FASCICULI MATHEMATICI

Nr 29 1999

MARIA GÓRZEŃSKA, MARIA LEŚNIEWICZ, CZESŁAW PRĘTKI

APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES IN EXPONENTIAL WEIGHTED SPACES

ABSTRACT: In this note we define some operators $L_{\tilde{n}}$ and $U_{\tilde{n}}$ of the Szasz-Mirakjan type in exponential weighted spaces of functions of several variables. In Sec. 2 we give some basic properties of these operators. The main theorems are given in Sec. 3.

The similar results for functions belonging to polynomial weighted spaces are given in [3]. Some properties of these operators for functions of one variable with exponential weighted spaces are given in [4].

KEY WORDS: linear positive operator, approximation theorem, function of several variables.

1. NOTATION

1.1. Let $N^1 \equiv N := \{1,2,...\}$, $N_0^1 \equiv N_0 := N \cup \{0\}$, $R^1 = R := (-\infty,+\infty)$, $R_+^1 \equiv R_+ := (0,+\infty)$, $R_0^1 \equiv R_0 := R_+ \cup \{0\}$ and, for every fixed $m \in N$, let $N^m := \{\tilde{n} = (n_1,...,n_m) : n_k \in N \text{ for } 1 \le k \le m\}$. Analogously are defined N_0^m, R^m, R_+^m and R_0^m. For $\tilde{x},\tilde{y} \in R_0^m$, $\tilde{x} = (x_1,...,x_m)$, $\tilde{y} = (y_1,...,y_m)$, and $\lambda \in R$ we define: $\tilde{x} + \tilde{y} := (x_1 + y_1,...,x_m + y_m)$, $\lambda\tilde{x} := (\lambda x_1,...,\lambda x_m)$, $\tilde{x} - \tilde{y} := (x_1 - y_1,...,x_m - y_m)$, $\tilde{x}/\tilde{y} := (x_1/y_1,...,x_m/y_m)$ if $\tilde{y} \in R_+^m$, $\tilde{x} < \tilde{y}$ if and only if $x_k < y_k$ for $1 \le k \le m$ (analogously $\tilde{x} \le \tilde{y}$) and $\tilde{\tilde{\lambda}} \in R^m$ if $\lambda_k = \lambda$ for $1 \le k \le m$. Moreover let $\int_{\tilde{x}}^{\tilde{y}} f(\tilde{t})d\tilde{t} := \int_{x_1}^{y_1} ... \int_{x_m}^{y_m} f(t_1,...,t_m)dt_1 dt_2...dt_m$ and, for $\tilde{k} \in N_0^m$, let $\sum_{\tilde{k} \ge \tilde{\tilde{0}}} := \sum_{k_1=0}^{\infty} ... \sum_{k_m=0}^{\infty}$ and $\tilde{k} \to \tilde{\infty}$ if and only if $k_j \to +\infty$ for $1 \le j \le m$.

1.2. Let for a fixed $p \in R_+$

$$(1) \qquad v_p(x) := e^{-px}, \qquad x \in R_0,$$

and let for a fixed $\tilde{p} \in R_+^m$ (with some $m \in N$)

$$(2) \qquad v_{\tilde{p}}(\tilde{x}) := \prod_{j=1}^{m} v_{p_j}(x_j), \qquad \tilde{x} \in R_0^m.$$

For a fixed $\tilde{p} \in R_+^m$ we define the exponential weighted space $C_{\tilde{p},m}$ of all real–valued functions f defined on R_0^m for which $v_{\tilde{p}}(\cdot)f(\cdot)$ is uniformly continuous and bounded on R_0^m and the norm is defined by the formula

$$(3) \qquad \|f\|_{\tilde{p}} := \sup_{\tilde{x} \in R_0^m} v_{\tilde{p}}(\tilde{x})|f(\tilde{x})|.$$

For $f \in C_{\tilde{p},m}$ we define the modulus of continuity

$$\omega(f, C_{\tilde{p},m}; \tilde{t}) := \sup_{\tilde{0} \le \tilde{h} \le \tilde{t}} \|\Delta_{\tilde{h}} f(\cdot)\|_{\tilde{p}}, \qquad \tilde{h}, \tilde{t} \in R_0^m,$$

where $\Delta_{\tilde{h}} f(\tilde{x}) := f(\tilde{x} + \tilde{h}) - f(\tilde{x})$ for $\tilde{h}, \tilde{x} \in R_0^m$. Next, for a fixed $\tilde{\alpha} \in R_+^m$ and $\tilde{\tilde{0}} < \tilde{\alpha} \le \tilde{\tilde{1}}$, we define the class $\mathrm{Lip}(C_{\tilde{p},m}; \tilde{\alpha})$ of all functions $f \in C_{\tilde{p},m}$ for which

$$\omega(f, C_{\tilde{p},m}; \tilde{t}) = 0(t_1^{\alpha_1} + t_2^{\alpha_2} + \ldots + t_m^{\alpha_m})$$

as $t_k \to 0_+$ for $0 \le k \le m$.

