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 (ψ, p) -INTEGRAL EQUIVALENCE OF MULTIVALUED DIFFERENTIAL SYSTEMS WITH DELAY

ABSTRACT: In our paper we deal with (ψ, p) -integral equivalence of systems:

$$(a) \quad x'(t) \in A(t)x(t) + F(t, x_t),$$

$$(b) \quad y'(t) = A(t)y(t).$$

To reach this aim we use Ky Fan fixed point theorem. To proof uniformly boundedness we use lemmas 6 and 7 which deal with g -compactness.

KEY WORDS: systems with delay, multifunction, integral equivalence, g -compactness, upper semicompact mappings.

The integral equivalence between two systems of ordinary differential equations is the subject of many recent investigations. The idea of integral equivalence appeared for the first time in [1]. The paper [4] contains some generalisations of those considerations. The main results of this paper were obtained with the use of the fixed point theorem, which is proved in [3]. The purpose of our paper is to generalise some results from [4] to the case of differential systems with delay. Our considerations are based on the idea of (φ, g) -compact operators as in [4], but instead of Haščák Fixed Point Theorem Ky-Fan Fixed Point Theorem for multivalued mappings we will be applied, which requires to work in other topological spaces.

More precise, we shall consider (ψ, p) -integral equivalence of systems:

$$(a) \quad x'(t) \in A(t)x(t) + F(t, x_t),$$

$$(b) \quad y'(t) = A(t)y(t),$$

where x, y are n -dimensional vectors, $t \geq 0$, $A(t)$ is an $n \times n$ matrix function defined on $J = [0, \infty)$, whose elements are integrable on compact subsets of J ; $F: [0, \infty) \times C_\tau \rightarrow 2^{R^n}$, where 2^{R^n} is the family of all nonempty subsets of R^n , $\tau > 0$, C_τ is space of continuous functions from $[-\tau, 0]$ to R^n with norm

$$|\phi| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|.$$

If $u(t)$ is any continuous function defined on $[t_0 - \tau, \infty)$ to R^n , then for every $t \in [t_0, \infty)$, u_t is element from C_τ , which is defined

$$u_t(\theta) = u(t + \theta), \quad -\tau \leq \theta \leq 0$$

By the solution of (a) on $[a, b)$, $b \leq \infty$ we mean the function $x(t)$ which is continuous on $[a - \tau, b)$, absolutely continuous on $[a, b)$ and satisfies (a) almost everywhere (a.e.) on the interval $[a, b)$.

Let denote

$$L_p := L_p([t_0, \infty)) := \left\{ x(t) : \int_{t_0}^{\infty} |x(s)|^p ds < \infty \right\}, \quad p \in [1, \infty).$$

Definition 1. [1] Let $\psi(t)$ be positive continuous function on an interval $[t_0, \infty)$ and let $p \geq 1$. We shall say that two systems (a) and (b) are (ψ, p) -integral equivalent on $[t_0, \infty)$ iff for each solution $x(t)$ of (a) there exists a solution $y(t)$ of (b) such that

$$(c) \quad \psi^{-1}(t) |x(t) - y(t)| \in L_p([t_0, \infty))$$

and conversely, for each solution $y(t)$ of (b) exists a solution $x(t)$ of (a) such that (c) holds. By a restricted (ψ, p) -integral equivalence between (a) and (b) we shall mean that the relation (c) is satisfied for some subsets of solutions of (a) and (b), it means particulary for the ψ -bounded solutions.

We shall say that a function $z(t)$ is ψ -bounded on the interval $[t_0, \infty)$ iff

$$\sup_{t \geq t_0} |\psi^{-1}(t) z(t)| < \infty.$$

NOTATIONS AND PRELIMINARIES

We shall write $|\cdot|$ for any convenient vector (matrix) norm. If A is a subset of R^n we define

$$|A| := \sup\{|a| : a \in A\}.$$

We take note of

$$E := (\delta_{ij})_{n \times n}, \quad \text{where } \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

$L_p^n(J)$ will denote n -th Cartesian product of $L_p(J)$ and let $B(t_0)$ ($= B([t_0, \infty))$) be the space of all continuous functions from $[t_0, \infty)$ to R^n . The topology on $B(t_0)$ will be that introduced by the family of semi-norm $\{p_n\}$ where for each $x \in B(t_0)$:

$$(1) \quad p_n(x) := \sup_{t_0 \leq t \leq t_0+n} |x(t)|.$$

Then a fundamental system of neighbourhoods of the function $x(t) = 0$, $t \in [t_0, \infty)$ is given by the sets v_n , $n = 1, 2, \dots$, where

$$v_n := \left\{ x \in B(t_0) : p_n(x) < \frac{1}{n} \right\}.$$

Under this topology $B(t_0)$ is a complete, locally convex and metrizable vector space. This topology is equivalent to the topology of uniform convergence on compact subsets of $[t_0, \infty)$.

