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APPROXIMATION IN ORLICZ-SLOBODECKII SPACE BY FUNCTIONS IN $C^\infty(\Omega)$

ABSTRACT: We wish to prove that $\{\varphi \in C^\infty(\Omega) : \|\varphi\|_{B^{k,M}} < \infty\}$ is dense in Orlicz-Slobodeckii space $B^{k,M}(\Omega)$ for some $\Omega \subset R^N$ of finite measure.

KEY WORDS: Orlicz space, Slobodeckii space, translation operator, distributional derivative.

1. PRELIMINARIES

Assume that Ω is a nonempty, open and convex set in N – dimensional real Euclidean space R^N. Let a function $M : \Omega \times [0,\infty) \to [0,\infty)$ satisfies the following conditions:

1. $M(t,0) = 0$ and $M(t,u) > 0$ for $u > 0$, and almost every $t \in \Omega$;
2. $M(t,\cdot)$ is convex and continuous at zero for almost every $t \in \Omega$;
3. $M(\cdot,u)$ is measurable for every $u \geq 0$.

The function M safisfies the condition Δ_2 if there exists a constant $K > 0$ such that $M(t,2u) \leq KM(t,u)$ for a.e. $t \in \Omega$ and every $u \geq 0$.

Denote by X the space of all real-valued and Lebesgue measurable functions defined on Ω, with equality almost everywhere on Ω.

Let k be an arbitrary positive, noninteger number and let $k = [k] + \lambda$, where $[k]$ denotes the integer part of k, $0 < \lambda < 1$. Then for any function M we define on X a functional I by

$$I(u) = \sum_{|\alpha| \leq [k]} \left(\int_\Omega M(x, |D^\alpha u(x)|)dx + \int_\Omega \int_\Omega M\left(\frac{x+y}{2}, \frac{|\Delta(x,y)D^\alpha u|}{|x-y|^\lambda} \right) \frac{dx\,dy}{|x-y|^N} \right),$$

where $\Delta(x,y)f = f(x) - f(y)$, $\alpha = (\alpha_1,\ldots,\alpha_N)$ is a multiindex with $\alpha_i \geq 0$, $D^\alpha u = \partial^{|\alpha|}u / \partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}$ is the distributional derivative of u, $|\alpha| = \sum_{i=1}^N \alpha_i$. The functional I is a convex modular on X.

Further, for any fixed k, we define

$$B^{k,M}(\Omega) = \{u \in X : I(\alpha u) < \infty \text{ for some } \alpha > 0\}.$$

The vector space $B^{k,M}(\Omega)$ is called the Orlicz-Slobodeckii space, [1]. The space $B^{k,M}(\Omega)$ with the Luxenburg norm $\|\cdot\|_{B^{k,M}}$ generated by the convex modular I is a Banach space, [1].

Let $G \subset R^N$. We shall write $G \subset\subset \Omega$ provided $\overline{G} \subset \Omega$ and $\overline{G}$ is a compact subset of R^N.

Denote:

$$B = \{(x,y) \in \Omega \times \Omega :\ x = y\}.$$

For any set A in the σ – algebra of Lebesgue measurable subsets of $\Omega \times \Omega$, $\Omega \subset R^N$, we define the nonegative measure ν by

$$\nu(A) = \iint_A |x-y|^{-N}\, dxdy \qquad \text{and} \qquad \nu(B) = 0.$$

The measure ν is separable σ – finite, [2]. Let $L^M(\Omega \times \Omega, \nu)$ be the Orlicz space of all real and measurable functions F defined on $\Omega \times \Omega$, generated by the modular

$$J(F) = \int_\Omega \int_\Omega M\left(\frac{x+y}{2}, |F(x,y)|\right) d\nu(x,y).$$

Let $\|\cdot\|_J$ be the Luxemburg norm in $L^M(\Omega \times \Omega, \nu)$ generated by J.