1.3. In the papers [2] – [4] were considered the following operators L_n and U_n for $f \in C_{p,1}$

$$(4) \qquad L_n(f; x) := \sum_{j=0}^{\infty} a_{n,j}(x) f\left(\frac{2j}{n}\right),$$

$$(5) \qquad U_n(f; x) := \sum_{j=0}^{\infty} a_{n,j}(x) \, \frac{n}{2} \int_{\frac{2j}{n}}^{\frac{2j+2}{n}} f(t)dt,$$

$n \in N$ and $x \in R_0$, where

$$(6) \qquad a_{n,j}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2j}}{(2j)!}, \qquad j \in N_0,$$

and $\cosh x$, $\sinh x$, $\tanh x$ are the elementary hyperbolic functions.

In [4] was proved that L_n and U_n, $n \in N$, are a linear positive operators from the space $C_{p,1}$ into $C_{q,1}$ for every $q > p > 0$, provided that $n > p(\ln q/p)^{-1}$. Moreover some approximation properties of these operators are given in [2] and [4].

1.4. In this note we introduce the operators $L_{\widetilde{n}}^{[i]}$, $\widetilde{n} \in N^m$, $i = 1,2$, in the space $C_{\widetilde{p},m}$ with some $m \in N$ and $\widetilde{p} \in R_+^m$. For $\widetilde{n} \in N^m$, $\widetilde{k} \in N_0^m$ and $\widetilde{x} \in R_0^m$ we set

(7) $$A_{\widetilde{n},\widetilde{k}}(\widetilde{x}) := \prod_{j=1}^{m} a_{n_j,k_j}(x_j),$$

(8) $$B_{\widetilde{n},\widetilde{k}}(\widetilde{x}) := \prod_{j=1}^{m} \frac{n_j}{2} a_{n_j,k_j}(x_j),$$

where $a_{n_j,k_j}(x_j)$ is defined by (6). Next, for $f \in C_{\widetilde{p},m}$, $\widetilde{n} \in N^m$ and $\widetilde{x} \in R_0^m$, we define the operators $L_{\widetilde{n}}^{\{i\}}$, $i = 1,2$, as follows

(9) $$L_{\widetilde{n}}^{\{1\}}(f;\widetilde{x}) := \sum_{\widetilde{k} \geq \widetilde{\widetilde{0}}} A_{\widetilde{n},\widetilde{k}}(\widetilde{x}) f\left(\frac{2\widetilde{k}}{\widetilde{n}}\right),$$

(10) $$L_{\widetilde{n}}^{\{2\}}(f;\widetilde{x}) := \sum_{\widetilde{k} \geq \widetilde{\widetilde{0}}} B_{\widetilde{n},\widetilde{k}}(\widetilde{x}) \int_{\frac{2\widetilde{k}}{\widetilde{n}}}^{\frac{2\widetilde{k}+\widetilde{\widetilde{2}}}{\widetilde{n}}} f(\widetilde{t})d\widetilde{t}.$$

From (6) – (10) we deduce that $L_{\widetilde{n}}^{\{i\}}$, $\widetilde{n} \in N^m$, $i = 1,2$, is well–defined on every $C_{\widetilde{p},m}$ and $L_{\widetilde{n}}^{\{i\}}$ is linear positive operator.

Since $\sum_{j=0}^{\infty} a_{n,k}(x) = 1$ for all $x \in R_0$ and $n \in N$, we have by (6) – (10)

(11) $$L_{\widetilde{n}}^{\{i\}}(1;\widetilde{x}) = 1 \quad \text{for all } \widetilde{x} \in R_0^m,\ \widetilde{n} \in N^m,\ i = 1,2.$$

Next, we observe that if $f(\widetilde{x}) = f_1(x_1) \cdot f_2(x_2) \cdot ... \cdot f_m(x_m)$ for $\widetilde{x} \in R_0^m$ and $f_k(x_k) \in C_{p_k,1}$, $1 \leq k \leq m$, with some $p_k \in R_+$, then $f \in C_{\widetilde{p},m}$ with $\widetilde{p} = (p_1,...,p_m)$ and moreover for all $\widetilde{x} \in R_0^m$ and $\widetilde{n} \in N^m$ holds

(12) $$L_{\widetilde{n}}^{\{1\}}(f;\widetilde{x}) = \prod_{j=1}^{m} L_{n_j}(f_j;x_j),$$

(13) $$L_{\widetilde{n}}^{\{2\}}(f;\widetilde{x}) = \prod_{j=1}^{m} U_{n_j}(f_j;x_j).$$

In this paper we shall denote by $M_{p,q}$ the suitable positive constants depending only on indicated parameters p, q.

