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A. KANANTHAI

ON THE TEMPERED DISTRIBUTION RELATED TO THE ULTRA-HYPERBOLIC EQUATIONS

ABSTRACT: In this paper, the distribution $e^{\alpha t}\square^k\delta$, where α is a complex number, $t=(t_1,t_2,\dots,t_n)\in R^n$ and $\square^k$ is the n-dimensional ultra-hyperbolic operator iterated k-times, is considered. Some properties of $e^{\alpha t}\square^k\delta$ are studied. Moreover, $e^{\alpha t}\square^k\delta$ is used to solving of the equation of the ultra-hyperbolic type.

KEY WORDS: tempered distribution, ultra-hyperbolic operator, homogeneous distribution.

1. INTRODUCTION

Let R^n be the Euclidean n-dimensional space. The n-dimensional ultrahyperbolic operator $\square^k$ iterated k-times is defined by

$$\square^k=\left(\frac{\partial^2}{\partial t_1^2}+\frac{\partial^2}{\partial t_2^2}+\cdots+\frac{\partial^2}{\partial t_p^2}-\frac{\partial^2}{\partial t_{p+1}^2}-\frac{\partial^2}{\partial t_{p+2}^2}-\cdots-\frac{\partial^2}{\partial t_{p+q}^2}\right)^k, \tag{1}$$

where $p+q=n$ and k is a nonnegative integer.

Consider the linear differential equation of the form

$$\square^k u(t)=f(t), \tag{2}$$

where $u(t)$ and $f(t)$ are generalized functions of variable $t=(t_1,t_2,\dots,t_n)\in R^n$.

I.M. Gelfand and G.E. Shilov in 1964 ([3] p. 279-282) obtained the elementary solution of (2). Their solution has a very complicated form. After that S. E. Trione [8] showed that the generalized function $R_{2k}(t)$ defined below by (3) is the unique elementary solution of (2). Recently, M. Aquirre Tellez ([1] p. 147-149) proved that $R_{2k}(t)$ exists only for even n with odd p.

In this paper, we study the distribution $e^{\alpha t}\square^k\delta$ (α is a complex number) being an extension of the distribution $e^{\alpha t}\delta^k$ introduced by A. Kananthai in [4]. Moreover, $e^{\alpha t}\square^k\delta$ is connected with some notions from [5]. The distribution $e^{\alpha t}\square^k\delta$ has interesting properties and plays an important role in solving some equations of ultra-hyperbolic type. Moreover, as it is mentioned above, their solutions are related to $R_{2k}(t)$.

2. PRELIMINARIES

Definition 2.1. Let $t=(t_1,t_2,\dots,t_n)$ *be a point of the n-dimensional Euclidean space* R^n. *Denote*

$$V=t_1^2+t_2^2+\dots+t_p^2-t_{p+1}^2-t_{p+2}^2-\dots-t_{p+q}^2,\quad p+q=n$$

The set

$$\Gamma_+=\{t\in R_n : t_1>0 \text{ and } V>0\}$$

*is called an **interior of the forward cone.** By* $\overline{\Gamma}_+$ *we denote its closure. For any complex number* α, *define*

$$R_\alpha(t)=\begin{cases}\dfrac{V_{(\alpha-n)/2}}{K_n(\alpha)} & \text{for} \quad t\in\Gamma_+,\\ 0 & \text{for} \quad t\in R_n/\Gamma_+,\end{cases} \tag{3}$$

where $K_n(\alpha)$ *is given by the formula*

$$K_n(\alpha)=\frac{\pi^{(n-1)/2}\,\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{4}$$

The function $R_\alpha(t)$ was introduced by Nozaki ([6] p. 72).

It is well known that $R_\alpha(t)$ is an ordinary function if $\mathrm{Re}(\alpha)\ge n$ and it is a distribution of α if $\mathrm{Re}(\alpha)<n$. Let $\operatorname{supp} R_\alpha(t)$ denote the support of $R_\alpha(t)$. Suppose that $\operatorname{supp} R_\alpha(t)\subset\overline{\Gamma}_+$.

Lemma 2.1 $R_\alpha(t)$ *is a homogeneous distribution of order* $\alpha-n$. *In particular, it is a tempered distribution.*

Proof. Since $R_\alpha(t)$ satisfies the Euler equation

$$\sum_{i=1}^{n}\frac{\partial R_\alpha(t)}{\partial t_i}=(\alpha-n)R_\alpha(t),$$

we conclude that $R_\alpha(t)$ is a homogeneous distribution of order $\alpha-n$. W.F. Donoghue ([2], p. 154-155) proved that every homogeneous distribution is a tempered distribution.