2. Results

For any measurable function F, defined on $\Omega \times \Omega$, we define the translation oparator τ_ν, $\nu \in \Omega$, by

$$(\tau_\nu F)(x,y) = \begin{cases} F(x+\nu, y+\nu) & \text{for } \ (x,y) \in [\Omega \cap (\Omega - \nu)] \times [\Omega \cap (\Omega - \nu)], \\ 0 & \text{elsewhere in } \ \Omega \times \Omega. \end{cases}$$

By τ we denote the family $\tau = (\tau_\nu)_{|\nu| \le \delta}$, $\delta > 0$, of translation operators.

The function M will be called τ – bounded, if there exist a constant $c > 0$ and a family nonnegative and measurable functions h_ν on $\Omega \times \Omega$, $\nu \in \Omega$, such that the inequality

$$(1) \qquad M\left(\frac{x+y}{2} - \nu, u\right) \le M\left(\frac{x+y}{2}, cu\right) + h_\nu(x,y)$$

holds for every $u \ge 0$, $|\nu| \le \delta$ and a.e. $(1/2)(x+y) \in \Omega \cap (\Omega + \nu)$, where

$$(2)\qquad h_v(x,y)\begin{cases} f_v(x,y) \text{ for } (x,y)\notin B \text{ and } \sup\limits_{|v|\le\delta}\int\limits_{\Omega}\int\limits_{\Omega} f_v(x,y)dv(x,y)<\infty, \\ g_v(x) \quad \text{for } (x,y)\in B \text{ and } \sup\limits_{|v|\le\delta}\int\limits_{\Omega} g_v(x)dx<\infty. \end{cases}$$

Let S_0 be the set of all simple function s of the form

$$\sum_{i=1}^{m} a_i \chi_{A_i}(x,y),$$

a_i being real numbers and A_i, $A_i \subset \Omega\times\Omega$ being bounded and $\operatorname{dist}(A_i,B)>0$ for $i=1,2,\ldots,m$.

Lemma 1. If $\int\limits_T M(t,u)dt<\infty$ *for every bounded set* T, $T\subset\Omega$, *and every* $u\ge 0$, *then* S_0 *is dense in modular sense in* $L^M(\Omega\times\Omega,v)$.

Proof. Let $F\in L^M(\Omega\times\Omega,v)$. Then $J(aF)<\infty$ for some $a>0$. Let (P_n) be the sequence of neighbourhoods of the B such that

$$(3)\qquad \iint\limits_{P_n} M\left(\frac{x+y}{2}, a\,|F(x,y)|\right)dv(x,y) < \frac{1}{n}.$$

Let $F\ge 0$ on $\Omega\times\Omega$. Then there exists the nondecreasing sequence of simple functions (F_n) such that $F_n(x,y)\to F(x,y)$ for every $(x,y)\in\Omega\times\Omega$.

Let us denote

$$\overline{F}_n(x,y)=\begin{cases} F_n(x,y) & \text{if } \ (x,y)\in(\Omega\times\Omega)_n, \\ 0 & \text{if } \ (x,y)\notin(\Omega\times\Omega)_n, \end{cases}$$

where $(\Omega\times\Omega)_n=\{(x,y)\in\Omega\times\Omega : |(x,y)|<n\}$. Then $\overline{F}_n(x,y)\to F(x,y)$, $n\to\infty$ for every $(x,y)\in\Omega\times\Omega$. We define a new suqence by the formula:

$$\widetilde{F}_n(x,y)=\begin{cases} \overline{F}_n(x,y) & \text{if } \ (x,y)\notin P_n, \\ 0 & \text{elsewhere in } \ \Omega\times\Omega. \end{cases}$$

Then $\widetilde{F}_n\in S_0$ for $n=1,2,\ldots$. By the dominated convergence theorem and (3) we obtain

$$J(a(\widetilde{F}_n-F))\le \int\limits_{\Omega}\int\limits_{\Omega} M\left(\frac{x+y}{2}, a\,|\overline{F}_n(x,y)-F(x,y)|\right)dv(x,y)+$$

$$+\iint\limits_{P_n} M\left(\frac{x+y}{2}, a\,|F(x,y)|\right)d\nu(x,y) \to 0 \quad \text{as} \qquad n\to\infty.$$

Let F be arbitrary. Then there exists the decomposition of F of the form $F = F_+ - F_-$ and we apply the above result to F_+ and F_-.