2. AUXILIARY RESULTS

2.1. First we shall give some basic properties of the operators L_n and U_n defined by (4) – (6). In [2], [4] and [5] are proved the following lemmas:

Lemma 1 ([2]). *For every $x \in R_0$ and $n \in N$ we have*

$$L_n(1;x) = 1 = U_n(1;x),$$

$$L_n((t-x)^2;x) \le \frac{3(x+1)}{n},$$

$$U_n((t-x)^2;x) \le \frac{19x+1}{4n}. \qquad \blacksquare$$

Lemma 2 ([4]). *For every $q > p > 0$ there exist positive constant $M_{p,q}$ and a natural number $n_0 > p(\ln q/p)^{-1}$ such that for all $x \in R_0$ and $n > n_0$*

$$\left.\begin{array}{l} \left\| L_n\left(\frac{1}{v_p(t)};\cdot\right)\right\|_q \\ \left\| U_n\left(\frac{1}{v_p(t)};\cdot\right)\right\|_q \end{array}\right\} \le M_{p,q},$$

$$\left.\begin{array}{l} v_q(x)\, L_n\left(\frac{(t-x)^2}{v_p(t)};x\right) \\ v_q(x)\, U_n\left(\frac{(t-x)^2}{v_p(t)};x\right) \end{array}\right\} \le M_{p,q}\frac{x+1}{n}. \qquad \blacksquare$$

Lemma 3 ([5]). *For every fixed $s \in N_0$ and $q > p > 0$ there exists positive constant $M_{p,q,s}$ and natural number n_0, $n_0 > p(\ln q/p)^{-1}$, such that for all $n > n_0$ holds*

$$\sup_{x\in R_0} v_q(x)\sum_{j=0}^{\infty}\left|\frac{d^s}{dx^s}a_{n,j}(x)\right|\frac{1}{v_p\left(\frac{2k}{n}\right)}\leq M_{p,q,s}n^s. \qquad \blacksquare$$

Using Lemma 1 and Lemma 2, we shall prove

Lemma 4. Let p,q *be fixed numbers and* $q>p$. *Then there exist a positive constant* $M_{p,q}$ *and natural number* $n_0>p(\ln q/p)^{-1}$ *such that for every* $x\in R_0$ *and* $n>n_0$

$$\left.\begin{array}{l} v_q(x)\,L_n\left(\left|\int\limits_x^t \frac{du}{v_p(u)}\right|;x\right) \\ v_q(x)\,U_n\left(\left|\int\limits_x^t \frac{du}{v_p(u)}\right|;x\right)\end{array}\right\}\leq M_{p,q}\sqrt{\frac{x+1}{n}}.$$

Proof. We shall prove only the above inequality for L_n because the proof for U_n is analogous. As in [4] by (1) we have

$$\left|\int\limits_x^t \frac{1}{v_p(u)}du\right|\leq\left(\frac{1}{v_p(t)}+\frac{1}{v_p(x)}\right)|t-x|, \qquad x,t\in R_0.$$

Hence for every $x\in R_0$, $n\in N$ and $q>p$ we get

$$v_q(x)\,L_n\left(\left|\int\limits_x^t \frac{du}{v_p(u)}\right|;x\right)\leq v_q(x)\,L_n\left(\frac{|t-x|}{v_p(t)};x\right)+L_n(|t-x|;x).$$

Using the Hölder inequality and Lemma 1 and Lamma 2 we obtain

$$L_n(|t-x|;x)\leq\{L_n((t-x)^2;x)\}^{1/2}\{L_n(1;x)\}^{1/2}\leq\sqrt{\frac{3(x+1)}{n}},$$

$$v_q(x)L_n\left(\frac{|t-x|}{v_p(t)};x\right)\leq v_q\left\{L_n\left(\frac{(t-x)^2}{v_p(t)};x\right)\right\}^{1/2}\left\{L_n\left(\frac{1}{v_p(t)};x\right)\right\}^{1/2}\leq M_{p,q}\sqrt{\frac{x+1}{n}},$$

for every $x \geq 0$ and $n > n_0$. where n_0 is given in Lemma 2. Summing up, we obtain the desired inequality for L_n. ■

