillation of the above equation are obtained. A comparison result is also established.

KEY WORDS: oscillation, neutral, differential equation.

1. INTRODUCTION

Interest in the study of oscillatory and asymptotic characteristics of neutral delay differential equations has been steadily increasing during the last ten years. Recently developments on the dynamical characteristics of neutral differential equations can be found in the monographs of Gopalsamy [5], Gyori and Ladas [6], Bainov and Mishev [7] and Erbe, Kong and Zhang [12]. The odd order neutral delay differential equation

$$\frac{d^n}{dt^n}(y(t)-py(t-\tau))+\sum_{i=1}^{m}q_i(t)y(t-\sigma_i)=0 \tag{1}$$

has been investigated in [3, 8, 9, 12]. One of the common features of the above investigations is the restriction that the coefficient satisfies $0\leq p<1$.

When $n=1$, Eq. (1) becomes

$$\frac{d}{dt}(y(t)-py(t-\tau))+\sum_{i=1}^{m}q_i(t)y(t-\sigma_i)=0. \tag{2}$$

The purpose of this paper is mainly to examine the oscillatory characteristics of (1) in the case $p\geq 1$. Our results presented as follows in in Section 2, we give some lemmas. Using the lemmas of Section 2, we establish in Section 3 sufficient conditions for the oscillation of (1) for $p\geq 1$, and in Section 4, we obtain

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corresponding results for the case $0 \le p < 1$, which extends and improves the results of Erbe and Kong [10]. In section 5, we conclude with a derivation of a comparison result.

A nontrivial solution of (1) defined on a half-line $[t_0,\infty)$ is said to be oscillatory if the solution has an unbounded sequence of zeros. The equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

For the sake of convenience, let

$$z(t) = y(t) - p\,y(t-\tau). \tag{3}$$

2. Some Lemmas

The main results in this section are following Lemmas 1-4 which will play key roles in the proofs of the theorems in Sections 3 and 4.

Lemma 1. Supose that $p \in [1,\infty)$, σ_i, $\tau \in (0,\infty)$, $\tau - \sigma_i > 0$, $i = 1,2,\ldots,m$. *Suppose further that* $q_i \in C([t_0,\infty),R_+)$, $\sum_{i=1}^{m} q_i(s)ds \not\equiv 0$ *on any subintervals of* $[t_0,\infty)$ *and*

$$\liminf_{t\to\infty} \int_t^{t+\frac{\tau-\max\sigma_i}{n}} \sum_{i=1}^{m} q_i(s)\,ds > 0. \tag{4}$$

Then

(a) *If* $y(t)$ *is eventually positive, then either* $z^{(i)}(t)$ *is decreasing with*

$$z^{(i)}(t) \to -\infty \quad as \quad t\to\infty \tag{5}$$

for $i = 0,1,2,\ldots,n-1$; *or* $z^{(i)}(t)$ *is monotonic and*

$$z^{(i)}(t) \to 0 \quad as \quad t\to\infty \quad and \quad z^{(i)}(t)\, z^{(i+1)}(t) < 0 \tag{6}$$

for $i = 0,1,2,\ldots,n-1$.

(b) *If* $y(t)$ *is eventually negative, then either* $z^{(i)}(t)$ *is increasing with*

$$z^{(i)}(t) \to \infty \quad as \quad t\to\infty$$

for $i = 0,1,2,\ldots,n-1$; *or* (6) *holds*.

(c) *If n is odd and* (6) *holds, then* $z(t) > 0$ $(z(t) < 0)$ *for* $y(t) > 0$ $(y(t) < 0)$.

Proof. Condition (4) implies that

$$\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(t)dt = \infty.$$

By Lemma 2 in [1], it is easy to see that (a), (b) and (c) hold. The proof is complete.