Let X, Y be a topological spaces. Let us denote by 2^Y the family of all nonempty subsets of the space Y and let $cf(Y)$ be the set of all nonempty convex and closed subsets of Y .

Definition 2. (C. Berge [7]) *A mapping $F : X \rightarrow 2^Y$ is upper semicontinuous at a point $x \in X$ iff for an arbitrary neighbourhood $O_{F(x)}$ of the set-image $F(x)$ there exists a neighbourhood O_x of the point x such that $F(O_x) \subset O_{F(x)}$, where*

$$F(O_x) := \bigcup_{x \in O_x} F(z).$$

This mapping is said to be upper semicontinuous iff it is upper semicontinuous at each point $x \in X$.

Let $\psi(t)$ be a positive continuous function on $[t_0, \infty)$. For $z \in B(t_0)$, we denote

$$|z|_\psi := \sup_{t \geq t_0} |\psi^{-1}(t)z(t)|.$$

Let $B_\psi(t_0) := \{z \in B(t_0) : |z|_\psi < \infty\}$. Then $B_\psi(t_0)$ with the norm $|\cdot|_\psi$ is a Banach space. For $\rho > 0$, we denote

$$B_{\psi, \rho}(t_0) := \{z \in B(t_0) : |z|_\psi \leq \rho\}.$$

Let $\varphi(t)$ be a positive continuous function defined on $[t_0, \infty)$. By $L_{p, \varphi}([t_0, \infty))$ ($1 \leq p < \infty$) we shall denote the set of all real-valued measurable functions $y(t)$ defined on $[t_0, \infty)$ such that

$$|y|_{p,\varphi} := \left(\int_0^{\infty} |\varphi^{-1}(s)y(s)|^p ds \right)^{1/p} < \infty.$$

$L_{p,\varphi}([t_0, \infty))$ with the norm $|\cdot|_{p,\varphi}$ is also Banach space.

Theorem 1. (Ky Fan [8]) If A is locally convex topological vector space and A_0 is a nonempty compact convex subset of A , then for every upper semicontinuous mapping $T: A_0 \rightarrow \text{cf}(A_0)$, there exists a point $x \in A_0$ such that $x \in T(x)$.

Definition 3. (W. Sobieszek [10]) A mapping $F: X \rightarrow 2^Y$ is upper semicompact (sequentially upper semicontinuous) at point $x \in X$ iff from the assumptions $x_n \rightarrow x$, $x_n \in X$, $y_n \in F(x_n)$ it follows that there exists a subsequence of the sequence $\{y_n\}$ which converges to some $y \in F(x)$.

Definition 4. Let $g: J \times J \rightarrow J$ be a function such that $g(t, x)$ is monotone nondecreasing in x for each fixed $t \in J$. Let $\varphi(t)$ be a positive continuous function defined on $[t_0, \infty)$. A set B of functions which are defined on J is (φ, g) -bounded iff there are two nonnegative constants c, K such that

$$|y(t)| \leq K \varphi(t) g(t, c) \quad \text{a.e. on } J$$

for each $y \in B$.

Definition 5. An operator T from $A \subset L_{p,\varphi}^n(J)$ into a Banach space Y is called (φ, g) -compact, iff it is continuous and maps each (φ, g) -bounded subset A into relatively compact subset of Y .

Lemma 1. (Banach-Saks [9]) Let x_n weakly converges to x in space L_p , $p \in (1, \infty)$. Then exists subsequence $\{x_{1n}\}$ of sequence $\{x_n\}$ such that

$$(1/k)(x_{11} + x_{12} + \dots + x_{1k})$$

converges to x in L_p -norm.

Lemma 2. (A. Haščák [3]) Let $K \subset L_1([t_0, \infty))$ and suppose that there exists $g: [t_0, \infty) \rightarrow [0, \infty)$, $g \in L_1([t_0, \infty))$ such that for each $f \in K$

$$|f(t)| \leq g(t) \text{ a.e. on } [t_0, \infty)$$

Then K is weakly relatively compact in $L_1([t_0, \infty))$.