Definition 2.2. A generalized function $u(t)$ *is called an elementary solution of the n-dimensional ultra-hyperbolic operator iterated k-times if* $u(t)$ *satisfies the equation* $\Box^k u(t) = \delta$, *where* $\Box^k$ *is defined by* (1).

Lemma 2.2. The function $u(t) = R_{2k}(t)$ *with* $\alpha = 2k$ *is the unique elementary solution of the equation* $\Box^k u(t) = \delta$.

The proof of this Lemma is given by S.E. Trione [8]. As it was mention above, M. Aquirre Tellez ([1] p. 147-149) proved that $R_{2k}(t)$ exists only for even n with odd p.

Lemma 2.3. If k is a nonnegative integer and α, β *are the positive even numbers with* $\alpha + \beta = 2k$, *then*

$$R_\alpha(t) * R_\beta(t) = R_{\alpha+\beta}(t).$$

Proof. By Lemma 2.1, $R_\alpha(t)$ is a tempered distribution. Let $\operatorname{supp} R_\alpha(t) = = K \subset \overline{\Gamma}_+$, where K is a compact set. Then $R_\alpha(t) * R_\beta(t)$ exists and it is well defined. By Lemma 2.2, $u(t) = R_{2k}(t)$ is a unique solution of $\Box^k u(t) = \delta$. We have

$$\Box^k u(t) = \Box^r \Box^{k-r} u(t) = \delta$$

for $r < k$. Then by Lemma 2.2,

$$\Box^{k-r} u(t) = R_{2r}(t).$$

Convolving both sides by $R_{2(k-r)}(t)$, we obtain

$$R_{2(k-r)}(t) * \Box^{k-r} u(t) = R_{2(k-r)}(t) * R_{2r}(t),$$

what is equivalent to

$$\Box^{k-r} R_{2(k-r)}(t) * u(t) = R_{2(k-r)}(t) * R_{2r}(t).$$

Using again Lemma 2.2 to the left hand side inequality, we get

$$\delta * u(t) = R_{2(k-r)}(t) * R_{2r}(t).$$

It follows that

$$u(t) = R_{2k-2r}(t) * R_{2r}(t).$$

Since $u(t)=R_{2k}(t)$, putting $\alpha=2k-2r$ and $\beta=2r$,, we obtain

$$R_\alpha(t)*R_\beta(t)=R_{\alpha+\beta}(t)$$

as desired.

3. PROPERTIES OF $e^{\alpha t}\Box^k\delta$

Lemma 3.1. The distribution $e^{\alpha t}\Box^k\delta$ *have the following properties:*

a) *We have*

$$(5)\qquad e^{\alpha t}\Box^k\delta=\Box\delta-2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\delta}{\partial t_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\delta}{\partial t_j}\right)+\left(\sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{p+q}\alpha_j^2\right)\delta$$

for $k=1$. *Moreover,* $e^{\alpha t}\Box^k\delta$ *is a tempered distribution of order 2 with support* $\{0\}$.

b) *Let* S *be a space of testing function of rapid descent and* S' *be a space of tempered distribution. For every* $\varphi\in S$ *and* $e^{\alpha t}\Box\delta\in S'$, *we have*

$$\left|\langle e^{\alpha t}\Box\delta,\varphi\rangle\right|\le CM,$$

where

$$(6)\qquad M=\max\left\{|\varphi(0)|,\left|\frac{\partial\varphi(0)}{\partial t_i}\right|,\left|\frac{\partial\varphi(0)}{\partial t_j}\right|,\ |\Box\varphi(0)|\right\},$$

and

$$(7)\qquad C=1+2\sum_{i=1}^{p}|\alpha_i|+2\sum_{j=p+1}^{p+q}|\alpha_j|+\sum_{i=1}^{p}\alpha_i^2+\sum_{j=p+1}^{p+q}\alpha_j^2.$$

Proof of a) Let D be the space of testing functions of infinitely differentiable with compact supports and D' be the space of distributions. Suppose that $\varphi(t)\in D'$. Then

$$\langle e^{\alpha t}\Box\delta,\varphi\rangle=\langle\delta,\Box e^{\alpha t}\varphi(t)\rangle$$

for $e^{\alpha t}\Box\delta\in D'$. We have

$$\Box e^{\alpha t}\varphi(t)=\sum_{i=1}^{p}\frac{\partial^2\delta}{\partial t_i^2}(e^{\alpha t}\varphi(t))-\sum_{j=p+1}^{p+q}\frac{\partial^2\delta}{\partial t_j^2}(e^{\alpha t}\varphi(t))=$$

$$= e^{\alpha t}\left(\sum_{i=1}^{p}\frac{\partial^2\varphi(t)}{\partial t_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2\varphi(t)}{\partial t_j^2}\right) + 2e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\varphi(t)}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\varphi(t)}{\partial t_j}\right) +$$

$$+ e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\varphi(t) =$$

$$= e^{\alpha t}\,\Box\,\varphi(t) + 2e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\varphi(t)}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\varphi(t)}{\partial t_j}\right) + e^{\alpha t}\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\varphi(t)$$