Lemma 2 [2]. *Let the* $\phi(t) \in C^n([t_0,\infty),R)$ *be of constant sign and let the following inequality hold:*

$$\phi(t)\phi^n(t) < 0 \qquad (\phi(t)\phi^n(t) > 0).$$

Then there exists $t^* \geq t_0$ *and a number* $l \in \{0,1,\dots,n\}$ *such that* $l+n$ *is odd (even) and for* $t \geq t^*$ *the following inequalities are valid:*

(7) $$\phi(t)\phi^{(i)}(t) > 0, \qquad i = 0,1,\dots,l\,,$$

(8) $$(-1)^{l+i}\phi(t)\phi^{(i)}(t) > 0, \qquad i = l+1,\dots,n.$$

Lemma 3. Suppose that $\phi(t) \in C^n([t_0,\infty),R)$, $t_0 \geq 0$, *such that* $\phi^{(i)}(t)$ $(i < n)$ *is of one sign in* $[t_0,\infty)$ *and* $\phi^n(t) \leq 0$, $t \geq t_0$. *Then* $\alpha > 0$ *implies that:*

(a) *If* $\phi(t) > 0$, *then*

(9) $$\phi(t-\alpha) \geq \frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t), \qquad t \geq t_0 + 2\alpha.$$

(b) *If* $\phi(t) < 0$, *then*

(10) $$\phi(t+\alpha) \leq \frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t), \qquad t \geq t_0.$$

Proof. Part (a) of Lemma 3 is Lemma 1 in [3]. Now we shall prove part (b) of Lemma 3.

Since $\phi^n(t) \leq 0$, by Lemma 2, there exists an integer k $(0 \leq k \leq n)$ such that

$$\phi^{(j)}(t) < 0 \quad \text{for} \quad j \leq k,$$

and

$$\phi^{(j)}(t)\phi^{(j+1)}(t) < 0 \quad \text{for} \quad k \leq j < n.$$

By Lemma 2, we have $k+n$ is even. Hence we may note that k is odd (even) if and only if n is odd (even).

If $k=n-1$ or $k=n$, then, expanding $\phi(t)$ by Taylor's theorem, there exists $x\in(t,t+\alpha)$ such that

$$\phi(t+\alpha)=\sum_{j=0}^{n-1}\frac{\alpha^j}{j!}\phi^{(j)}(t)+\frac{\alpha^n}{n!}\phi^{(n)}(x)\le\frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t).$$

This shows that (10) holds.

If $k<n-1$, then, by Lemma 2, we have $\phi^{(n-1)}(t)>0$. Obviously, (10) holds also. The proof is complete.

Lemma 4. Assume that $0\le p<1$, *and* (4) *holds. Then* $z(t)$ *satisfies* (6).

Proof. Since (4) holds, then

$$\int_{t_0}^{\infty}\sum_{i=1}^{m}q_i(s)\,ds=\infty.$$

By part (d) of Lemma 2 in [1], we have $\lim_{t\to\infty}z(t)=0$. The proof is complete.

3. Oscillation of (1) with $p\ge1$

Theorem 1. Suppose that $p\in[1,\infty)$; σ_i, $\tau\in(0,\infty)$, $\tau-\sigma_i>0$, $i=1,2,\dots,m$, *n is odd Suppose further that* (4) *holds for* $\mu>0$, $l=\tau$, $l=(\tau-\sigma_i)/n$, $i=1,2,\dots,m$

$$\text{(11)}\quad \liminf_{t\to\infty}\left\{\frac{1}{p}e^{\mu\tau}+\frac{1}{lp\mu(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m}(\tau-\sigma_i)^{n-1}e^{\mu\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l}q_i(s)ds\right\}>1,$$

and

$$\text{(12)}\qquad q_i(t)\le q_i(t-\tau),\qquad i=1,2,\dots,m.$$

Then every solution of (1) *oscillates.*

Proof. Suppose the contrary, let $y(t)$ be an eventually positive solution of (1). Then from (1) $z^{(n)}(t)\le0$ eventually. By part (a) of Lemma 1, we have that (5) or (6) holds. If (6) holds, then, by part (c) of Lemma 1, we have $z(t)>0$

eventually. Then there exist $M>0$ and $T \geq t_0$ such that $y(t-\sigma_i) \geq M$, for $i=1,2,\ldots,m$ and $t \geq T$. From (1), we have

$$z^{(n)}(t)+M\sum_{i=1}^{m} q_i(t) \leq 0. \tag{13}$$

Condition (4) implies that

$$\int_T^\infty \sum_{i=1}^{m} q_i(t)\,dt=\infty.$$

Integrating (13) from T to t $(t \geq T)$, we have

$$z^{(n-1)}(t)-z^{(n-1)}(T) \leq -M\int_T^t \sum_{i=1}^{m} q_i(s)\,ds. \tag{14}$$

Letting $t \to \infty$, we obtain $\lim_{t\to\infty} z^{(n-1)}(t)=-\infty$ and hence $\lim_{t\to\infty} z(t)=-\infty$, which is a contradiction. Hence (5) holds.