Lemma 3. (A. Haščák and M. Švec [1]) Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t \geq 0$, $Y(t)$ a nonsingular matrix and P a projection ($P^2 = P$). Further suppose that for $t \geq 0$ exist such constants $K > 0$, $p \geq 1$, that

$$\left(\int_0^t |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \leq K$$

and

$$\int_0^\infty \exp \left(-K^{-p} \int_0^t \varphi^p(s)\psi^{-p}(s)ds \right) dt < \infty$$

Then

$$\lim_{t \rightarrow \infty} \psi^{-1}(t)|Y(t)P| = 0 \quad \text{and} \quad |\psi^{-1}(t)Y(t)P| \in L_p([0, \infty)).$$

Lemma 4. (A. Haščák [2]) Let $p \geq 1$ and $f(t)$ be an integrable and nonnegative function for $t \geq 0$. Then

$$\left(\int_0^\infty \left(\int_t^\infty f(s) ds \right)^p dt \right)^{1/p} \leq \int_0^\infty s^{1/p} f(s) ds.$$

Lemma 5. (A. Haščák and M. Švec [1]) Let $g(t) \geq 0$ be continuous on $0 \leq t < \infty$ and such that

$$\int_0^\infty s g(s) ds < \infty.$$

Then

$$\int_t^\infty g(s) ds \in L_p([0, \infty)), \quad p \geq 1.$$

Theorem 2. (A. Haščák [5]) Let $w - \lim_{n \rightarrow \infty} x_n = x_0$ (i.e. $x_n \rightarrow x_0$ weakly) in $L_1([a, \infty))$ and let there exists a function $g \in L_1([a, \infty))$ such that

$$|x_n(t)| \leq g(t) \text{ a.e. on } [a, \infty) \quad n = 1, 2, \dots$$

Then there exists a subsequence $\{x_{1_n}\}$ of the sequence $\{x_n\}$ such that

$$(1/k)(x_{1_1} + x_{1_2} + \dots + x_{1_k})$$

converges to x_0 in the norm of $L_1([a, \infty))$.

MAIN RESULTS

Further throughout this paper we shall assume that

(H_0) the functions $\psi(t)$, $\varphi(t)$ are positive continuous functions on $[-\tau, \infty)$ and $J := [0, \infty)$ respectively,

(H_1) $F(t, \phi)$ is nonempty, compact and convex subset of R^n for $(t, \phi) \in J \times C_\tau$,

(H_2) for each fixed $t \in J$, the mapping $F(t, \phi)$ is upper semicontinuous,

(H_3) for each $x \in B_\psi(-\tau)$, there is measurable function $f_x : J \rightarrow R^n$ such that

$$f_x(t) \in F(t, x_t) \text{ a.e. on } J.$$

Further suppose that there exists $g : J \times J \rightarrow J$ such that

i) $g(t, u)$ is monotone nondecreasing in $u \in J$ for each fixed $t \in J$,

ii) $g(t, c) \in L_{p'}(J)$ for any constant $c \geq 0$ and some $p' \in [1, \infty)$,

iii) for each $\phi \in B_\psi(-\tau)$, holds

$$|F(t, \phi_t)| \leq \varphi(t) g(t, |(\psi^{-1}\phi)_t|) \text{ a.e. on } J, \text{ where } (\psi^{-1}\phi)(t) = \psi^{-1}(t)\phi(t).$$

Remark 1. For each $x \in B_\psi(-\tau)$ holds

$$|(\psi^{-1}x)_t| \leq |\psi^{-1}x|$$

where

$$|(\psi^{-1}x)_t| = \sup_{-\tau \leq \theta \leq 0} |\psi^{-1}(t+\theta)x(t+\theta)|$$

and

$$|\psi^{-1}x| = \sup_{t \in [-\tau, \infty)} |\psi^{-1}(t)x(t)|.$$

In the following we can substitute condition iii) by condition:

For each $x \in B_\psi(-\tau)$ holds

$$|F(t, x_t)| \leq \varphi(t) g(t, |\psi^{-1}x|).$$

Especially for g we can take

$$g(t, u) = m(t)u,$$

where $m(t)$ is positive function from $L_{p'}$ for some $p' \geq 1$.

Given a function $x \in B_\psi(-\tau)$ denote by $M(x)$ the set of all measurable functions $y: J \rightarrow R^n$ such that

$$y(t) \in F(t, x_t) \text{ a.e. on } J.$$

Theorem 3. Let the hypotheses $(H_0) - (H_3)$, i) - iii) be satisfied. Then the correspondence $x \rightarrow M(x)$ defines a bounded mapping of $B_{\psi, \rho}(-\tau)$ into $cf(L_{p', \varphi}^n(J))$.