2.2. Applying the above lemmas, we shall prove two lemmas on the operators $L_{\tilde{n}}^{\{i\}}$.

Lemma 5. For every fixed $m \in N$, $\tilde{p}, \tilde{q} \in R_+^m$ *and* $\tilde{q} > \tilde{p}$ *there exist positive constant* $M_{\tilde{p},\tilde{q}}$ *and* $\tilde{n}^*=(n_1^*,\dots,n_m^*) \in N^m$ *satisfying the condition*

$$n_j^* > p_j\left(\ln \frac{q_j}{p_j}\right)^{-1} \quad \text{for} \quad 1 \leq j \leq m, \tag{14}$$

such that for all $\tilde{n} > \tilde{n}^*$, $\tilde{n} \in N^m$, *and* $i = 1,2$ *we have*

$$\left\| L_{\tilde{n}}^{\{i\}}\left(\frac{1}{v_{\tilde{p}}(\tilde{t})};\cdot\right)\right\|_{\tilde{q}} \leq M_{\tilde{p},\tilde{q}}. \tag{15}$$

Proof. By (1) – (12) we have for $\tilde{x} \in R_0^m$ and $n \in N^m$

$$v_{\tilde{q}}(\tilde{x})\,L_{\tilde{n}}^{\{1\}}\left(\frac{1}{v_{\tilde{p}}(\tilde{t})};\tilde{x}\right) = \prod_{j=1}^{m} v_{q_j}(x_j)\,L_{n_j}\left(\frac{1}{v_{p_j}(t_j)};x_j\right),$$

$$v_{\tilde{q}}(\tilde{x})\,L_{\tilde{n}}^{\{2\}}\left(\frac{1}{v_{\tilde{p}}(\tilde{t})};\tilde{x}\right) = \prod_{j=1}^{m} v_{q_j}(x_j)\,U_{n_j}\left(\frac{1}{v_{p_j}(t_j)};x_j\right).$$

From this and by Lemma 2 we immediately obtain the desired assertion (15). ■

Lemma 6. Let $f \in C_{\tilde{p},m}$ *with some* $m \in N$, $\tilde{p} \in R_+^m$ *and let* $\tilde{q} \in R_+^m$ *and* $\tilde{q} > \tilde{p}$. *Then there exist positive constant* $M_{\tilde{p},\tilde{q}}$ *and* $\tilde{n}^* \in N^m$ *satisfying the condition (14) such that for all* $\tilde{n} > \tilde{n}^*$, $\tilde{n} \in N^m$, *and* $i = 1,2$

$$\left\| L_{\tilde{n}}^{\{i\}}(f;\cdot)\right\|_{\tilde{q}} \leq M_{\tilde{p},\tilde{q}} \|f\|_{\tilde{p}}. \tag{16}$$

This inequality and (6) – (10) *show that* $L_{\tilde{n}}^{\{i\}}$, $\tilde{n} \in N^m$, $i = 1,2$, *is a linear positive operator from the space* $C_{\tilde{p},m}$ *into* $C_{\tilde{q},m}$ *with* $\tilde{q} > \tilde{p}$, *provided that* $\tilde{n} > \tilde{n}^*$.

Proof. From (1) – (10) we get for $\tilde{x} \in R_0^m$, $\tilde{n} \in N^m$ and $i = 1,2$

$$v_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}(f;\tilde{x}) \right| \leq \|f\|_{\tilde{p}} \, v_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}\left(\frac{1}{v_{\tilde{p}}(\tilde{t})}; \tilde{x} \right) \right|,$$

which by (3) and Lemma 5 implies (16) for $\tilde{n} > \tilde{n}^*$ and $i = 1,2$. ■

3. The main theorems

3.1. In this part we shall prove two theorems on the degree of approximation of functions $f \in C_{\tilde{p},m}$ by $L_{\tilde{n}}^{\{i\}}$. For a fixed $m \in N$ and $\tilde{p} \in R_+^m$ we define the space

$$C_{\tilde{p},m}^1 := \left\{ f \in C_{\tilde{p},m} : \frac{\partial f}{\partial x_j} \in C_{\tilde{p},m} \quad \text{for} \quad 1 \leq k \leq m \right\}.$$

Theorem 1. Suppose that $f \in C_{\tilde{p},m}^1$ *with some fixed* $m \in N$ *and* $\tilde{p} \in R_+^m$. *Then for every fixed* $\tilde{q} \in R_+^m$ *and* $\tilde{q} > \tilde{p}$ *there exist positive constant* $M_{\tilde{p},\tilde{q}}$ *and* $\tilde{n}^* \in N^m$ *satisfying the condition (14) such that for all* $\tilde{x} \in R_0^m$, $\tilde{n} > \tilde{n}^*$ $(\tilde{n} \in N)$ *and* $i = 1,2$ *we have*

$$w_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}(f;\tilde{x}) - f(\tilde{x}) \right| \leq M_{\tilde{p},\tilde{q}} \sum_{j=1}^{m} \left\| \frac{\partial f}{\partial x_j} \right\|_{\tilde{p}} \sqrt{\frac{x_j + 1}{n_j}}. \tag{17}$$