From (1) and (2) we have

$$z^{(n)}(t)=-\sum_{i=1}^{m} q_i(t)\,y(t-\sigma_i)=-\sum_{i=1}^{m} q_i(t)\,z(t-\sigma_i)-p\sum_{i=1}^{m} q_i(t)\,z(t-\tau-\sigma_i) \geq$$

$$\geq -\sum_{i=1}^{m} q_i(t)\,z(t-\sigma_i)-p\sum_{i=1}^{m} q_i(t-\tau)\,z(t-\tau-\sigma_i)=$$

$$=-\sum_{i=1}^{m} q_i(t)\,z(t-\sigma_i)+pz^{(n)}(t-\tau).$$

Hence

$$z^{(n)}(t) \leq \frac{z^{(n)}(t+\tau)}{p}+\frac{1}{p}\sum_{i=1}^{m} q_i(t)\,z(t+\tau-\sigma_i). \tag{15}$$

Dividing both sides of (15) by $z^{(n-1)}(t)$ and noting that $z^{(n-1)}(t)$ is negative, we have

$$\frac{z^{(n)}(t)}{z^{(n-1)}(t)} \geq \frac{z^{(n)}(t+\tau)}{p\,z^{(n-1)}(t)}+\frac{1}{p\,z^{(n-1)}(t)}\sum_{i=1}^{m} q_i(t)\,z(t+\tau-\sigma_i). \tag{16}$$

Using part (b) of Lemma 3 for $\alpha=((n-1)/n)(\tau-\sigma_i)$ for the term $z(t+\tau-\sigma_i)$ $(i=1,2,\ldots,m)$, we have

$$(17)\qquad z(t+\tau-\sigma_i)\le \frac{(\tau-\sigma_i)^{(n-1)}}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} z^{(n-1)}\left(t+\frac{\tau-\sigma_i}{n}\right).$$

Let

$$(18)\qquad \lambda(t)=\frac{z^{(n)}(t)}{z^{(n-1)}(t)}>0.$$

By (16), (17) and (18) we obtain

$$(19)\qquad \lambda(t)\ge \frac{\lambda(t+\tau)}{p}\exp\left(\int_t^{t+\tau}\lambda(s)ds\right)+$$
$$+\frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m} q_i(t)(\tau-\sigma_i)^{n-1}\exp\left(\int_t^{t+\frac{\tau-\sigma_i}{n}}\lambda(s)ds\right).$$

Define $\{\lambda_k(t)\}$, $k=1,2,\ldots$ and $t\ge T$ and a sequence of numbers $\{\mu_k\}$, $k=1,2,\ldots$ as follows:

$$\lambda_1(t)=0,\quad t\ge T,$$

$$(20)\qquad \lambda_{k+1}(t)=\frac{\lambda_k(t+\tau)}{p}\exp\left(\int_t^{t+\tau}\lambda(s)ds\right)+$$
$$+\frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m} q_i(t)(\tau-\sigma_i)^{n-1}\exp\left(\int_t^{t+\frac{\tau-\sigma_i}{n}}\lambda_k(s)ds\right),$$
$$k=1,2,\ldots.$$

$$\mu_1=0,$$

$$(21)\qquad \mu_{k+1}=\inf_{t\ge T}\left\{\min_{\substack{l=\tau,\frac{\tau-\sigma_i}{n}\\ i=1,\ldots,m}}\left[\frac{\mu_k}{p}e^{\mu_k\tau}+\frac{1}{pl(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\times\right.\right.$$
$$\left.\left.\times\sum_{i=1}^{m}(\tau-\sigma_i)^{n-1}e^{\mu_k\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l}q_i(s)ds\right]\right\},\qquad k=1,2,\ldots$$

