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ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF NONLINEAR DIFFERENCE SYSTEMS

ABSTRACT. In this paper we give sufficient conditions for bounded on $N(n_0)$ and approched finite limit as $n \to \infty$ solutions of the equation (1) with initial condition $x(n_0) = x_0$. We also give conditions for approaches finite limit only part components of the solutions of (1)

KEY WORDS. Difference system, asymptotically constant solution.

INTRODUCTION

On the analogy of differential equations [2] we say that the system of difference equations has asymptotic equilibrium if every solution of this equation with arbitrary initial condition tends to a finite limit as n tends to infinity, and conversely, for every constant vector, there exists solution of the system which tends to this vector together with $n \to \infty$.

The asymptotic behavior of solution of systems of ordinary nonlinear differential equations is treated in many papers by various authors, as *Brauer* and *Wong* [1], *Hallam* and *Heidel* [3], *Švec* [5], *Trench* [6] and others. The asymptotic behavior of difference systems was studied by *Ved* and *Golovina* [7], *Ved* and *Kaparov* [8] dealing with the above problem for special case.

The purpose of this paper is to study behavior of the solutions of the difference equation of the form

(1) $$\Delta x(n) = P(n)F(n, Q(n)x(n), T(Q(n)x(n))),$$

where $n \in N(n_0) = \{n_0, n_0 + 1, \ldots\}$, n_0 is a natural number or zero.

In this paper we give sufficient conditions for bounded on $N(n_0)$ and approched finite limit as $n \to \infty$ solutions of the equation (1) with initial condition $x(n_0) = x_0$. We also give conditions for approaches finite limit only part components of the solutions of (1).

PRELIMINARIES

Δ is the forward difference operator i.e.

$\Delta x(n) = x(n+1) - x(n)$ for any function $x : N(n_0) \to R^k$ (R^k is the k-dimensional real euclidean space with norm $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1,...,x_k)^T$); $P,Q : N(n_0) \to M^k$ (M^k is the space of all $k \times k$ matrices $A = (a_{ij})$ with norm $|A| = \max_j \sum_{i=1}^{k} |a_{ij}|$); $F : N(n_0) \times D \times D \to R^k$ (D is a region in R^k) is for any $n \in N(n_0)$ continuous in the last two arguments, and T is a continuous operator from $\Phi(N,D)$ into $\Phi(N,D)$ ($\Phi = \Phi(N,R^k)$ is the space of all functions from $N(n_0)$ into R^k).

The topology of Φ is the topology of uniform convergence on every set $N_m(n_0) = \{n_0, n_0+1,...,n_0+m\}$, $m = 0,1,2,...$, that is $x_i \to x$ as $i \to \infty$ in Φ iff $\lim_{i\to\infty} |x_i(n) - x(n)| = 0$ uniformly on every set $N_m(n_0)$, $m = 0,1,2,...$.

Note also that Φ is a locally convex space [8] with topology defined by $|x(n)|_m = \sup\{|x(n)| : n \in N_m(n_0)\}$, $m = 0,1,2,...$.

Let $\Phi_1 = \Phi_1(N,R^k)$ be the Banach space in Φ of all bounded functions from $N_m(n_0)$ to R^k with norm $|x|_{\Phi_1} = |x(n)|_{\Phi_1} = \sup\{|x(n)| : n \in N(n_0)\}$.

Denote by Z the set of all solutions $x(n)$ of (1) with initial condition $x(n_0) = x_0$.

We will say that a solution $x \in Z$ has property (0), if

(0) $$x(n) \to c \quad \text{as} \quad n \to \infty,$$

where $c \in R^k$ is a constant vector.

Asymptotic Behavior

1. We now prove our main results.

Theorem 1. Suppose that

(i) $\omega_1, \omega_2 : <0,\infty) \to (0,\infty)$ *are nondecreasing and bounded functions,*

(ii) *there exist two nonnegative functions* $\Psi_1(n)$, $\Psi_2(n)$ *defined on* $N(n_0)$ *such that*

$$\sum_{n=n_0}^{\infty} \Psi_1(n) < \infty, \qquad \sum_{n=n_0}^{\infty} \Psi_2(n) < \infty,$$

(iii) $|P(n)[F(n,Q(n)x,T(Q(n)x)) - F(n,Q(n)y,T(Q(n)y))]| \le$
$\le \Psi_1(n)\omega_1(|x-y|) + \Psi_2(n)\omega_2(|x-y|)$ *for* $n \in N(n_0)$ *and* $x, y \in R^k$,

(iiii) $\sum_{n=n_0}^{\infty} |P(n)F(n,0,0)| < K < \infty.$

Then every solution $x(n) \in Z$ *is defined on* $N(n_0)$ *and for this solution there exists a constant vector* $c \in R^k$ *such that (0) holds.*

Conversely, for every constant vector $c \in R^k$ *there exists a solution* $x(n) \in Z$ *on* $N(n_0)$ *such that (0) holds.*

Proof. a) Every solution $x(n) \in Z$ of (1) can be written in the form

$$(2) \qquad x(n) = x(n_0) + \sum_{s=n_0}^{n-1} P(s)F(s,Q(s)x(s),T(Q(s)x(s))), \qquad n \in N(n_0)$$

where $x(n_0) = x_0$ is a initial condition.