Proof. Let $\tilde{x} = (x_1, \ldots, x_m)$ be a fixed point in R_0^m. Then by our assumption we can write for every $\tilde{t} = (t_1, \ldots, t_m) \in R_0^m$

$$f(\tilde{t}) - f(\tilde{x}) = \sum_{k=1}^{m} \int_{x_k}^{t_k} \frac{\partial}{\partial u_k} f(\tilde{y}_k) du_k,$$

where $\tilde{y}_k = (x_1, \ldots, x_{k-1}, u_k, t_{k+1}, \ldots, t_m)$. From this and by (11), for $\tilde{n} \in N^m$ and $i = 1,2$, follows

$$L_{\widetilde{n}}^{\{i\}}(f(\widetilde{t});\widetilde{x})-f(\widetilde{x})=\sum_{k=1}^{m}L_{\widetilde{n}}^{\{i\}}\left(\int_{x_k}^{t_k}\frac{\partial}{\partial u_k}f(\widetilde{y}_k)\,du_k\,;\widetilde{x}\right)$$

and consequently for $\widetilde{q}>\widetilde{p}$

$$v_{\widetilde{q}}(\widetilde{x})\left|L_{\widetilde{n}}^{\{i\}}(f(\widetilde{t});\widetilde{x})-f(\widetilde{x})\right|\le\sum_{k=1}^{m}v_{\widetilde{q}}(\widetilde{x})\,L_{\widetilde{n}}^{\{i\}}\left(\left|\int_{x_k}^{t_k}\frac{\partial}{\partial u_k}f(\widetilde{y}_k)\,du_k\right|;\widetilde{x}\right).$$

But by (1) – (3) we have

$$\left|\int_{x_k}^{t_k}\frac{\partial}{\partial u_k}f(\widetilde{y}_k)\,du_k\right|\le\left\|\frac{\partial f}{\partial x_k}\right\|_{\widetilde{p}}\left|\int_{x_k}^{t_k}\frac{du_k}{v_{\widetilde{p}}(\widetilde{y}_k)}\right|=$$

$$=\left\|\frac{\partial f}{\partial x_k}\right\|_{\widetilde{p}}\left(\prod_{j=1}^{k-1}\frac{1}{v_{p_j}(x_j)}\right)\left(\prod_{j=k+1}^{m}\frac{1}{v_{p_j}(t_j)}\right)\left|\int_{x_k}^{t_k}\frac{du_k}{v_{p_k}(u_k)}\right|\quad\text{for}\quad 2\le k\le m-1$$

and

$$\left|\int_{x_1}^{t_1}\frac{\partial}{\partial u_1}f(\widetilde{y}_1)\,du_1\right|\le\left\|\frac{\partial f}{\partial x_1}\right\|_{\widetilde{p}}\left(\prod_{j=2}^{m}\frac{1}{v_{p_j}(t_j)}\right)\left|\int_{x_1}^{t_1}\frac{du_1}{v_{p_1}(\widetilde{y}u_1)}\right|,$$

$$\left|\int_{x_m}^{t_m}\frac{\partial}{\partial u_m}f(\widetilde{y}_m)du_m\right|\le\left\|\frac{\partial f}{\partial x_m}\right\|_{\widetilde{p}}\left(\prod_{j=1}^{m-1}\frac{1}{v_{p_j}(x_j)}\right)\left|\int_{x_m}^{t_m}\frac{du_m}{v_{p_m}(u_m)}\right|.$$

Hence, using (1), (2), (12), Lemma 2 and Lemma 4, we get for $i=1$

$$v_{\widetilde{q}}(\widetilde{x})\,L_{\widetilde{n}}^{\{1\}}\left(\left|\int_{x_k}^{t_k}\frac{\partial}{\partial u_m}f(\widetilde{y}_m)du_m\right|;\widetilde{x}\right)\le$$

$$\le\left\|\frac{\partial f}{\partial x_m}\right\|_{\widetilde{p}}\left(\prod_{j=1}^{m-1}L_{n_j}(1;x_j)\right)v_{q_m}(x_m)\,L_{n_m}\left(\left|\int_{x_m}^{t_m}\frac{du_m}{v_{p_m}(u_m)}\right|;x_m\right)\le$$

$$\le M_{\widetilde{p},\widetilde{q}}\left\|\frac{\partial f}{\partial x_m}\right\|_{p}\sqrt{\frac{x_m+1}{n_m}}\qquad\text{for}\qquad n_m>n_m^*,$$