It is easy to see that

(i) $0 = \mu_1 < \mu_2 < \ldots,$

(ii) $\lambda_k(t) \le \lambda(t), \; k = 1,2,\ldots;$

(iii) $\frac{1}{l}\int_t^{t+l} \lambda_k(s)ds \ge \mu_k, \; k = 1,2,\ldots, \; t \ge T, \; l = \tau, \; \frac{\tau - \sigma_i}{n}, \; i = 1,2,\ldots,m.$

(i) and (ii) are obvious and hence we shall consider only (iii). It is evident for $k = 1$, (iii) holds; suppose that it is true for $k > 1$; then

$$\frac{1}{l}\int_t^{t+l} \lambda_{k+1}(s)ds = \frac{1}{pl}\int_t^{t+l} \lambda_k(s+\tau)\,\exp\left(\int_s^{s+\tau} \lambda_k(u)du\right)ds +$$

$$+\frac{1}{pl(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\int_t^{t+l}\sum_{i=1}^m (\tau-\sigma_i)^{n-1} q_i(s)\,\exp\left(\int_s^{s+\frac{\tau-\sigma_i}{n}} \lambda_k(u)du\right)ds \ge$$

$$\ge \frac{\mu_k}{p}e^{\mu_k\tau} + \frac{1}{pl(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^m (\tau-\sigma_i)^{n-1} e^{\mu_k\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l} q_i(s)ds \ge u_{k+1}.$$

Thus (iii) holds for $k = 1,2,\ldots$. From (11) and (21), it is easy to see that there exists α such that $\alpha > 1$ and $\mu_{k+1} \ge \alpha\mu_k$. Hence $\lim_{k\to\infty} \mu_k = +\infty$. From propeties (ii) and (iii), we have

$$\frac{z^{(n-1)}(t+l)}{z^{(n-1)}(t)} = \exp\left(\int_t^{t+l} \lambda\;(s)ds\right) \to +\infty. \tag{22}$$

On the other hand, since $y(t) > 0$, we have $z(t) > -p\,y(t-\tau)$. Substituting the last inequality into (1) we have

$$z^{(n)}(t) = -\sum_{i=1}^m y(t-\sigma_i)q_i(s) < \frac{1}{p}\sum_{i=1}^m z(t+\tau-\sigma_i)q_i(t).$$

Furthermore, by part (b) of Lemma 3, we have

$$z^{(n)}(t) < \frac{1}{p}\sum_{i=1}^m z(t+\tau-\sigma_i)q_i(t) \le$$

$$\le \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(t)\, z^{(n-1)}\left(t+\frac{\tau-\sigma_i}{n}\right)$$

and hence

$$z^{(n)}(t) < \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \times \left(\sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(t)\right) z^{(n-1)}\left(t+\frac{\tau-\max_{i=1,2,\ldots,m}\sigma_i}{n}\right). \tag{23}$$

Set

$$w(t) = z^{(n-1)}(t),$$

then

$$w'(t) < \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(t)\, w\left(t+\frac{\tau-\max_{i=1,2,\ldots,m}\sigma_i}{n}\right). \tag{24}$$

Integrating (24) from t to $t+(\tau-\max\sigma_i)/(2n)$, we have

$$w\left(t+\frac{\tau-\max\sigma_i}{2n}\right) - w(t) \le$$

$$\le \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \int_t^{t+\frac{\tau-\max\sigma_i}{2n}} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(s)\, w\left(s+\frac{\tau-\max\sigma_i}{n}\right) ds \le$$

$$\le w\left(t+\frac{\tau-\max\sigma_i}{n}\right) \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \int_t^{t+\frac{\tau-\max\sigma_i}{2n}} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(s) ds.$$

Hence

$$\frac{w\left(t+\frac{\tau-\max\sigma_i}{n}\right)}{w\left(t+\frac{\tau-\max\sigma_i}{2n}\right)} \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \int_t^{t+\frac{\tau-\max\sigma_i}{2n}} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} q_i(s) ds \le 1 \tag{25}$$

From (4) and (25), we obtain that

$$\frac{w\left(t+\dfrac{\tau-\max\sigma_i}{n}\right)}{w\left(t+\dfrac{\tau-\max\sigma_i}{2n}\right)}$$

is bounded, and this contradicts (22). This completes the proof.