Lemma 2. Let $\int\limits_T M(t,u)dt < \infty$ *for every bounded set* T, $T \subset \Omega$, *and every* $u \geq 0$. *If* M *is* $\tau-$*bounded, then the family* $\tau = (\tau_\nu)_{|\nu|\leq\delta}$, $\delta > 0$, *of translation operators is bounded and* $\sup\limits_{|\nu|\leq\delta} \|\tau_\nu F\|_J \leq c\|F\|_J$. *If* M *satisfies the condition* Δ_2, *then* $\|\tau_\nu F - F\|_J \to 0$, *as* $\nu \to 0$ *for every* $F \in L^M(\Omega\times\Omega,\nu)$.

Proof. Boundedness of τ follows from the inequalities

$$J(\tau_\nu F)) = \int\limits_\Omega\int\limits_\Omega M\left(\frac{x+y}{2}, |F(x+\nu, y+\nu)\right)d\nu(x,y) \leq$$

$$\leq \int\limits_\Omega\int\limits_\Omega M\left(\frac{x+y}{2}, k\,|F(x,y)|\right)d\nu(x,y) +$$

$$+\int\limits_\Omega\int\limits_\Omega h_\nu(x,y)d\nu(x,y) \leq J(kF) + c.$$

Hence, for any function $F \in L^M(\Omega\times\Omega,\nu)$, we have

$$J\left(\frac{\tau_\nu F}{k\|F\|_J}\right) \leq J\left(\frac{1}{\|F\|_J}F\right) + c \leq 1 + c$$

for every $|\nu| \leq \delta$. Hence

$$\sup_{|\nu|\leq\delta} \|\tau_\nu F\|_J \leq (1+c)k\|F\|_J\,.$$

Now, we prowe that for any function $F \in L^M(\Omega\times\Omega,\nu)$, $\|\tau_\nu F - F\|_J$ tends to zero as $\nu \to 0$. Taking the function $G \in S_0$, $G(x,y) = \sum\limits_{i=1}^m a_i \chi_{A_i-\nu}(x,y)$ we have

$$(\tau_\nu G)(x,y) = \sum_{i=1}^m a_i \chi_{A_i}(x+\nu, y+\nu) = \sum_{i=1}^m a_i \chi_{A_i-\nu}(x,y).$$

Let ν be chosen in such manner, that $\operatorname{dist}(A_i - \nu, B) > 0$ for $i = 1,2,\ldots,m$. Then $\tau_\nu G \in S_0$ and we have

$$|(\tau_\nu G)(x,y)| - G(x,y) = \sum |a_i| \chi_{A_i \dot{-} (A_i - \nu)}(x,y).$$

Hence

$$J(\tau_\nu G - G) \le \frac{1}{m}\sum_{i=1}^{m} \iint\limits_{A_i \dot{-} (A_i - \nu)} M\left(\frac{x+y}{2}, m|a_i|\right) d\nu(x,y) \to 0 \quad \text{as} \quad \nu \to 0_+,$$

because $|A_i \dot{-} (A_i - \nu)| \to 0$ as $\nu \to 0_+$ for $i = 1,2,\ldots,m$. Here $|\cdot|$ denotes Lebesgue measure. Hence $\|\tau_\nu G - G\|_J \to 0$ as $\nu \to 0_+$.

Let $\varepsilon > 0$ be given and let $F \in L^M(\Omega \times \Omega, \nu)$. Applying Lemma 1, we conclude that there exists a sequence (F_n), $F_n \in S_0$, such that $\|F_n - F\|_J \to 0$ as $n \to \infty$. Choosing n_0 sufficiently large and writing $G = F_{n_0}$ we obtain

$$\|F - G\|_J \le \frac{\varepsilon}{2(1+c)}.$$

Applying boundedness of τ and the last inequality we have

$$\|\tau_\nu F - F\|_J \le \|\tau_\nu F - \tau_\nu G\|_J + \|\tau_\nu G - G\|_J + \|G - F\|_J \le$$

$$\le (1+C)\|F - G\|_J + \|\tau_\nu G - G\|_J < \varepsilon$$

for sufficiently small ν.