$$v_{\widetilde{q}}(\widetilde{x})\,L_{\widetilde{n}}^{\{1\}}\left(\left|\int_{x_k}^{t_k}\frac{du_k}{v_{\widetilde{p}}(\widetilde{y}_k)}\right|;\widetilde{x}\right)\le$$

$$\leq \left\| \frac{\partial f}{\partial x_k} \right\|_{\widetilde{p}} \left\{ \prod_{j=k+1}^{m} v_{q_j}(x_j) L_{n_j}\left(\frac{1}{v_{p_j}(t_j)}; x_j \right) \right\} v_{q_k}(x_k) L_{n_k}\left(\left| \int_{x_k}^{t_k} \frac{du_k}{v_{p_k}(u_k)} \right|; x_k \right) \leq$$

$$\leq M_{\widetilde{p},\widetilde{q}} \left\| \frac{\partial f}{\partial x_k} \right\|_{\widetilde{p}} \sqrt{\frac{x_k+1}{n_k}} \qquad \text{for} \quad n_k > n_k^* \quad \text{and} \quad 1 \leq k \leq m-1.$$

The identical inequalities we get for $i = 2$, by (13), Lemma 2 and Lemma 4 for U_n.

Using the above inequalities to (18), we obtain the desired estimation (17). ■

Theorem 2. Suppose that $f \in C_{\widetilde{p},m}$ *with some* $m \in N$ *and* $\widetilde{p} \in R_+^m$. *Then for every fixed* $\widetilde{q} \in R_+^m$ *and* $\widetilde{q} > \widetilde{p}$ *there exist positive constant* $M_{\widetilde{p},\widetilde{q}}$ *and* $\widetilde{n}^* \in N^m$ *satisfying the condition (14) such that for all* $\widetilde{x} \in R_0^m$, $\widetilde{n} > \widetilde{n}^*$ *and* $\widetilde{n} \in N^m$, *and* $i = 1,2$ *holds*

$$v_{\widetilde{q}}(\widetilde{x}) \left| L_{\widetilde{n}}^{\{i\}}(f(\widetilde{t}); \widetilde{x}) - f(\widetilde{x}) \right| \leq M_{\widetilde{p},\widetilde{q}}\, \omega\left(f, C_{\widetilde{p},m}; \sqrt{\frac{x_1+1}{n_1}}, \ldots, \sqrt{\frac{x_m+1}{n_m}} \right). \tag{19}$$

Proof. Let as in [3] $f_{\widetilde{h}}$ be the Steklov mean of $f \in C_{\widetilde{p},m}$ defined by the formula

$$f_{\widetilde{h}}(\widetilde{x}) := \frac{1}{h_1 h_2 .. h_m} \int_{\widetilde{0}}^{\widetilde{h}} f(\widetilde{x} + \widetilde{u})\, d\widetilde{u}$$

for $\widetilde{h} \in R_+^m$ and $\widetilde{x} \in R_0^m$. Then

$$f_{\widetilde{h}}(\widetilde{x}) - f(\widetilde{x}) = \frac{1}{h_1 h_2 .. h_m} \int_{\widetilde{0}}^{\widetilde{h}} (f(\widetilde{x} + \widetilde{u}) - f(\widetilde{x}))\, d\widetilde{u}$$

and for $1 \leq k \leq m$

$$\frac{\partial}{\partial x_k} f_{\widetilde{h}}(\widetilde{x}) = \frac{1}{h_1 h_2 .. h_m} \int_0^{h_1} \ldots \int_0^{h_{k-1}} \int_0^{h_{k+1}} \ldots \int_0^{h_m} (f(\widetilde{x} + \widetilde{u}') - f(\widetilde{x} + \widetilde{u}''))\, d\widetilde{u}''$$

where $\widetilde{u}' := (u_1, \ldots, u_{k-1}, h_k, u_{k+1}, \ldots, u_m)$, $\widetilde{u}'' := (u_1, \ldots, u_{k-1}, 0, u_{k+1}, \ldots, u_m)$, $d\widetilde{u}'' = du_1 \ldots du_{k-1} du_{k+1} \ldots du_m$. From this we get for $\widetilde{h} \in R_+^m$

$$\| f_h - f \|_{\widetilde{p}} \leq \omega(f, C_{\widetilde{p},m}; \widetilde{h}), \tag{20}$$

(21) $$\left\| \frac{\partial f_{\tilde{h}}}{\partial x_k} \right\|_{\tilde{p}} \le 2h_k^{-1}\, \omega(f, C_{\tilde{p},m}; \tilde{h}), \qquad 1 \le k \le m,$$

which implies $f_{\tilde{h}} \in C^1_{\tilde{p},m}$. Hence we can write for every $\tilde{x} \in R_0^m$, $\tilde{n} \in N^m$, $\tilde{h} \in R_+^m$, $\tilde{q} > \tilde{p}$ and $i = 1,2$