Remark 1. The application of Lemma 3 with $\alpha=((n-1)/n)(\tau-\sigma_i)$ in (16) is not totally unjustified. Suppose that Lemma 3 is applied to (16) with t replaced by $t-\alpha_i$ in $z(t+\tau-\sigma_i)$. Then the reduced inequality will be

$$\frac{(\alpha_i)^{n-1}}{(n-1)!}z^{n-1}(t-\alpha_i+\tau-\sigma_i)\geq z(t+\tau-\sigma_i).$$

If we choose α_i such that $0<\alpha_i<\tau-\sigma_i$, the function

$$G(\alpha_i)=(\alpha_i)^{n-1}(\tau-\sigma_i-\alpha_i)$$

will attain its maximum value at $\alpha_i=((n-1)/n)(\tau-\sigma_i)$. Similar remark will hold for Theorem 2 in Section 4.

Remark 2. Condition (4) can be replaced by

$$\liminf_{t\to\infty}\int_t^{t+\frac{\tau-\sigma_i}{n}}q_i(s)\,ds>0\quad\text{for some } i. \tag{26}$$

Remark 3. When $n=1$, condition (11) becomes

$$\liminf_{t\to\infty}\left\{\frac{1}{p}e^{\mu\tau}+\frac{1}{pl}\sum_{i=1}^{m}e^{\mu(\tau-\sigma_i)}\int_t^{t+l}q_i(s)ds\right\}>1. \tag{27}$$

This is a known result due to Zhang and Gopalsamy [4].

Corollary 1. Assume that for $l=\tau,\ (\tau-\sigma_i)/n,\ i=1,2,\dots,m$

$$\liminf_{t\to\infty}\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m}(\tau-\sigma_i)^{n-1}\left(\frac{1}{l}\int_t^{t+l}q_i(s)ds\right)\times \tag{28}$$

$$\times \sum_{k=0}^{\infty} \frac{e\left[\left(k+\frac{1}{n}\right)\tau - \frac{\sigma_i}{n}\right]}{p^{k+1}} > 1.$$

Then every solution of (1) is oscillatory.

Proof. In fact, for $(1/p)e^{\mu\tau} < 1$, we have

$$\liminf_{t\to\infty} \frac{1}{\mu p l(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \times$$

$$\times \sum_{i=1}^{m} e^{\mu(\frac{\tau-\sigma_i}{n})}(\tau-\sigma_i)^{n-1} \int_t^{t+l} q_i(s)ds \left(1-\frac{1}{p}e^{\mu\tau}\right)^{-1} =$$

$$= \liminf_{t\to\infty} \frac{1}{\mu p l(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \times$$

$$\times \sum_{i=1}^{m} e^{\mu\left(\frac{\tau-\sigma_i}{n}\right)}(\tau-\sigma_i)^{n-1} \int_t^{t+l} q_i(s)ds \sum_{k=0}^{m} \frac{e^{k\mu\tau}}{p^k} \geq$$

$$\geq \liminf_{t\to\infty} \frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \times$$

$$\times \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} \sum_{k=0}^{\infty} \left(\frac{1}{l}\int_t^{t+l} q_i(s)ds\right) \frac{e\left[\left(k+\frac{1}{n}\right)\tau - \frac{\sigma_i}{n}\right]}{p^{k+1}} > 1,$$

i.e. (11) holds. By Theorem 1, every solution of (1) oscillates. If $(1/p)e^{\mu\tau} \geq 1$, then (11) is satisfied also. Therefore the corollary holds.

Example 1. Consider the equation

$$\frac{d^3}{dt^3}(y(t) - py(t-\tau)) + \left[1+\frac{1}{t}\right]y(t-\sigma) = 0, \qquad t \geq 1,$$

where $p \geq 1$, $\tau > 0$, $\sigma > 0$ and $\tau - \sigma > 0$. Let $q(t) = 1 + (1/t)$. It is easy to see that for all $l > 0$

$$\lim_{t\to\infty} \frac{1}{l}\int_t^{t+l} q_i(s)ds = \lim_{t\to\infty} \frac{1}{l}\int_t^{t+l}\left[1+\frac{1}{s}\right]ds = \lim_{t\to\infty}\left(1+\ln\left(1+\frac{l}{t}\right)\right) = 1.$$

According to Theorem 1, the above equation is oscillatory if for all $\mu > 0$

$$\frac{1}{p}e^{\mu\tau} + \frac{2}{\vartheta p\mu}(\tau-\sigma)^2 e^{\mu(\frac{\tau-\sigma}{3})} > 1.$$

4. OSCILLATION OF (1) WITH $0 \le p < 1$

In this section we estabilish a new sufficient condition for the oscillation of (1) with $0 \le p < 1$.