Proof. We have to show that for every $x \in B_{\psi, \rho}(-\tau)$ $M(x)$ is

- a) nonempty
- b) convex
- c) closed
- d) included in $L_{p', \varphi}^n(J)$
- e) $\forall_{\delta > 0} \exists_{K > 0} \forall_{x \in B_{\psi, \delta}} \Rightarrow \forall_{y \in M(x)} |y|_{p', \varphi} \leq K.$

The statements a) and b) are implied by (H_3) and (H_1) .

e) Let $\delta > 0$, $x \in B_{\psi, \delta}(-\tau)$. Let $y \in M(x)$, which means that

$$y(t) \in F(t, x_t) \text{ a.e. on } J.$$

Then by iii), i) and using the inequality $|(\psi^{-1}x)_t| \leq |\psi^{-1}x|$ we have

$$|y(t)| \leq \varphi(t) g(t, |(\psi^{-1}x)_t|) \leq \varphi(t) g(t, \delta),$$

i.e.

$$|\varphi^{-1}(t) y(t)| \leq g(t, \delta).$$

Now to prove e) it suffices to choose for

$$K \geq |g(t, \delta)|_{p'}.$$

d) is implied by e).

c) Let $\{y_n\}$, $y_n \in M(x)$ be a sequence such that $|y_n - y|_{p', \varphi} \rightarrow 0$ as $n \rightarrow \infty$, $p' \in [1, \infty)$.

By this fact that convergence in L_p -norm on interval J implies that there is a subsequence $\{y_{1n}\}$ of sequence $\{y_n\}$ such that $\{y_{1n}(t)\}$ converges a.e. on J to $y(t)$ as $n \rightarrow \infty$. On the other hand

$$y_{1n}(t) \in F(t, x_t) \text{ a.e. on } J.$$

Because of (H_1)

$$y(t) \in F(t, x_t) \text{ a.e. on } J.$$

Thus $y(t) \in M(x)$ and the proof of Theorem 3 is complete.

Theorem 4. Let the hypotheses of Theorem 3 be satisfied. Then the mapping

$$M : B_{\psi, \rho}(-\tau) \rightarrow \text{cf}(L_{p', \varphi}^n(J))$$

is weakly upper semicontinuous.

Proof. Let $x_n, x \in B_{\psi, \rho}(-\tau)$, $x_n \rightarrow x$ in the topology induced by the family of seminorm (1) and let $y_n \in M(x_n)$. The existence of a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ which converges weakly to some $y \in L_{p', \varphi}^n(J)$ is implied by Lemma 2, ii) and iii) in the case $p' = 1$. For $p' > 1$ it follows from

$$|y|_{p', \varphi} \leq C := \left(\int_0^\infty g^{p'}(s, c) ds \right)^{1/p'} < \infty.$$

Thus we have only to prove that $y \in M(x)$. By Theorem 2 (in the case $p' = 1$) or by Lemma 1 (in the case $p' > 1$), there is a subsequence $\{y_{2n}\}$ of the sequence $\{y_{1n}\}$ such that

$$\left| \frac{1}{n} \sum_{k=1}^n y_{2k} - y \right|_{p', \varphi} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, by this fact that convergence in L_p -norm on interval J implies convergence of some subsequence almost everywhere on J , exists a sequence $\{\sigma_n\}$ $\sigma_n \in N$, $\sigma_n \geq n$, such that

$$\frac{1}{\sigma_n} \sum_{k=1}^{\sigma_n} y_{2k}(t) \rightarrow y(t) \text{ a.e. on } J$$

for $n \rightarrow \infty$. On the other hand $(x_n)_t, x_t \in C_\tau, ((x_n)_t := x_n(t+s)$ where $s \in [-\tau, 0])$ and $(x_n)_t \rightarrow x_t$ (since x_n converges uniformly on every compact subset of $[-\tau, \infty)$).

By (H_2) for almost every fixed $t \in J$ and any $\varepsilon > 0$ there is an integer $N(\varepsilon, t)$ such that

$$F(t, (x_n)_t) \subset F(t, x_t) + K_\varepsilon := \{u + v : u \in F(t, x_t), |v| \leq \varepsilon\}, \text{ for } n \geq N(\varepsilon, t).$$

Thus

$$y_{2k}(t) \in F(t, x_t) + K_\varepsilon, \quad 2k \geq N(\varepsilon, t)$$

and by the convexity of $F(t, \phi)$ we get

$$\frac{1}{\sigma_n} \sum_{k=1}^{\sigma_n} y_{2k}(t) \in F(t, x_t) + K_\varepsilon,$$

so that $y(t) \in F(t, x_t)$ a. e. on J . The proof is complete.