Moreover,

(8) $\langle\delta, \Box\, e^{\alpha t}\varphi(t)\rangle =$

$$= \Box\,\varphi(0) + 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\varphi(0)}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\varphi(0)}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\varphi(0) =$$

$$= \left\langle \delta + 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial\delta}{\partial t_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial\delta}{\partial t_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right), \varphi(t)\right\rangle.$$

By equality of distribution, we have immediatelly (5). Since δ, $\partial\delta/\partial t_i$, $\partial\delta/\partial t_j$ and $\Box\,\delta$ have support $\{0\}$, by (5), $e^{\alpha t}\,\Box\,\delta$ has support $\{0\}$. Hence, by L. Shwartz theorem (see [7]), $e^{\alpha t}\,\Box\,\delta$ is a tempered distribution. Since $e^{\alpha t}\,\Box\,\delta$ is a finite linear combination of Dirac-delta distributions and its derivatives up to order 2, by A.H. Zemanian theorem (see Theorem 3 in [9], p. 98), $e^{\alpha t}\,\Box\,\delta$ is pf order 2 with the point support $\{0\}$.

Proof of b). Since

$$\langle e^{\alpha t}\,\Box\,\delta, \varphi\rangle = \langle\delta, \Box\, e^{\alpha t}\varphi(t)\rangle,$$

by (8), we have

$$|\langle e^{\alpha t}\,\Box\,\delta,\varphi\rangle| \le |\Box\,\varphi(0)| + 2\sum_{i=1}^{p}|\alpha_i|\left|\frac{\partial\varphi(0)}{\partial t_i}\right| + \sum_{j=p+1}^{p+q}|\alpha_j|\left|\frac{\partial\varphi(0)}{\partial t_j}\right| +$$

$$+\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)|\varphi(0)|.$$

Taking into account M and C defined respectively by formulas (6) and (7), we get

$$|\langle e^{\alpha t}\,\Box\,\delta,\varphi\rangle|\le\left(1+2\sum_{i=1}^{p}|\alpha_i|+2\sum_{j=p+1}^{p+q}|\alpha_j|+\sum_{i=1}^{p}\alpha_i^2+\sum_{j=p+1}^{p+q}\alpha_j^2\right)M\le CM.$$

Lemma 3.2. If $u(t)$ is a tempered distribution, then

$$(e^{\alpha t}\,\Box\,\delta)*u(t)=\Box\,u(t)-2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial u(t)}{\partial t_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial u(t)}{\partial t_j}\right)+ \tag{9}$$

$$+\left(\sum_{i=1}^{p}\alpha_i^2+\sum_{j=p+1}^{p+q}\alpha_j^2\right)u(t).$$

Proof. Convolving both sides of (5) by $u(t)$, we get immediately (9).

4. Main results

Theorem 4.1. Let L be the partial differential operator defined by

$$L\equiv\Box\,-2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial t_i}-\sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial t_j}\right)+\left(\sum_{i=1}^{p}\alpha_i^2+\sum_{j=p+1}^{p+q}\alpha_j^2\right).$$

Then tha equation of ultra-hyperbolic type

$$Lu(t)=\delta \tag{10}$$

for $u\in S'$ has the unique elementary solution given by the formula

$$u(t)=e^{\alpha t}R_2(t),$$

where $R_2(t)$ is defined by (3) *with $\alpha=2$.*

Proof. Suppose that $u\in S'$ satisfies the equation (10). Taking into accout the definition of L, and (9), we can write

$$(e^{\alpha t}\,\Box\,\delta)*u(t)=Lu(t).$$

Hence, by (10), we have

$$(e^{\alpha t}\,\Box^k\,\delta)*u(t)=\delta.$$

Convolving both sides by $e^{\alpha t}R_2(t)$, we get

$$(e^{\alpha t}R_2(t))*((e^{\alpha t}\,\Box\,\delta)*u(t))=e^{\alpha t}R_2(t)*\delta=e^{\alpha t}R_2(t).$$

Hence, applying Lemma 2.2 with $k = 1$, we obtain

$$e^{\alpha t}R_2(t) = e^{\alpha t}(R_2(t) * \Box\,\delta) * u(t) =$$
$$= (e^{\alpha t}\Box R_2(t)) * u(t) = (e^{\alpha t}\delta) * u(t),$$

$$\delta * u(t) = u(t).$$

Therefore $u(t) = e^{\alpha t}R_2(t)$ is the unique elementary solution of the equation (10).

