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A NOTE ON UNSTABLE NEUTRAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

ABSTRACT: The aim of this paper is to improve an existing sufficient conditions for all bounded solutions of the second order neutral differential equation

$$(x(t)-px(t-r))''-q(t)x(\sigma(t))=0$$

to be oscillatory.

KEY WORDS: neutral equation, delayed argument.

We consider the second order neutral differential equation of the form:

$$(x(t)-px(t-r))''-q(t)x(\sigma(t))=0 \tag{1}$$

under the assumptions:

(i) p and τ are positive numbers;

(ii) $q,\sigma \in C(R_+,R_+)$, $\lim\limits_{t\to\infty}\sigma(t)=\infty$, $\sigma(t)\le t$;

(iii) σ is nondecreasing.

We put $z(t)=x(t)-px(t-\tau)$. By a proper solution of Eq. (1) we mean a function $x:[T_x,\infty)\to R$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)|: t\ge T\}>0$ for any $T\ge T_x$ so that $z(t)$ is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, many papers devoted differential equations with neutral terms has appeared. Many good results knowing for differential equations without neutral terms have been extended to neutral equations. The recent books by D.D. Bainov and D.P. Mishev [1], by I. Győri and G. Ladas [4], and by L.H. Erbe, Q. Kong and B.G. Zhang [3], numerize some important work in this area and reflect the overall new developments in the theory of neutral equations.

The purpose of this papaer is to improve the following sufficient conditions presented in [3, Theorem 4.6.1]:

Theorem A. Assume that (i)-(iii) *hold and* $0<p<1$.

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s-\sigma(t))q(s)ds > 1.$$

Then every bounded solution of Eq. (1) *is oscillatory.*

The objective of this paper is to provide such criterion which includes coefficient p explicitly. It is known that Eq. (1) always has un unbounded nonoscillatory solution (see e.q. [3]). Therefore really we only need to find conditions for all bounded solutions of (1) to be oscillatory.

Theorem 1. Assume that (i)-(iii) *hold and* $0 < p < 1$. *Let there exist an integer number* $n \geq 0$ *such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s-\sigma(t))q(s)ds > \frac{1-p}{1-p^{n+1}}. \tag{3}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Without loss of generality we may assume that $x(t)$ is an eventually positive bounded solution of Eq. (1). Define

$$z(t) = x(t) - px(t-\tau). \tag{4}$$

We have $z''(t) > 0$ for all large t, say $t \geq t_0$. If $z'(t) > 0$ eventually, then $\lim_{t\to\infty} z(t) = \infty$, which contradicts the boundedness of x. Therefore $z'(t) < 0$. We get two possibilities for $z(t)$:

(a) $z(t) > 0$ and $t \geq t_1 \geq t_0$,

(b) $z(t) < 0$ and $t \geq t_1$.

In case (a), Eq. (1) and be written in the form

$$z''(t) = q(t)x(\sigma(t)).$$

Using (4) we have

$$z''(t) = q(t)z(\sigma(t)) + pq(t)x(\sigma(t)-\tau).$$

Repeated this procedure we arrive at

$$z''(t) = q(t)\sum_{i=0}^{n} p^i z(\sigma(t)-i\tau) + p^{n+1}q(t)x(\sigma(t)-(n+1)\tau).$$

Therefore

$$z''(t) \geq q(t) \sum_{i=0}^{n} p^i z(\sigma(t) - i\tau).$$

For simplicity denote $\sum_{i=0}^{n} p^i = k$. Then using monotonicity of $z(t)$ one gets

$$z''(t) \geq k q(t) z(\sigma(t)). \tag{5}$$

An integration of (5) from s to t yields

$$z'(t) - z'(s) \geq \int_s^t k q(u) z(\sigma(u))\, du.$$

Then integrating in s from $\sigma(t)$ to t we see that

$$\begin{aligned} z'(t)(t - \sigma(t)) - z(t) + z(\sigma(t)) &\geq \int_{\sigma(t)}^{t} \int_s^t k q(u) z(\sigma(u))\, du\, ds \geq \\ &\geq \int_{\sigma(t)}^{t} k q(s)(s - \sigma(t)) z(\sigma(s))\, ds \geq \\ &\geq z(\sigma(t)) \int_{\sigma(t)}^{t} k q(s)(s - \sigma(t))\, ds. \end{aligned}$$

Hence for $t \geq t_1$ we have

$$z(t) + z(\sigma(t)) \left(k \int_{\sigma(t)}^{t} q(s)(s - \sigma(t))\, ds - 1 \right) \leq 0 \tag{6}$$

which contradicts (3).

For the case (b) we obtain

$$x(t) < p x(t - \tau) < p^2 x(t - 2\tau) < \ldots < p^n x(t - n\tau)$$

for $t \geq t_1 + n\tau$ and we are led to that $\lim_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$. This is a contradiction.