Lemma 6. Let $Y(t)$ be a continuous matrix for $t \geq 0$ with $\det Y(t) \neq 0$ for each $t \geq 0$ and P be an $n \times n$ constant matrix with $P^2 = P$. Suppose that there exist constants $K_1 > 0$ and $1 < p < \infty$ such that

$$(2) \quad \left(\int_0^t |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \leq K_1, \quad \text{for all } t \geq 0,$$

$$(3) \quad \int_0^\infty \exp \left(-K_1^{-p} \int_0^t \varphi^p(s)\psi^{-p}(s)ds \right) dt \leq \infty,$$

$$(4) \quad \int_0^\infty |PY^{-1}(s)\varphi(s)| g(s, c) ds < \infty, \quad \text{for any } c \geq 0.$$

Then the operator $T_1 : L_{p', \varphi}^n(J) \rightarrow B_\psi(-\tau)$, $1/p + 1/p' = 1$ defined by the formula

$$(T_1 y)(t) := \begin{cases} \int_0^t Y(t)PY^{-1}(s)y(s)ds, & t \geq 0, \\ 0, & -\tau \leq t < 0, \end{cases}$$

is (φ, g) -compact.

Proof. This proof is basing on the proof of Lemma 5 from [4]. For each $y \in L_{p', \varphi}^n(J)$ we have

$$|T_1 y|_\psi := \sup_{t \geq -\tau} |\psi^{-1}(t)(T_1 y)(t)| \leq K_1 |y|_{p', \varphi}$$

which implies that T_1 is bounded and hence continuous on $L_{p', \varphi}^n(J)$.

Further, take any (φ, g) -bounded sequence $\{y_k\}$ from $L_{p', \varphi}^n(J)$. We have to show that the sequence $\{T_1 y_k\}$ contains a subsequence which is convergent in the topology induced by the family of seminorm (1) in $B_\psi(-\tau)$.

Let

$$z_i(t) := \int_0^t Y(t) P Y^{-1}(s) y_i(s) ds, \quad i = 1, 2, \dots$$

Since $\{y_i, i = 1, 2, \dots\}$ is (φ, g) -bounded set in $L_{p', \varphi}^n(J)$, $p' \in (1, \infty)$ there is a subsequence $\{y_{i_k}\}$ of $\{y_i\}$ which converges weakly to an element $y \in L_{p', \varphi}^n(J)$ i.e.

$$z_{i_k}(t) := (T_1 y_{i_k})(t) \rightarrow (T_1 y)(t) = \int_0^t Y(t) P Y^{-1}(s) y(s) ds =: z(t).$$

The functions z_{i_k} , $i = 1, 2, \dots$ are uniformly ψ -bounded.

Moreover the functions $\psi^{-1}(t) z_{i_k}(t)$, $i = 1, 2, \dots$ are equicontinuous on every compact subinterval of J .

By Ascoli Theorem as well as by Cantor's diagonalization process, the sequence $\{z_{i_k}\}$ contains a subsequence $\{z_{2i_k}\}$ such that $\{\psi^{-1}(t) z_{2i_k}(t)\}$ is uniformly convergent on every compact subinterval of J .

Lemma 7. Let $Y(t)$ be a continuous matrix for $t \geq 0$ with $\det Y(t) \neq 0$ for each $t \geq 0$ and P be an $n \times n$ constant matrix with $P^2 = P$.

If there exist constants $K_3 > 0$ and $1 < p < \infty$ such that

$$(5) \quad \left(\int_0^\infty |\psi^{-1}(t) Y(t) P Y^{-1}(s) \varphi(s)|^p ds \right)^{1/p} \leq K_3, \quad \text{for all } t \geq 0$$

then the linear operator $T_3: L_{p', \varphi}^n(J) \rightarrow B_\psi(-\tau)$, $1/p + 1/p' = 1$ defined by the formula:

$$(T_3 y)(t) := \begin{cases} \frac{\psi(t)}{\psi(0)} \int_0^\infty Y(0) P Y^{-1}(s) y(s) ds, & -\tau \leq t < 0, \\ \int_t^\infty Y(t) P Y^{-1}(s) y(s) ds, & t \geq 0, \end{cases}$$

is (φ, g) -compact.

Proof. For each $y \in L^n_{p',\varphi}(J)$ we have

$$|T_3 y|_\psi := \sup_{t \geq -\tau} |\psi^{-1}(t)(T_3 y)(t)| = \sup_{t \geq 0} |\psi^{-1}(t)(T_3 y)(t)| \leq \int_0^\infty |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)||\varphi^{-1}(s)y(s)| ds \leq K_3 |y|_{p',\varphi},$$

which implies that T_3 is (φ, g) -bounded and hence continuous on $L^n_{p',\varphi}(J)$. Further, take any (φ, g) -bounded sequence $\{y_k\}$ from $L^n_{p',\varphi}(J)$. We have to show that the sequence $\{T_3 y_k\}$ contains a subsequence which is convergent in topology of seminorm (1) in $B_\psi(-\tau)$.