From the assumption of Theorem and (2) we obtain

$$|x(n)| \le |x_0| + \sum_{s=n_0}^{n-1} \{\Psi_1(s)\omega_1(|x(s)|) + \Psi_2(s)\omega_2(|x(s)|)\} + \sum_{s=n_0}^{n-1} |P(s)F(s,0,0)|.$$

Hence it follows that $x(n)$ is bounded on $N(n_0)$. This guarantees that $\sum_{n=n_0}^{\infty} P(n)F(n,Q(n)x(n),T(Q(n)x(n))) < \infty$. Therefore from (2) it follows that $x(n) \to c$ as $n \to \infty$. Hence (0) holds.

b) Let $c \in R^k$ be a constant vector.

Denote $B_\delta = \{u \in R^k : |u|_{\Phi_1} \le \delta\}$, where

$$\delta \ge |c| + K_1 + K_2 + K, \qquad K_1 > \omega_1(a)\sum_{n=n_0}^{\infty} \Psi_1(n), \qquad K_2 > \omega_2(a)\sum_{n=n_0}^{\infty} \Psi_2(n)$$

for every real number $a \ge 0$.

For $x \in B_\delta$ we define the operator

$$(3) \qquad Gx(n) = c - \sum_{s=n}^{\infty} P(s)F(s,Q(s)x(s),T(Q(s)x(s))), \qquad n \in N(n_0).$$

It is evident that B_δ is a convex and closed subset of Φ_1.

The mapping G satisfies the assumptions of the Shauder Theorem. Namely, it satisfies the following:

(i) $GB_\delta \subset B_\delta$. Indeed, if $x \in B_\delta$, then for all $n \in N(n_0)$

$$(4)\quad \left|Gx(n)\right| \le |c| + \sum_{s=n_0}^{\infty}\left|P(s)F(s,Q(s)x(s),T(Q(s)x(s)))\right| \le$$

$$\le |c| + \sum_{n=n_0}^{\infty}\left[\Psi_1(n)\omega_1(|x(n)|) + \Psi_2(n)\omega_2(|x(n)|)\right] +$$

$$+ \sum_{n=n_0}^{\infty}\left|P(n)F(n,0,0)\right| \le$$

$$\le |c| + \omega_1(\delta)\sum_{n=n_0}^{\infty}\Psi_1(n) + \omega_2(\delta)\sum_{n=n_0}^{\infty}\Psi_2(n) + \sum_{n=n_0}^{\infty}\left|P(n)F(n,0,0)\right| \le \delta.$$

(ii) G is continuous on B_δ.

Let $\varepsilon > 0$, and choose $n_1 \in N(n_0)$ so large that

$$\sum_{n=n_1}^{\infty}\Psi_1(n)\omega_1(2\delta) < \varepsilon, \qquad \sum_{n=n_1}^{\infty}\Psi_2(n)\omega_2(2\delta) < \varepsilon.$$

Let $\{x_i(n)\}_{i=1}^{\infty}$ be a sequence in B_δ such that $x_i(n) \to x(n)$ as $i \to \infty$. Since B_δ is a closed set, it follows that $x(n) \in B_\delta$ and for $n \in N(n_0)$ we have

$$|Gx_i(n) - Gx(n)| =$$

$$= \left|\sum_{s=n}^{\infty}P(s)F(s,Q(s)x(s),T(Q(s)x(s))) - \sum_{s=n}^{\infty}P(s)F(s,Q(s)x_i(s),T(Q(s)x_i(s)))\right| \le$$

$$\le \sum_{s=n_0}^{n_1-1}\left|P(s)\left[F(s,Q(s)x(s),T(Q(s)x(s))) - F(s,Q(s)x_i(s),T(Q(s)x_i(s)))\right]\right| +$$

$$+ \sum_{s=n_1}^{\infty}\left|P(s)[F(s,Q(s)x(s),T(Q(s)x(s))) - F(s,Q(s)x_i(s),T(Q(s)x_i(s)))]\right|.$$

From which, by the continuity of F, it follows that

$$\lim_{i\to\infty}\sup_{n\ge n_1}\left|Gx_i(n) - Gx(n)\right| = 0.$$

(iii) GB_δ is precompact.