Theorem 2. Suppose that $p \in [0,1)$; $\tau, \sigma_i \in (0,\infty)$, $i = 1,2,\dots,m$; *n is odd. Assume further that*

$$\text{(29)} \qquad \liminf_{t\to\infty} \int_t^{t+\frac{\max\sigma_i}{n}} \sum_{i=1}^m q_i(s)\,ds > 0$$

holds and for $\mu > 0$, $l = \sigma_i/n$, $i = 1,2,\dots,m$,

$$\text{(30)} \qquad \liminf_{t\to\infty} \left\{ pe^{\mu\tau} + \frac{1}{\mu l(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \sum_{i=1}^m (\sigma_i)^{n-1} e^{\mu\left(\frac{\sigma_i}{n}\right)} \int_t^{t+l} q_i(s)ds \right\} > 1,$$

and

$$\text{(31)} \qquad q_i(t) \ge q_i(t-\tau), \qquad i = 1,2,\dots,m.$$

Then every solution of (1) *oscillates.*

Proof. Suppose the contrary, let $y(t)$ be an eventually positive solution of (1). Then from (1) $z^{(n)}(t) \le 0$ eventually. By Lemmas 1 and 4 we have $z(t) > 0$ eventually and $\lim_{t\to\infty} z(t) = 0$. Hence $z'(t) < 0$ eventually. In view of Lemma 2 we obtain $z^{(n-1)}(t) > 0$ eventually. From (1) and (31) we have

$$z^{(n)}(t) = -\sum_{i=1}^m q_i(t)\,y(t-\sigma_i) = -\sum_{i=1}^m q_i(t)\,z(t-\sigma_i) - p\sum_{i=1}^m q_i(t)\,z(t-\tau-\sigma_i) \le$$

$$\le -\sum_{i=1}^m q_i(t)\,z(t-\sigma_i) - p\sum_{i=1}^m q_i(t-\tau)\,z(t-\tau-\sigma_i) =$$

$$= -\sum_{i=1}^m q_i(t)\,z(t-\sigma_i) + p\,z^{(n)}(t-\tau).$$

Hence

$$z^{(n)}(t) \le p z^{(n)}(t-\tau) - \sum_{i=1}^{m} q_i(t) z(t-\sigma_i). \tag{32}$$

Dividing both sides of (32) by $-z^{(n-1)}(t)$ and noting that $z^{(n-1)}(t) > 0$ eventually, we have

$$\frac{z^{(n)}(t)}{-z^{(n-1)}(t)} \ge \frac{p z^{(n)}(t-\tau)}{-z^{(n-1)}(t)} + \frac{1}{z^{(n-1)}(t)} \sum_{i=1}^{m} q_i(t) z(t-\sigma_i). \tag{33}$$

Using part (a) of Lemma 3 for $\alpha = ((n-1)/n)\sigma_i$ for the term $z(t-\sigma_i)$ $(i = 1,2,\dots,m)$, we have

$$z(t-\sigma_i) \ge \frac{(\sigma_i)^{n-1}}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} z^{(n-1)}\left(t - \frac{\sigma_i}{n}\right). \tag{34}$$

Let

$$\lambda(t) = -\frac{z^{(n)}(t)}{z^{(n-1)}(t)} > 0. \tag{35}$$

By (33), (34) and (35) we obtain

$$\lambda(t) \ge p\lambda(t-\tau) \exp\left(\int_{t-\tau}^{t} \lambda(s) ds\right) + \tag{36}$$
$$+ \frac{1}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} \sum_{i=1}^{m} q_i(t) \sigma_i^{n-1} \exp\left(\int_{t-\frac{\sigma_i}{n}}^{t} \lambda(s)\, ds\right).$$

Repeating the proof of Theorem 1, we can obtain that the conclusion is valid. The proof is complete.