Let

$$z_i(t) := \begin{cases} \frac{\psi(t)}{\psi(0)} \int_0^\infty Y(0)PY^{-1}(s)y_i(s) ds, & -\tau \leq t < 0, \\ \int_t^\infty Y(t)PY^{-1}(s)y_i(s) ds, & t \geq 0. \end{cases}$$

Since $\{y_i, i = 1, 2, \dots\}$ is (φ, g) -bounded subset of $L^n_{p',\varphi}(J)$, $p' \in (1, \infty)$ there is a subsequence $\{y_{i_i}\}$ which converges weakly to an element $y \in L^n_{p',\varphi}(J)$ a.e.

$$z_{i_i}(t) := (T_3 y_{i_i})(t) \rightarrow (T_3 y)(t) = \int_t^\infty Y(t)PY^{-1}(s)y(s) ds =: z(t), \quad t \geq 0.$$

From $z_{i_i}(0) \rightarrow z(0)$ and the definition of T_3 we also have:

$$z_{i_i}(t) \rightarrow z(t), \quad t \in [-\tau, 0).$$

Further, there are nonnegative constants c, K , such that

$$|y_{i_i}(t)| \leq K\varphi(t)g(t, c) \text{ a.e. on } J, \quad i = 1, 2, \dots$$

Using this fact, the Holder inequality and (5), we have

$$|\psi^{-1}(t)z_{i_i}(t)| \leq KK_3 \left(\int_0^\infty g^{p'}(s, c) ds \right)^{1/p'}, \quad i = 1, 2, \dots$$

Thus the functions $z_{i_i}, i = 1, 2, \dots$ are uniformly ψ -bounded and from the inequalities for $0 \leq t_1 \leq t_2$:

$$\begin{aligned}
& |\psi^{-1}(t_2)z_{1i}(t_2) - \psi^{-1}(t_1)z_{1i}(t_1)| = \\
& = \left| \int_{t_2}^{\infty} (\psi^{-1}(t_2)Y(t_2) - \psi^{-1}(t_1)Y(t_1))PY^{-1}(s)y_{1i}(s)ds + \right. \\
& \quad \left. + \int_{t_1}^{t_2} \psi^{-1}(t_2)Y(t_2)PY^{-1}(s)y_{1i}(s)ds \right| \leq \\
& \leq \int_{t_1}^{t_2} |\psi^{-1}(t_2)PY^{-1}(s)\varphi(s)| |\varphi^{-1}(s)y_{1i}(s)| ds + \\
& \quad + |\psi^{-1}(t_2)Y(t_2) - \psi^{-1}(t_1)Y(t_1)| \int_{t_2}^{\infty} |PY^{-1}(s)\varphi(s)| |\varphi^{-1}(s)y_{1i}(s)| ds \leq \\
& \leq \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2)PY^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \left(\int_{t_1}^{t_2} |\varphi^{-1}(s)y_{1i}(s)|^{p'} ds \right)^{1/p'} + \\
& \quad + |\psi^{-1}(t_2)Y(t_2) - \psi^{-1}(t_1)Y(t_1)| |P| |\psi(t_1)Y^{-1}(t_1)| \times \\
& \quad \times \left(\int_{t_2}^{\infty} |\psi^{-1}(t_1)Y(t_1)PY^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \left(\int_{t_2}^{\infty} |\varphi^{-1}(s)y_{1i}(s)|^{p'} ds \right)^{1/p'} \leq \\
& \leq K \left(\int_0^{\infty} g^{p'}(s,c)ds \right)^{1/p'} \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2)Y(t_2)PY^{-1}(s)\varphi(s)|^p ds \right)^{1/p} + \\
& \quad + |\psi^{-1}(t_2)Y(t_2) - \psi^{-1}(t_1)Y(t_1)| |\psi(t_1)Y^{-1}(t_1)| KK_3 \left(\int_0^{\infty} g^{p'}(s,c)ds \right)^{1/p'} ,
\end{aligned}$$

it follows that the functions $\psi^{-1}(t)z_{1i}(t)$, $i = 1, 2, \dots$ are equicontinuous on every compact subinterval of J . Further from the definition of T_3 we have that the functions $\psi^{-1}(t)z_{1i}(t)$, $i = 1, 2, \dots$ are equicontinuous on $[-\tau, 0)$ also.