Corollary. For any function $u \in B^{k,M}(\Omega)$

$$\sup_{|\nu|\le\delta} \iint\limits_{\Omega\,\Omega} M\left(\frac{x+y}{2}, \frac{|\Delta(x,y)(D^\alpha u(\cdot+\nu) - D^\alpha u(\cdot))|}{|x-y|^\lambda}\right) \frac{dx\,dy}{|x-y|^N} \to 0 \ \text{ as } \delta \to 0.$$

Proof. Let $u \in B^{k,M}(\Omega)$. Then

$$F_\alpha(x,y) = \frac{D^\alpha u(x) - D^\alpha u(y)}{|x-y|^\lambda}$$

belongs to $L^M(\Omega \times \Omega, \nu)$ for every α, $|\alpha| \le [k]$. Let $0 < \varepsilon < 1$. Then $\|\tau_\nu F_\alpha - F_\alpha\|_J < \varepsilon$ for sufficiently small ν.
Hence

$$J\left(\frac{\tau_\nu F_\alpha - F_\alpha}{\varepsilon}\right) < 1 \qquad \text{for sufficiently small } \nu \text{ and}$$

$$I_v = \int_\Omega\int_\Omega M\left(\frac{x+y}{2}, \frac{|\Delta(x,y)(D^\alpha u(\cdot+v) - D^\alpha u(\cdot))}{|x-y|^\lambda}\right)\frac{dx\,dy}{|x-y|^N} \leq$$

$$\leq \varepsilon \int_\Omega\int_\Omega M\left(\frac{x+y}{2}, \frac{|\Delta(x,y)(D^\alpha u(\cdot+v) - D^\alpha u(\cdot))}{\varepsilon|x-y|^\lambda}\right)\frac{dx\,dy}{|x-y|^N} < \varepsilon$$

for sufficiently small v. Hence $\sup_{|v|\leq\delta} I_v \to 0$ as $v \to 0_+$.

Let ρ be a nonnegative, real-valued function belonging to $C_0^\infty(R^N)$ and having the properties

1. $\rho(x) = 0$ if $|x| \geq 1$, and
2. $\int_{R^N} \rho(x)dx = 1.$

If $\varepsilon > 0$, the function

$$\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon) \tag{4}$$

is nonnegative, belongs to $C_0^\infty(R^N)$ and satisfies

1. $\rho_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$,
2. $\int_{R^N} \rho_\varepsilon(x)dx = 1.$

For any Lebesgue measurable function u the convolution of u and ρ_ε is the function defined by

$$(\rho_\varepsilon * u)(x) = \int_{R^N} \rho_\varepsilon(x-y)u(y)dy$$

if the right side of above equality makes sense.

If $\operatorname{supp} u \subset\subset \Omega$, then $\rho_\varepsilon * u \in C_0^\infty(\Omega)$ provided $\varepsilon < \operatorname{dist}(\operatorname{supp} u, \partial\Omega)$, [4].

Lemma 3. Let M be τ-bounded, satisfies the condition Δ_2 and $\int_T M(t,u)dt < \infty$ for arbitrary bounded set $T \subset \Omega$ and every $u \geq 0$. If $u \in B^{k,M}(\Omega)$, then $\rho_\varepsilon * u \to u$ in $B^{k,M}(\Omega')$ as $\varepsilon \to 0$ for any $\Omega' \subset\subset \Omega$.

Proof. Let $\Omega' \subset\subset \Omega$ and $0 < \varepsilon < dist(\Omega', \partial\Omega)$. Then for every $|\alpha| \leq [k]$ there holds

$$D^\alpha \rho_\varepsilon * u = \rho_\varepsilon * D^\alpha u \quad \text{on the set } \Omega'.$$

Hence

$$D^\alpha(\rho_\varepsilon * u)(x) = \int_{R^N} \rho_\varepsilon(x-z)D^\alpha u(z)dz = \int_{R^N} \rho_\varepsilon(z)D^\alpha u(x+z)dz.$$