5. A COMPARISON RESULT

We shall consider the equation

$$\frac{d^n}{dt^n}(y(t) - P(t)y(t-\tau)) + Q(t)y(t-\sigma) = 0 \tag{37}$$

and the equation

(38) $$\frac{d^n}{dt^n}(y(t)-P^*(t)y(t-\tau))+Q^*(t)y(t-\sigma)=0$$

Lemma 4 [11]. *Assume that* $P(t)\geq 1$; $Q(t)\in C([t_0,\infty),R^+)$ *such that*

(39) $$\int_{t_0}^{\infty} s^n Q(s)\int_{s}^{\infty}(u-s)^{n-1}Q(u)\,du\,ds=\infty.$$

If $y(t)$ *is an eventually positive solution of* (37), *and let* $z(t)=y(t)-$ $-P(t)y(t-\tau)$, *then* $z(t)<0$ *eventually.*

Theorem 3. Suppose that $P(t)\geq 1$; $Q^*(t)\in C([t_0,\infty),R^+)$ *satisfies (39). Suppose further that*

(40) $$P(t)\leq P^*(t) \qquad \text{and} \qquad Q(t)\geq Q^*(t)..$$

Then oscillation of (38) implies that of (37).

Proof. Suppose the conclusion is not valid; let $y(t)$ be an eventually positive solution of (37). Let $z(t)=y(t)-P(t)y(t-\tau)$. Then, by Lemma 4, we have $z(t)<0$, $z^{(n)}(t)\leq 0$ eventually.

Since n is odd, then, by Lemma 5.2.2 in [2], we have either

(41) $$z^{(i)}(t)<0, \qquad i=0,1,\ldots,n$$

or

(42) $$z^{(i)}(t)<0, \quad i=0,1,\ldots,l \quad \text{and} \quad (1)^i z^{(i)}(t)\leq 0, \quad i=l+1,\ldots,n.$$

where l is odd.

Let $T_1\geq t_0$ be such that

(43) $$z^{(n)}(t)=-Q(t)\,y(t-\sigma)$$

holds for all $t\geq T_1$. If (41) holds, then integrating (43) from T_1 to t we have

(44) $$z(t)\leq-\int_{T_1}^{t}\int_{T_1}^{s_{n-1}}\ldots\int_{T_1}^{s_1}Q(u)\,y(u-\sigma)\,du\,ds_1\ldots ds_{n-1}.$$

That is,

$$(45)\qquad P(t)y(t-\tau)\geq y(t)+\int_{T_1}^{t}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1}Q(u)\,y(u-\sigma)\,du\,ds_1\dots ds_{n-1}.$$

Since $P(t)\geq 1$, then

$$(46)\qquad y(t)\geq\frac{1}{P(t+\tau)}\left[y(t+\tau)+\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1}Q(u)y(u-\sigma)du\,ds_1\dots ds_{n-1}\right].$$

By condition (40), we have

$$(47)\qquad y(t)\geq\frac{1}{P^*(t+\tau)}\Big[y(t+\tau)+$$

$$+\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1}Q^*(u)y(u-\sigma)\,du\,ds_1\dots ds_{n-1}\Big],\qquad t\geq T_1+\tau.$$

Let $T\geq T_1+\tau$ such that (47) holds for all $t\geq T$. Set

$$(48)\qquad T_0=\max\{\tau,\sigma\}.$$

Now we consider the set of functions:

$$\Omega=\{w\in C([T-T_0,\infty),R^+):0\leq w(t)\leq 1,\ \text{for}\ t\geq T-T_0\},$$

and define a mapping F on Ω as

$$(Fw)(t)=\begin{cases}\dfrac{1}{y(t)P^*(t+\tau)}\Big[w(t+\tau)y(t+\tau)+\\ \qquad+\displaystyle\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1}Q^*(u)w(u-\sigma)y(u-\sigma)du\,ds_1\dots ds_{n-1}\Big], & t\geq T,\\ \dfrac{t-T+T_0}{T_0}(Fw)(T)+\left(1-\dfrac{t-T+T_0}{T_0}\right), & T-T_0\leq t\leq T.\end{cases}$$

It is easy to see by using (47) that F maps Ω into itself, and for any $w\in\Omega$, we have $(Fw)(t)>0$ for $T-T_0\leq t\leq T$.