The next theorem is a generalization of Lemma 3.1.

Theorem 4.2. The distribution $e^{\alpha t}\Box^k\delta$ *is a linear combination of Dirac-delta and its partial derivative up to order 2k and* $(e^{\alpha t}\Box^k\,\delta) * u(t)$ *is the partial differential operator of order 2k defined on* S'.

Proof. We have

$$\langle e^{\alpha t}\Box^k\,\delta, \varphi(t)\rangle = \langle \delta, \Box^k\, e^{\alpha t}\varphi(t)\rangle.$$

Repeating k times this same procedure as in Lemm 3.1, we obtain that $e^{\alpha t}\Box^k\delta$ is a linear combination of Dirac-delta and its partial derivative up to order $2k$. Similarly as in Lemma 3.2, we get the second part of the thesis.

Theorem 4.3. The partial differential equation

$$(e^{\alpha t}\Box^k\,\delta) * u(t) = \delta \qquad (11)$$

for $u \in S'$ *has an elementary solution given by the formula*

$$u(t) = e^{\alpha t}R_{2k}(t),$$

where $R_{2k}(t)$ *is defined by* (3) *with* $\alpha = 2k$.

Proof. Convolving both sides of (11) by $e^{\alpha t}R_{2k}(t)$, we have

$$(e^{\alpha t}R_{2k}(t)) * (e^{\alpha t}\Box^k\,\delta) * u(t) = e^{\alpha t}R_{2k}(t) * \delta = e^{\alpha t}R_{2k}(t).$$

Consequently,

$$e^{\alpha t}(\Box^k\, R_{2k}(t)) * u(t) = e^{\alpha t}R_{2k}(t).$$

Since $\Box^k R_{2k}(t) = \delta$, by Lemma 2.2, we have

$$e^{\alpha t}R_{2k}(t) = (e^{\alpha t}\delta) * u(t) = \delta * u(t).$$

Hence the thesis follows immediately.

Corollary 4.1. The convolution

$$\underbrace{e^{\alpha t}R_2(t) * e^{\alpha t}R_2(t) * \cdots * e^{\alpha t}R_2(t)}_{k-times}$$

in an elementary solution of (11).

Proof. By Lemma 2.3

$$\underbrace{e^{\alpha t}R_2(t) * e^{\alpha t}R_2(t) * \cdots * e^{\alpha t}R_2(t)}_{k-times} =$$

$$= e^{\alpha t}\underbrace{\left(R_2(t) * R_2(t) * \cdots * R_2(t)\right)}_{k-times} = e^{\alpha t}R_{2k}(t).$$

Therefore, in view of Theorem 4.3,

$$\underbrace{e^{\alpha t}R_2(t) * e^{\alpha t}R_2(t) * \cdots * e^{\alpha t}R_2(t)}_{k-times}$$

is an elementary solution of (11).

REFERENCES

[1] M. Aguirre Tellez, The distributional Hankel transform of Marcel Riesz's ultra-huperbolic Kernel, *Studies in Applied Math.* 93(1994) 133-162, Massachusetts Inst. of Technology, Elsevier Sc. Inc.

[2] W.F. Donoghue, *Distributions and Fourier transform*, Academic Press 1969.

[3] I.M. Gelfand, G.E. Shilov, *Generalized functions*, Vol. 1, Academic Press 1964.

[4] A. Kananthai, *On the distribution* $e^{\alpha t}\square^k\delta$ *and its applications*, Proceedings in Annual Conference in Mathematics, 1996, King Mongkut's Institute of Technology, Ladkrabang, Thailand.

[5] A. Kananthai, *On the solutions of the n-Dimensional Diamond Operator*, to appear in Applied Mathemathics and Computation, Massachusetts Inst. of Technology, Elsevier Sc. Inc.

[6] Y. Nozaki, On Riemann-Liouville integral of ultra-hyperbolic type, *Kodai Mathematical Seminar Reports*, Vol 6.2 (1964), 69-87.

[7] L. Schwartz, *Theorie des distribution,* Vol. 1 and 2 Actualite's, Scientifiques Industrial, Paris, 1957, 1959.

[8] S.E. Trione, *On Marcel Riesz's ultra-hyperbolic Kernel*, Trabajos de Matematica 116 preprint, (1987).

[9] A.H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill 1965.

(Department of Mathematics, Chiangmai University Chiangmai, 50200 Thailand)

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