Let $\eta > 0$ be fixed. Applying Jensen's inequality we have for arbitrary $|\alpha| \leq [k]$

$$\int_{\Omega'}\int_{\Omega'} M\left(\frac{x+y}{2}, \frac{1}{\eta|x-y|^{\lambda}}|\Delta(x,y)D^{\alpha}(\rho_{\varepsilon} * u - u)|\right)\frac{dxdy}{|x-y|^{N}} \leq$$

$$= \int_{\Omega'}\int_{\Omega'} M\left(\frac{x+y}{2}, \frac{1}{\eta|x-y|^{\lambda}}\left|\int_{|z|<\varepsilon} \rho_{\varepsilon}(z)\Delta(x,y)(D^{\alpha}u(\cdot+z) - D^{\alpha}u(\cdot))dz\right|\right)\frac{dxdy}{|x-y|^{N}} \leq$$

$$\leq \int_{|z|<\varepsilon} \rho_{\varepsilon}(z)dz \cdot$$

$$\cdot\left(\sup_{|z|<\varepsilon}\int_{\Omega'}\int_{\Omega'} M\left(\frac{x+y}{2}, \frac{1}{\eta|x-y|^{\lambda}}|\Delta(x,y)(D^{\alpha}u(\cdot+z) - D^{\alpha}u(\cdot))|\right)\right)\frac{dxdy}{|x-y|^{N}} <$$

$< \frac{1}{2l}$ for sufficiently small ε, where $l = \sum_{|\alpha|\leq[k]} 1.$

Moreover

$$\int_{\Omega'} M\left(x, \frac{1}{\eta}|D^{\alpha}\rho_{\varepsilon} * u - D^{\alpha}u)(x)|\right)dx =$$

$$= \int_{\Omega'}\left(x, \frac{1}{\eta}\left|\int_{|z|<\varepsilon} \rho_{\varepsilon}(z)[D^{\alpha}u(x+z) - D^{\alpha}u(x)]dz\right|\right)dx \leq$$

$$\leq \int_{|z|<\varepsilon} \rho_{\varepsilon}(z)dz \sup_{|z|<\varepsilon}\int_{\Omega'} M\left(x, \frac{1}{\eta}|D^{\alpha}u(x+z) - D^{\alpha}u(x)|\right)dx \leq \frac{1}{2l}$$

for sufficiently small $\varepsilon > 0$, [3].

Hence

$$I\left(\frac{1}{\eta}(\rho_{\varepsilon} * u - u\right) \leq \sum_{|\alpha|\leq[k]}\left(\frac{1}{2l} + \frac{1}{2l}\right) = 1.$$

We conclude

$$\|\rho_{\varepsilon} * u - u\|_{B^{k,M}(\Omega')} < \eta \quad \text{for sufficiently small } \varepsilon > 0.$$

In the sequel let us assume, that the form function M is τ – bounded with the family $(g_{\nu})_{|\nu|\leq\delta}$ of functions of the $g_{\nu}(x) = |\nu| s(x)$, where s is nonnegative and measurable on Ω and $\int_{\Omega} s(x)dx < \infty$.

Lemma 4. Let M be τ *– bounded and let* $\Omega \subset R^{N}$ *be a set of finite measure. If* $u \in B^{k,M}(\Omega)$, *then* $u\psi \in B^{k,M}(\Omega)$ *for any function* $\psi \in C_0^{\infty}(\Omega)$.

Proof. The following identities

(5) $$\Delta(x,y)(f\cdot g)=g(x)\Delta(x,y)f+f(y)\Delta(x,y)g$$

and

(6) $$\Delta(x,y)D^{\alpha}(f\cdot g)=\sum_{\beta\le\alpha}C_{\alpha,\beta}\,\Delta(x,y)(D^{\beta}f\,D^{\alpha-\beta}g)$$

are true arbitrary functions f and g. Here $\beta\le\alpha$ denotes that $\beta_i\le\alpha_i$ for every $i=1,2,\dots,N$.

Let $u\in B^{k,M}(\Omega)$. Then $\mathrm{I}(au)<\infty$ for some $a>0$.