Next we define the sequence $w_k(t)$ in Ω,

$$w_0(t)=1,\qquad t\geq T-T_0,$$

$$w_{k+1}(t)=(Fw_k)(t),\qquad \text{for}\qquad k=0,1,\dots.$$

Then, by using (47) and a simple induction, we can easily see that

$$0 \le w_{k+1}(t) \le w_k(t) \le 1 \quad \text{for} \quad t \ge T - T_0, \quad k = 0,1,\dots.$$

Set

$$w(t) = \lim_{k\to\infty} w_k(t), \qquad t \ge T - T_0.$$

Then it follows from Lebesgue's dominated convergence theorem that $w(t)$ satisfies

$$w(t) = \frac{1}{y(t)P^*(t+\tau)}\Bigg[w(t+\tau)y(t+\tau) + \int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1} Q^*(u)\,w(u-\sigma)y(u-\sigma)\,du\,ds_1\dots ds_{n-1}\Bigg], \qquad t \ge T,$$

and

$$w(t) = \frac{t-T+T_0}{T_0}(Fw)(T) + \left(1 - \frac{t-T+T_0}{T_0}\right) > 0, \qquad T - T_0 \le t < T.$$

Again set

$$v(t) = w(t)\,y(t).$$

Then $v(t)$ satisfies $v(t) > 0$ for $T - T_0 \le t < T$ and

$$v(t) = \frac{1}{P^*(t+\tau)}\Bigg[v(t+\tau) + \int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\dots\int_{T_1}^{s_1} Q^*(u)\,v(u-\sigma)\,du\,ds_1\dots ds_{n-1}\Bigg], \qquad t \ge T. \tag{49}$$

Clearly, $v(t)$ is continuous on $[T-T_0, T)$. Then by the method of steps we see, in view of (49), that $v(t)$ is continuous on $[T-T_0, \infty)$.

Since $v(t) > 0$ for $T - T_0 \le t < T$, it is easy to see that $v(t) > 0$ for $t \ge T - T_0$. Hence, $v(t)$ is positive solution of (38). This is a contradiction.

If (42) holds, then, integrating (43) from T_1 to t we get

$$z^{(l)}(t) \ge -\int_{T_1}^{t}\int_{T_1}^{s_{n-l-1}}\dots\int_{T_1}^{s_1} Q(s)\,y(s-\sigma)\,ds\,ds_1\dots ds_{n-l-1}, \qquad t \ge T_1. \tag{50}$$

Again integrating (50) from T_1 to t, we have

$$(51) \qquad y(t) \geq \frac{1}{P(t+\tau)}\left[y(t+\tau) + \int_{T_1}^{t} \frac{(t-s)^{l-1}}{(l-1)!} \int_{T_1}^{s} \int_{T_1}^{s_{n-l-1}} \ldots \int_{T_1}^{s_1} Q(u) y(u-\sigma) du\, ds_1 \ldots ds_{n-l-1} \right].$$

By condition (40), we have

$$(52) \qquad y(t) \geq \frac{1}{P^*(t+\tau)}\left[y(t+\tau) + \int_{T_1}^{t} \frac{(t-s)^{l-1}}{(l-1)!} \int_{T_1}^{s} \int_{T_1}^{s_{n-l-1}} \ldots \int_{T_1}^{s_1} Q^*(u)\, y(u-\sigma)\, du\, ds_1 \ldots ds_{n-l-1} \right],$$

which, using a method similar to the proof of the former case, yields that Eq. (38) has a positive solution. This is also a contradiction. The proof of Theorem 3 is complete.

Remark 4. When $n = 1$, condition (39) becomes

$$(53) \qquad \int_{t_0}^{\infty} s\, Q(s) \int_{s}^{\infty} Q(u)\, du\, ds = \infty.$$

Clearly, condition (53) is weaken than $\int_{t_0}^{\infty} Q(s)\, ds = \infty.$ Therefore, our comparison result extends and improves that of Zhang and Gopalsamy [4].

Remark 5. It would not be difficult to extend our comparison result to nonlinear equations.

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