(22) $$v_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}(f(\tilde{t}); \tilde{x}) - f(\tilde{x}) \right| \le v_{\tilde{q}}(\tilde{x}) \Big\{ \left| L_{\tilde{n}}^{\{i\}}\big(f(\tilde{t}) - f_{\tilde{h}}(\tilde{t}); \tilde{x}\big) \right| + \left| L_{\tilde{n}}^{\{i\}}\big(f_{\tilde{h}}(\tilde{t}); \tilde{x}\big) - f_{\tilde{h}}(\tilde{x}) \right| + \left| f_{\tilde{h}}(\tilde{x}) - f(\tilde{x}) \right| \Big\} := S_1 + S_2 + S_3.$$

Applying Lemma 6 and (20), we get for $\tilde{n} > \tilde{n}^*$ and $\tilde{h} \in R_+^m$

$$S_1 \le M_{\tilde{p},\tilde{q}} \left\| f - f_{\tilde{h}} \right\|_{\tilde{p}} \le M_{\tilde{p},\tilde{q}}\, \omega(f, C_{\tilde{p},m}; \tilde{h}),$$

and

$$S_3 \le \left\| f - f_{\tilde{h}} \right\|_{\tilde{p}} \le \omega(f, C_{\tilde{p},m}; \tilde{h}).$$

In view of Theorem 1 and (21) we have

$$S_2 \le M_{\tilde{p},\tilde{q}} \sum_{j=1}^{m} \left\| \frac{\partial f_{\tilde{h}}}{\partial x_j} \right\|_{\tilde{p}} \sqrt{\frac{x_j + 1}{n_j}} \le 2M_{\tilde{p},\tilde{q}}\, \omega(f, C_{\tilde{p},m}; \tilde{h}) \sum_{j=1}^{m} h_j^{-1} \sqrt{\frac{x_j + 1}{n_j}}$$

for $\tilde{n} > \tilde{n}^*$ and $\tilde{h} \in R_0^m$. Consequently we get from (22)

(23) $$v_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}(f(\tilde{t}); \tilde{x}) - f_{\tilde{h}}(\tilde{x}) \right| \le \le M_{\tilde{p},\tilde{q}}\, \omega(f, C_{\tilde{p},m}; h_1, \dots, h_m) \left\{ 1 + \sum_{j=1}^{m} h_j^{-1} \sqrt{\frac{x_j + 1}{n_j}} \right\},$$

for $\tilde{x} \in R_0^m$, $\tilde{n} > \tilde{n}^*$, $\tilde{h} \in R_0^m$ and $i = 1,2$. Now, for fixed $\tilde{x}$ and $\tilde{n}$, setting $\tilde{h} = (\sqrt{(x_1+1)/n_1}, \sqrt{(x_2+1)/n_2}, \dots, \sqrt{(x_m+1)/n_m})$ to (23), we obtain the desired estimation (19). Thus the proof is finished. ■

From Theorem 2 we derive the following two corollaries.

Corollary 1. Let $f \in C_{\tilde{p},m}$ *with some* $m \in N$ *and* $\tilde{p} \in R_+^m$. *Then for every* $\tilde{x} \in R^m$ *and* $i = 1,2$ *holds*

$$\lim_{\tilde{n} \to \tilde{\infty}} L_{\tilde{n}}^{\{i\}}(f; \tilde{x}) = f(\tilde{x}).$$

Corollary 2. Let $f \in \mathrm{Lip}(C_{\tilde{p},m}, \tilde{\alpha})$ *with some* $m \in N$, $\tilde{p} \in R_+^m$, $\tilde{\alpha} \in R_+^m$ *and* $\tilde{\alpha} \leq \tilde{\tilde{1}}$. *Then for every* $\tilde{q} \in R_+^m$, $\tilde{q} > \tilde{p}$, *there exist positive constant* $M_{\tilde{p},\tilde{q}}$ *and* $\tilde{n}^* \in N^m$ *as in Theorem 1 such that for all* $\tilde{x} \in R_0^m$, $\tilde{n} > \tilde{n}^*$ *and* $i = 1,2$ *holds*

$$v_{\tilde{q}}(\tilde{x}) \left| L_{\tilde{n}}^{\{i\}}(f;\tilde{x}) - f(\tilde{x}) \right| \leq M_{\tilde{p},\tilde{q}} \sum_{k=1}^{m} \left(\frac{x_k + 1}{n_k} \right)^{\frac{\alpha_k}{2}},$$