The conclusion of Theorem 1 can be strengthend as follows:

Theorem 2. Assume that (i)-(iii) *hold and* $0 < p < 1$. *Further assume that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s - \sigma(t)) q(s)\, ds > 1 - p. \tag{7}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Denote $a = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s-\sigma(t))\,q(s)\,ds$. Let integer n be chosen such that

$$a > \frac{1-p}{1-p^{n+1}}.$$

Then the assertion of this theorem follows immediately from Theorem 1.

Remark 1. It is easy to see that we have obtained better result than Theorem A provides. There is the constant p includes in our criterion.

Remark 2. Theorem 2 is also true for $p = 0$. This result is due to Koplatadze and Čanturia [2] (see also [5, Theorem 4.3.1]).

Example 1. Consider the following neutral differential equation

(8) $$(x(t) - p\,x(t-\tau))'' - (1/t^2)\,x(\lambda t) = 0, \quad p \in (0,1), \quad \lambda \in (0,1), \quad \tau > 0.$$

Condition (7) for Eq. (8) reduces to

(9) $$\ln(1/\lambda) + \lambda \geq 2 - p$$

and so for example for $p = 1/2$ and $\lambda = 1/4$ condition (8) is fulfilled and therefore all bounded solutions of Eq.(7) are oscillatory. On the other hand criterion (2) fails.

Theorem 3. Assume that (i)-(iii) *hold and* $p = 1$. *Let there exist an integer number* $n > 0$ *such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s-\sigma(t))\,q(s)\,ds > \frac{1}{n}.$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. Assume that $x(t)$ is an eventually positive bounded solution of Eq. (1). We can proceed exactly as in the proof of Theorem 1 to see that there are two possibilities for $z(t)$:

(a) . $z(t) > 0, \quad z'(t) < 0, \quad z''(t) > 0$ for $t \geq t_1 \geq t_0$,

(b) $z(t) < 0, \quad z'(t) < 0, \quad z''(t) > 0$ for $t \geq t_1$.

In case (a) we are led to (6) with constant $k = n$, which contradicts the assumptions.

In the case (b) we have $\lim_{t\to\infty} z(t) = -\alpha$, where $\alpha > 0$ is a finite number. So there exists $t_2 \geq t_1$ such that $-\alpha < z(t) < -\alpha/2$, $t \geq t_2$. Thus

$$-\alpha < x(t) - x(t-\tau) < -\alpha/2, \quad t \geq t_2.$$

Consequently

$$x(t) < -(\alpha/2) + x(t-\tau) < -2(\alpha/2) + x(t-2\tau) < \cdots < -n(\alpha/2) + x(t-n\tau)$$

for $t \geq t_2 + n\tau$. Chose a sequence $\{t_n\}$ such that $t_n = t_2 + n\tau$. Then

$$x(t_2 + n\tau) < -n(\alpha/2) + x(t_2)$$

and therefore $\lim_{n\to\infty} x(t_n) = -\infty$. This is a contradiction with boundedness of $x(t)$.

Theorem 4. Assume that (i)-(iii) *fold and* $p = 1$. *Further assume that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s - \sigma(t))\, q(s)\, ds > 0. \tag{10}$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Proof. The proof is similar to the proof of Theorem 2 and so it can be omitted.

Combining our previous results we have

Corollary 1. Assume that (i)-(iii) *hold and* $0 \leq p \leq 1$. *Further assume that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s - \sigma(t))\, q(s)\, ds > 1 - p.$$

Then every bounded solution of Eq. (1) *is oscillatory.*

Our previous results can be easily extended to mor general neutral differential equation with variable coefficient p. In the sequel we consider the second order neutral differential equation

$$(x(t) - p(t)x(t-\tau))'' - q(t)x(\sigma(t)) = 0. \tag{11}$$

Theorem 5. Assume that (ii)-(iii) *hold. Assume that eventually*

$$0 < p_1 \leq p(t) \leq p_2 < 1.$$

Further assume that

$$\text{(12)} \qquad \limsup_{t\to\infty} \int_{\sigma(t)}^{t} (s-\sigma(t))\, q(s)\, ds > 1 - p_1 .$$

Then every bounded solution of Eq. (11) *is oscillatory.*

Proof. Let $x(t)$ be a bounded eventually positive solution of (11). Set

$$z(t) = x(t) - p(t)\, x(t-\tau),$$

then $z''(t) > 0$ for all large t. Proceeding as in the proof of Theorem 1 we can see that $z'(t) > 0$. Then again there are two possibilities for $z(t)$:

(a) $z(t) > 0$ for $t \geq t_1 \geq t_0$,

(b) $z(t) < 0$ for $t \geq t_1$.

Proceeding as in the proof of Theorem 1 we can show that condition (12) eliminates case (a) and case (b) is inaccessible with respect to condition $p(t) \leq p_2 < 1$.

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