By the Ascoli Theorem as well as Cantor's diagonalization process, the sequence $\{z_{1i}\}$ contains the subsequence $\{z_{2i}\}$ such that $\{\psi^{-1}(t)z_{2i}(t)\}$ is uniformly convergent on every compact subinterval of J .

Theorem 5. Let the hypotheses of Theorem 3 be satisfied and D be a matrix space. Suppose that $T: L_{p', \varphi}^n(J) \rightarrow D$ is a (ψ, g) -compact linear operator. Then the operator TM defined by

$$TMx := \{z \in D : z = Ty \text{ and } y \in M(x)\}$$

maps $B_{\psi, \rho}(J)$ into $\text{cf}(D)$ and is upper semicontinuous.

Proof. This proof is the same as the proof of Theorem 6 in [4].

Lemma 8. Let the mappings $F_i : X \rightarrow Z^Y, i = 1, 2$ are upper semicontinuous. Then the mappings $-F_i$ ($i = 1, 2$) and $F_1 + F_2$ are upper semicontinuous.

Theorem 6. Let the hypotheses (H_0) – (H_3) be satisfied and let $Y(t)$ be a fundamental matrix of (b).

Suppose that

- a) there exist supplementary projectors P_1, P_2 (i.e. $P_1 + P_2 = E, P_i^2 = P_i, i = 1, 2$) and constants $K > 0$ and $2 \leq p < \infty$ such that

$$\int_t^\infty |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p ds + \int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p ds \leq K^p$$

for all $t \geq 0$.

- b) there exists $g : J \times J \rightarrow J$ such that

i) $g(t, u)$ is monotone nondecreasing in $u \in J$ for each fixed $t \in J$,

ii) $\int_0^\infty s^{p'/p} g^{p'}(s, c) ds < \infty$ for any constant $c \geq 0$ and some $1/p + 1/p' = 1$,

iii) for each $\phi \in B_\psi(-\tau)$, holds

$$|F(t, \phi_t)| \leq \varphi(t) g(t, |(\psi^{-1}\phi)_t|) \text{ a.e on } J, \text{ where } (\psi^{-1}\phi)(t) = \psi^{-1}(t)\phi(t),$$

c) $\int_0^\infty \exp\left\{-K^{-p} \int_0^t \varphi^p(s)\psi^{-p}(s) ds\right\} dt < \infty$,

d) $\forall_{c \geq 0} \int_0^\infty |P_1Y^{-1}(s)\varphi(s)| g(s, c) ds < \infty$.

Then there exists $t_0 \geq 0$ such that the set of ψ -bounded on $[t_0, \infty)$ solutions of (a) and (b) are (ψ, p) -integral equivalent on $[t_0, \infty)$.

Proof. Let $y(t)$ be a ψ -bounded solution of (b) on $[t_0, \infty)$, $t_0 \geq 0$. Then there is $\rho > 0$ such that $y \in B_{\psi, \rho}(t_0)$. Let choose t_0 so that

$$K \left(\int_{t_0}^{\infty} g^{p'}(s, 2\rho) ds \right)^{1/p'} \leq \rho.$$

Define for $x \in B_{\psi, 2\rho}(t_0 - \tau)$ the operator

$$TMx := \left\{ z : z(t) := \begin{cases} \psi(t)\psi^{-1}(t_0) \times \\ \times \left[y(t_0) - \int_{t_0}^{\infty} Y(t_0)P_2Y^{-1}(s)f_x(s)ds \right], & t_0 - \tau \leq t < t_0, \\ y(t) + \int_{t_0}^{t_1} Y(t)P_1Y^{-1}(s)f_x(s)ds + & t \geq t_0. \\ - \int_t^{\infty} Y(t)P_2Y^{-1}(s)f_x(s)ds, & \end{cases} \right. , f_x \in M_x \}.$$

Let $X = TM(B_{\psi, 2\rho}(t_0 - \tau))$. The functions in $TM(B_{\psi, 2\rho}(t_0 - \tau))$ are uniformly bounded, because $X \subset B_{\psi, 2\rho}$. Further, for each $z \in TMx$, $x \in B_{\psi, 2\rho}(t_0 - \tau)$ we have

$$1) \quad |\psi^{-1}(t)z(t)| = |\psi^{-1}(t_0)z(t_0)|, \quad t \in [t_0 - \tau, t_0],$$

$$2) \quad |\psi^{-1}(t)z(t)| \leq |\psi^{-1}(t)y(t)| +$$

$$+ \int_{t_0}^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|g(s, 2\rho)ds +$$

$$+ \int_t^{\infty} |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|g(s, 2\rho)ds \leq$$

$$\leq \rho + \left(\int_{t_0}^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \left(\int_{t_0}^t g^{p'}(s, 2\rho) ds \right)^{1/p'} +$$

$$+ \left(\int_t^{\infty} |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p ds \right)^{1/p} \left(\int_t^{\infty} g^{p'}(s, 2\rho) ds \right)^{1/p'} \leq$$

$$\leq \rho + K \left(\int_{t_0}^{\infty} g^{p'}(s, 2\rho) ds \right)^{1/p'}, \quad t \geq t_0.$$