Now, from the fact that $GB_\delta \subset B_\delta$ it follows that the elements of GB_δ are uniformly bounded (in the norm). It suffices to prove that the elements of GB_δ satisfy Cauchy's condition uniformly on GB_δ.

If $x \in B_\delta$ and $n > m \in N(n_1)$, then we have the following estimate for G:

$$|Gx(n) - Gx(m)| =$$

$$= \left| \sum_{s=m}^{\infty} P(s)F(s,Q(s)x(s),T(Q(s)x(s))) - \sum_{s=n}^{\infty} P(s)F(s,Q(s)x(s),T(Q(s)x(s))) \right| =$$

$$= \left| \sum_{s=m}^{n-1} P(s)F(s,Q(s)x(s),T(Q(s)x(s))) \right| \leq$$

$$\leq \sum_{s=m}^{n-1} \Psi_1(s)\omega_1(\delta) + \sum_{s=m}^{n-1} \Psi_2(s)\omega_2(\delta) + \sum_{s=m}^{n-1} |P(s)F(s,0,0)|.$$

By the assumptions of Theorem the right hand side of this inequality does not dependent on x and tends to zero as $m \to \infty$, so given $\varepsilon > 0$, there exists $n_2 \in N(n_1)$ such that for all $x \in B_\delta$, $n, m \in N(n_2)$

$$|Gx(n) - Gx(m)| < \varepsilon.$$

The Shauder Fixed Point Theorem implies that G has fixed point $\bar{x}(n) \in B_\delta$, which means that $\bar{x}(n)$ is a solution of the equation (3) and also a solution of (1) with initial condition $x(n_0) = x_0$ and

$$\bar{x}(n) = c - \sum_{s=n}^{\infty} P(s)F(s,Q(s)\bar{x}(s),T(Q(s)\bar{x}(s))), \qquad n \in N(n_0). \tag{5}$$

Hence it follows that $x(n) \to c$ as $n \to \infty$. The proof is complete.

Now we consider two systems

$$x(n+1) = A(n)x(n) + f(n, x(n)) \tag{6}$$

and

$$y(n+1) = A(n)y(n) \tag{7}$$

where $\det A(n) \neq 0$ for $n \in N(n_0)$, $x, y \in R^k$, A is $k \times k$ matrix function on $N(n_0)$, $f : N(n_0) \times D \to R^k$ for any $n \in N(n_0)$ is continuous in the last argument.

Denote by Z_1 the set of all solutions $x(n)$ of (6) with initial conditon $x(n_0) = x_0$.

We will say that systems (6) and (7) are asymptotically equivalent if for each solution $x(n) \in Z_1$ defined on $N(n_0)$ there exists a solution $y(n)$ of (7) such that

(8) $|x(n)-y(n)|\to 0$ as $n\to\infty$ and conversely.

In particular, if the system (7) is of the form $\Delta y(n)=0$ and (8) holds, we will say that the system (6) has an asymptotic equilibrium.

Let $X(n)$ be a fundamental matrix for system (7) such that $X(n_0)=I$.

Denote by $X^{-1}(n)$ the inverse matrix to $X(n)$.

Let $Q(n)=X(n)$, $P(n)=X^{-1}(n+1)$, $(P(n)=X^{-1}(n)A^{-1}(n))$.

Theorem 2. Assume that for the matrices P, Q $(Q(n)=X(n);\ P(n)= =X^{-1}(n+1)$ *or* $P(n)=X^{-1}(n)A^{-1}(n))$ *and function f from (6) the hypotheses* (i), (ii) *of Theorem1 hold, and*

(iii) $|P(n)[f(n,Q(n)x)-f(n,Q(n)y)]|\le \Psi_1(n)\omega_1(|x-y|)+\Psi_2(n)\omega_2(|x-y|)$ *for* $n\in N(n_0)$ *and* $x,y\in R^k$,

(iv) $\sum_{n=n_0}^{\infty}|P(n)f(n,0)|<K<\infty.$

Then for every solution $x(n)\in Z_1$ *there exists a constant vector* $c\in R^k$ *such that*

(9) $$x(n)=X(n)z(n),\qquad z(n)\to c \text{ as } n\to\infty$$

holds.

Conversely, for every constant vector $c\in R^k$ *there exists a solution* $x(n)\in Z_1$ *such that (9) holds.*

Proof. By substitution

(10) $$x(n)=X(n)z(n)$$

we can transform every initial problem (6) to the problem (1) of the form

(11) $$\Delta z(n)=X^{-1}(n+1)f(n,X(n)z(n)),\quad z(n_0)=x(n_0)=x_0$$

or

(12) $$\Delta z(n)=X^{-1}(n)A^{-1}(n)f(n,X(n)z(n)).$$

From Theorem 1 and (10) it follows that every solution $x(n)\in Z_1$ is defined on $N(n_0)$ and for this solution there exists a constant vector $c\in R^k$ such that (9) holds, and conversely.