Let $\psi\in C_0^{\infty}(\Omega)$ and let $K=\operatorname{supp}\psi$, $K\subset\Omega$. For arbitrary α, $|\alpha|\le[k]$, we have

$$\int_\Omega\int_\Omega M\left(\frac{x+y}{2},\frac{|\Delta(x,y)D^{\alpha}(u\psi)|}{|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^{N}}\le$$

$$\le\int_K\int_K M\left(\frac{x+y}{2},\frac{|\Delta(x,y)D^{\alpha}(u\psi)|}{|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^{N}}+$$

$$+2\int_K\int_{\Omega\setminus K} M\left(\frac{x+y}{2},\frac{|\Delta(x,y)D^{\alpha}(u\psi)|}{|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^{N}}=I_1+2I_2.$$

Taking into acount the identities (5) and (6) we obtain

$$I_1\le C_1\sum_{\beta\le\alpha}\left\{\int_K\int_K M\left(\frac{x+y}{2},\frac{C_2|D^{\beta}u(y)||\Delta(x,y)D^{\alpha-\beta}\psi|}{|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^{N}}+\right.$$

$$\left.+\int_K\int_K M\left(\frac{x+y}{2},\frac{C_2|D^{\alpha-\beta}\psi(x)||\Delta(x,y)D^{\beta}u|}{|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^{N}}\right\}=$$

$$=C_1\sum_{\beta\le\alpha}(I_1^{\beta}+I_2^{\beta}).$$

We estimate I_1^{β}. Applying τ-boundedness of M and substituting $x=y-2h$ we have

$$I_1^{\beta}\le$$

$$\le C_3\int_K\left[\int_{\frac{1}{2}(K-K)} M\left(y-h,\frac{C_2|D^{\beta}u(y)||D^{\alpha-\beta}\psi(y-2h)|-D^{\alpha-\beta}\psi(y)|}{|2h|^{\lambda}}\right)\frac{dh}{|h|^{N}}\right]dy\le$$

$$\le C_3\int_K\left[\int_{S(0,r)}\left(M\left(y,\frac{C_2|D^{\beta}u(y)||D^{\alpha-\beta}\psi(y-2h)|-D^{\alpha-\beta}\psi(y)|}{|2h|^{\lambda}}\right)+\right.\right.$$

$$+g_h(y))\frac{dh}{|h|^N}\Bigg]dy \le$$

$$\le C_3\int_K\Bigg[\Bigg(\int_{|h|\le 1}+\int_{1<|h|\le r}\Bigg)\Bigg(M\Bigg(y,\frac{C_2|D^\beta u(y)||D^{\alpha-\beta}\psi(y-2h)|-D^{\alpha-\beta}\psi(y)|}{|2h|^\lambda}\Bigg)+$$

$$+g_h(y))\frac{dh}{|h|^N}\Bigg]dy \le C_3(I_{1,1}^\beta+I_{1,2}^\beta),$$

where $S(0,r)$ denotes the ball with centre at zero and radius r such that $\frac{1}{2}(K-K)\subset S(0,r)$.

Applying the mean – value theorem to the function $D^{\alpha-\beta}\psi$ and by convexity of M we obtain

$$\text{(7)}\qquad I_{1,1}^\beta \le \int_K\Bigg(\int_{|h|\le 1}M(y,C_4|D^\beta u(y)|)\frac{dh}{|h|^{N-1+\lambda}}\Bigg)dy+\int_{|h|\le 1}\frac{dh}{|h|^{N-1}}\int_K s(y)dy \le$$

$$\le A\int_\Omega M(y,C_4|D^\beta u(y)|)dy+B.$$

In the integral $I_{1,2}^\beta$ we use boundedness of $D^{\alpha-\beta}\psi$.