3.2. Finaly we shall prove the Berstein inequality for the operators $L_{\tilde{n}}^{\{i\}}$ (see. [5]).

Theorem 3. Suppose that $f \in C_{\tilde{p},m}$ *with some* $m \in N$ *and* $\tilde{p} > R_+^m$. *Then for every fixed* $\tilde{q} \in R_+^m$, $\tilde{q} > \tilde{p}$, *and* $\tilde{s} = (s_1, \ldots, s_m) \in N^m$ *there exist positive constant* $M_{\tilde{p},\tilde{q},\tilde{s}}$ *and* $\tilde{n}^* \in N^m$ *satisfying the condition (14) such that for all* $\tilde{n} > \tilde{n}^*$ *and* $i = 1,2$ *holds the following Berstein inequality*

$$\left\| \frac{\partial^{s_1+s_2+\ldots+s_m}}{\partial x_1^{s_1} \partial x_2^{s_2} \ldots \partial x_m^{s_m}} L_{\tilde{n}}^{\{i\}}(f;\tilde{x}) \right\|_{\tilde{q}} \leq M_{\tilde{p},\tilde{q},\tilde{s}} \left(\prod_{k=1}^{m} n_k^{s_k} \right) \| f \|_{\tilde{p}}. \tag{24}$$

Proof. Let $i = 1$. In view of (9), (7) and (2) we have for $\tilde{s} \in N_0^m$, $\tilde{x} \in R_0^m$ and $\tilde{n} \in N^m$

$$\left| \frac{\partial^{s_1+\ldots+s_m}}{\partial x_1^{s_1} \ldots \partial x_m^{s_m}} L_{\tilde{n}}^{\{1\}}(f;\tilde{x}) \right| \leq$$

$$\leq \sum_{k_1=0}^{\infty} \ldots \sum_{k_m=0}^{\infty} \left| a_{n_1,k_1}^{(s_1)}(x_1) \right| \ldots \left| a_{n_m,k_m}^{(s_m)}(x_m) \right| \left| f\left(\frac{2k_1}{n_1}, \ldots, \frac{2k_m}{n_m} \right) \right| \leq$$

$$\leq \| f \|_{\tilde{p}} \prod_{j=1}^{m} \left(\sum_{k_j=0}^{\infty} \left| \frac{d^{s_j}}{dx_j^{s_j}} a_{n_j,k_j}(x_j) \right| \frac{1}{v_{p_j}\left(\frac{2k_j}{n_j} \right)} \right),$$

which by (1) – (3) and Lemma 3 implies

$$\left\| \frac{\partial^{s_1+\ldots+s_m}}{\partial x_1^{s_1}\ldots\partial x_m^{s_m}} L_{\tilde{n}}^{\{1\}}(f;\tilde{x}) \right\|_{\tilde{q}} \le$$

$$\le \|f\|_{\tilde{p}} \prod_{j=1}^{m} \left(\sup_{x_j \in R_0} v_{q_j}(x_j) \sum_{k_j=0}^{\infty} \left| a_{n_j,k_j}^{(s_j)}(x_j) \right| \frac{1}{v_{p_j}\left(\frac{2k_j}{n_j}\right)} \right) \le$$

$$\le M_{\tilde{p},\tilde{q},\tilde{s}} \|f\|_{\tilde{p}} \prod_{j=1}^{m} n_j^{s_j} \qquad \text{for} \quad \tilde{n} > \tilde{n}^*.$$

The proof of (24) for $i = 2$ is similar.

REFERENCES

[1] M. Becker, D. Kucharski, R.J. Nessel, Global approximation theorems for the Szasz – Mirakjan operators in exponential weight spaces, In: Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1997) *Birkhäuser Verlag, Basel, ISNM*, 40(1978), 319-333.

[2] B. Firlej, L. Rempulska, Approximation of functions by some linear positive operators, Grant KBN, 61–255/DS/1994 (in Polish).

[3] B. Firlej, L. Rempulska, Approximation of functions of several variables by some operators of the Szasz–Mirakjan type, *Fasc. Math.* 27(1997), ...

[4] M. Leśniewicz, L. Rempulska, Approximation by some operators of the Szasz – Mirakjan type in exponential weight spaces, *Glasnik Matematički*, 32(52)(1997), 57-69.

[5] L. Rempulska, M. Skorupka, The Bernstein inequality for some operators of the Szasz–Mirakjan type, *Note di Matematica*, 14(2)(1994), 277-290.

[6] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, New York 1963.

(Institute of Mathematics, Poznan University of Technology, 60-965 Poznań, Poland)

Received on 19.02.1997 and, in revised form, on 25.06.1997.