Let $x \in B_{\psi, 2\rho}(t_0 - \tau)$ and $z \in TM x$. Then there is $f_x \in M(x)$ such that

$$z'(t) = A(t)z(t) + f_x(t) \text{ a.e. on } [t_0, \infty).$$

Therefore by iii) of b) we have for $t_0 \leq t_1 \leq t_2$:

$$\begin{aligned} |z(t_2) - z(t_1)| &\leq \int_{t_1}^{t_2} |A(s)| |z(s)| ds + \int_{t_1}^{t_2} |f_x(s)| ds \leq \\ &\leq 2\rho \int_{t_1}^{t_2} \psi(s) |A(s)| ds + \int_{t_1}^{t_2} \varphi(s) g(s, 2\rho) ds. \end{aligned}$$

Thus the function in X are equicontinuous. Moreover $\bar{X} \subset B_{\psi, 2\rho}$, in other words $TM(\bar{X}) \subset TM(B_{\psi, 2\rho}) = X \subset \bar{X}$. Then by this and by Lemmas 6 and 7, Theorem 5 and Lemma 8 the operator TM maps \bar{X} into $cf(\bar{X})$ and is upper semicontinuous.

Thus by Ky Fan Theorem, there exists $x \in B_{\psi, 2\rho}(t_0 - \tau)$ such that $x \in TM x$. Clearly such an x is a solution of (a).

Conversely, let $x(t)$ be a ψ -bounded solution of (a). Define:

$$y(t) := x(t) - \int_{t_0}^t Y(t) P_1 Y^{-1}(s) f_x(s) ds + \int_t^{\infty} Y(t) P_2 Y^{-1}(s) f_x(s) ds,$$

where

$$f_x(t) := x'(t) - A(t)x(t) \in F(t, x_t) \text{ a.e. on } [t_0, \infty).$$

It is easy to prove that $y(t)$ is a ψ -bounded solution of (b). It remains to prove that

$$\psi^{-1}(t) |x(t) - y(t)| \in L_p([t_0, \infty)).$$

Since

$$\begin{aligned} \psi^{-1}(t)(x(t) - y(t)) &= \\ &= \int_{t_0}^t \psi^{-1}(t) Y(t) P_1 Y^{-1}(s) f_x(s) ds - \int_t^{\infty} \psi^{-1}(t) Y(t) P_2 Y^{-1}(s) f_x(s) ds, \end{aligned}$$

it is sufficient to show that the terms on the right-hand side belong to $L_p([t_0, \infty))$. By the assumptions of this theorem and from Hölder inequality we get:

$$\begin{aligned} \left| \int_{t_0}^t \psi^{-1}(t) Y(t) P_1 Y^{-1}(s) f_x(s) ds \right| &\leq \int_{t_0}^t |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s)| \varphi(s) g(s, c) ds \leq \\ &\leq |\psi^{-1} Y(t) P_1| \int_{t_0}^t |P_1 Y^{-1}(s)| \varphi(s) |g(s, c)| ds. \end{aligned}$$

From a) and Lemma 3 we have

$$|\psi^{-1} Y(t) P_1| \in L_p([t_0, \infty)).$$

From d) we obtain that this first term belongs to $L_p([t_0, \infty))$.

For the second term we have

$$\begin{aligned} \int_t^\infty |\psi^{-1} Y(t) P_2 Y^{-1}(s)| |f_x(s)| ds &\leq \int_t^\infty |\psi^{-1} Y(t) P_2 Y^{-1}(s)| \varphi(s) g(s, c) ds \leq \\ &\leq \left(\int_t^\infty |\psi^{-1} Y(t) P_2 Y^{-1}(s)|^p ds \right)^{1/p} \left(\int_t^\infty g^{p'}(s, c) ds \right)^{1/p'} \leq \\ &\leq K \left(\int_t^\infty g^{p'}(s, c) ds \right)^{1/p'}. \end{aligned}$$

Thus from ii) of b) and from Lemma 4 we see that this term also belongs to $L_p([t_0, \infty))$.

The proof of the theorem is thus complete.

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