Then

$$\text{(8)}\qquad I_{1,2}^\beta \le \int_{1<|h|\le r}dh\Bigg(\int_K M(y,C_5|D^\beta u(y)|)\Bigg)dy+\int_K s(y)dy \le$$

$$\le C\int_\Omega M(y,|D^\beta u(y)|)dy+D.$$

The integral I_2^β is estimated as follows

$$I_2^\beta \le \int_\Omega\int_\Omega M\Bigg(\frac{x+y}{2},\frac{C_6|\Delta(x,y)|D^\beta u}{|x-y|^\lambda}\Bigg)\frac{dxdy}{|x-y|^N}.$$

Arguing in like manner we obtain similar estimations for the expression I_2. Thus

$$I_2 \le C_1\sum_{\beta\le\alpha}\int_K\Bigg[\int_{\Omega\setminus K}M\Bigg(\frac{x+y}{2},\frac{C_2|D^\beta u(y)||\Delta(x,y)D^{\alpha-\beta}\psi|}{|x-y|^\lambda}\Bigg)\frac{dx}{|x-y|^N}\Bigg]dy \le$$

$$\le C_3 \sum_{\beta\le\alpha} \int_K \left[\int_{\Omega y} \left(M\left(y, \frac{C_7 |D^\beta u(y)| |D^{\alpha-\beta}\psi(y-2h)| - D^{\alpha-\beta}\psi(y)|}{|2h|^\lambda} \right) + \right. \right.$$

$$\left. \left. + g_h(y)\right) \frac{dh}{|h|^N} \right] dy =$$

$$\le C_3 \sum_{\beta\le\alpha} \int_K \left[\left(\int_{\substack{\Omega y \\ |h|\le 1}} + \int_{\substack{\Omega y \\ |h|>1}} \right) \left(M\left(y, \frac{C_7 |D^\beta u(y)| |D^{\alpha-\beta}\psi(y-2h)| - D^{\alpha-\beta}\psi(y)|}{|2h|^\lambda} \right) + \right. \right.$$

$$\left. \left. + g_h(y) \right) \frac{dh}{|h|^N} \right] dy = C_3 \sum_{\beta\le\alpha} (I_{2,1}^\beta + I_{2,2}^\beta),$$

where $\Omega_y = \frac{1}{2}[y - (\Omega \setminus K)]$ for $y \in K$.

We have

(9) $$I_{2,1}^\beta \le E \int_\Omega M(y, C_8 |D^\beta u(y)|) dy + F.$$

Now we estimate $I_{2,2}^\beta$ using the fact $|\Omega| < \infty$. Then

(10) $$I_{2,2}^\beta \le C_9 \int_K \left(\int_{\substack{\Omega y \\ |h|>1}} M(y, C_{10} |D^\beta u(y)|) \frac{dh}{|h|^{N+\lambda}} + \int_{\substack{\Omega y \\ |h|>1}} \frac{s(y) dh}{|h|^{N-1}} \right) dy \le$$

$$\le C_9 \int_K \left(\int_{\substack{\Omega_K \\ |h|>1}} \frac{dh}{|h|^{N+\lambda}} \int_\Omega M(y, C_{10} |D^\beta u(y)|) dy + |\tfrac{1}{2}(\Omega \setminus K)| \int_K s(y) dy \right) =$$

$$= G \int_\Omega M(y, C_{10} |D^\beta u(y)|) dy + H$$

where $|\frac{1}{2}(\Omega \setminus K)| < \infty$ and the constants A, B, C, D, E, F, G, H are finite. Moreover, we have

(11) $$\int_\Omega M(x, |D^\alpha (u\psi)(x)|) dx \le \sum_{\beta\le\alpha} \int_\Omega M(x, c_{11} |D^\beta u(x)|) dx.$$

Let us denote

$$b = \frac{a}{\max(C_4, C_5, C_7, C_8, C_{10}, C_{11})}.$$

Then, from the inequalities (7), (8), (9), (10) and (11) we obtain $I(bu\psi)<\infty$. Finally, we conclude $u\psi \in B^{k,M}(\Omega)$.

Theorem. Let M be τ-bounded, satisfies the condition Δ_2 and $\int_T M(t,u)dt<\infty$ for any bounded measurable set $T\in\Omega$ and every $u\geq 0$. If the set Ω has finite measure, then $C^{\infty}(\Omega)\cap B^{k,M}(\Omega)$ is dense in $B^{k,M}(\Omega)$.

Proof. Let Ω_i be the open set defined by $\Omega_i=\{t\in\Omega:|t|<i,\operatorname{dist}(t,\partial\Omega)>\frac{1}{i}\}$, $i=1,2,\dots$, and $\Omega_0=\Omega_{-1}=\varnothing$. Then

$$R=\{U_i:U_i=\Omega_{i+1}-\Omega_{i-1}, i=1,2,\dots\}$$

is a collection of open subsets of Ω that covers Ω, [4].

Let $(\psi_i)_{i=1}^{\infty}$ be a partition of unity on Ω subordinate to R. Then

(12) $$\operatorname{supp}\psi_i\subset U_i,\qquad i=1,2,\dots.$$

Let $u\in B^{k,M}(\Omega)$. By Lemma 4 we have $u\psi_i\in B^{k,M}(\Omega)$ for $i=1,2,\dots$. If $0<\delta_i<\frac{1}{(i+1)(i+2)}$ then, by (12), we have that $\rho_{\delta_i}*(D^{\alpha}u\psi_i)$ has the support in $\Omega_{i+2}\setminus\Omega_{i-2}$ for every $|\alpha|\leq[k]$ and $i=1,2,\dots$. Let us denote $\rho_{\delta_i}\equiv\rho_i$. Hence the series

$$\sum_{i=1}^{\infty}\rho_i*(u\psi_i)$$

converges and defines a function $v\in C^{\infty}(\Omega)$. By Lemma 3 we have

(13) $$\|\rho_i*(u\psi_i)-u\psi_i\|_{B^{k,M}}<\frac{\varepsilon}{2^i}.$$

Let n be an arbitrary nonnegative integer number and let $\varepsilon>0$. Then

$$\int_{\Omega_n}\int_{\Omega_n}M\left(\frac{x+y}{2},\frac{|\Delta(x,y)(D^{\alpha}v-D^{\alpha}u)|}{\varepsilon|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^N}=$$

$$=\int_{\Omega_n}\int_{\Omega_n}M\left(\frac{x+y}{2},\frac{|\Delta(x,y)(D^{\alpha}\sum_{i=1}^{\infty}(\rho_i*(u\psi_i))-D^{\alpha}\sum_{i=1}^{\infty}(u\psi_i))|}{\varepsilon|x-y|^{\lambda}}\right)\frac{dxdy}{|x-y|^N}=$$

$$= \int_{\Omega_n}\int_{\Omega_n} M\left(\frac{x+y}{2}, \frac{\left|\Delta(x,y)\left[\sum_{i=1}^{n+1}(D^\alpha(\rho_i*(u\psi_i)) - D^\alpha(u\psi_i))\right]\right|}{\varepsilon|x-y|^\lambda}\right)\frac{dxdy}{|x-y|^N} \le$$

$$= \sum_{i=1}^{n+1}\frac{1}{2^i}\int_{\Omega_n}\int_{\Omega_n} M\left(\frac{x+y}{2}, \frac{2^i|\Delta(x,y)(D^\alpha(\rho_i*(u\psi_i)) - D^\alpha(u\psi_i))|}{\varepsilon|x-y|^\lambda}\right)\frac{dxdy}{|x-y|^N}.$$

Moreover

$$\int_{\Omega_n} M\left(x, \frac{1}{\varepsilon}|D^\alpha v - D^\alpha u)(x)|\right)dx \le$$

$$\le \sum_{i=1}^{n+1}\frac{1}{2^i}\int_{\Omega_n} M\left(x, \frac{2^i}{\varepsilon}|D^\alpha(\rho_i*(u\psi_i) - u\psi_i)(x)|\right)dx.$$

Then, from (13) we obtain

$$\int_{\Omega_n} M\left(x, \frac{1}{\varepsilon}|D^\alpha v - D^\alpha u)(x)|\right)dx +$$

$$+ \int_{\Omega_n}\int_{\Omega_n} M\left(\frac{x+y}{2}, \frac{|\Delta(x,y)(D^\alpha v - D^\alpha u)|}{\varepsilon|x-y|^\lambda}\right)\frac{dxdy}{|x-y|^N} \le 1.$$

Hence, letting $n \to \infty$, by the monotone convergence theorem we have

$$I((v-u)/\varepsilon) \le 1 \text{ and hence } \|v-u\|_{B^{k,M}} < \varepsilon.$$

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