Corollary 1. Assume that the hypotheses of Theorem 2 are satisfied and let all solutions of the system (7) be bounded on $N(n_0)$. Then the systems (6)-(7) are asymptotically equivalent.

Proof. Since all solutions $y(n)$ of (7) are bounded on $N(n_0)$, then for the fundamental matrix $X(n)$ of (7) we have

$$|X(n)| \leq M \text{ for } n \in N(n_0),$$

where M is a constant.

If a is a constant vector, then the vector valued function $y(n) = X(n)a$ is a solution of (7).

Let $x(n) \in Z_1$ be a solution of (6) on $N(n_0)$, Theorem 2 implies that for the solution x of (6) there exists a constant vector c such that (9) holds for $n \in N(n_0)$.

Consider the solution $y(n)$ of (7) in the form $y(n) = X(n)c$. Then we get

$$|x(n) - y(n)| = |X(n)z(n) - X(n)c| \leq |X(n)||z(n) - c| \leq \leq M|z(n) - c| \to 0 \quad \text{as} \quad n \to \infty. \tag{13}$$

Conversely, for each solution $y(n)$ of (7) there exists a constant vector $c \in R^k$ such that $y(n) = X(n)c$, and for c there exists $x(n) \in Z_1$ such that (9) holds. Therefore (13) holds.

Definition. If (7) has a constant vector function as soıution and (8) holds, then the system (6) has an asymptotical equilibrium.

Corollary 2. Assume that the hypotheses of Theorem 2 *are satisfied and furthermore the system (7) has a constant vector function as solution . Then the system (6) has an asymptotical equilibrium.*

Proof. System (7) has asymptotical equilibrium. We can find a fundamen-tal matrix $X(n)$ such that $X(n) \to I$ as $n \to \infty$. Then from (9) $x(n) \to Ic = c$ as $n \to \infty$. Hence $|x(n) - c| \to 0$ as $n \to \infty$.

Corollary 3. Assume that the hypotheses of Theorem 2 are satisfied and solution $y(n) = 0$ of (7) is exponentially asymptotically stable. Then for every solution $x \in Z_1$ we have

$$x(n) \to 0 \text{ as } n \to \infty.$$

Proof. The solution $y(n)=0$ of (7) is exponentially asymptotically stable. There exists constants $M>0$ and η, $0<\eta<1$ such that

$$|X(n)|\leq M\eta^{n-n_0} \quad \text{for} \quad n\in N(n_0)\,.$$

From (9) for $n\in N(n_0)$ we get

$$|x(n)|\leq|X(n)||z(n)|\leq M\eta^{n-n_0}|z(n)|\,.$$

Hence it follows that $x(n)\to 0$ as $n\to\infty$.

Remark. If the solution $y(n)$ of (7) is asymptotically stable, then the assertion of Corollary 3 is true.

2. In this section we give the conditions under which for each solution $x(n)$ of (6) there exists a solution $y(n)$ of (7) with property

$$x(n)=y(n)+o(|\Phi(n,n_0)|) \quad \text{for} \quad n\in N(n_0) \tag{14}$$

where $\Phi(n,n_0)$ is a fundamental matrix of (7).

Remark. The matrix

$$\Phi(n,s)=X(n)X^{-1}(s)$$

satisfies the same equation as $X(n)$, i.e. $\Phi(n+1,s)=A(n)\,\Phi(n,s)$.
Moreover, $\Phi(n,n)=I$ for all $n\geq n_0$. We shall call Φ the fundamental matrix. Other properties of the matrix Φ are

(i) $\Phi(n,s)\Phi(s,t)=\Phi(n,t)$,

(ii) if $\Phi^{-1}(n,s)$ exists then $\Phi^{-1}(n,s)=\Phi(s,n)$.

Lemma 1. Let $u(n)$, $h(n)$ and $\gamma(n)$ be real-valued nonnegative functions defined on $N(n_0)$, for which the ineqality

$$u(n)\leq\gamma(n)\left[c+\sum_{s=n_0}^{n-1}h(s)u(s)\right] \tag{15}$$

holds for all $n\in N(n_0)$, where c is a nonnegative constant, then

$$u(n)\leq\gamma(n)c\exp\left(\sum_{s=n_0}^{n-1}h(s)\gamma(s)\right) \quad \textit{for all } n\in N(n_0). \tag{16}$$

We omit the proof of this Lemma because of its simplicity.

Theorem 3. Suppose that

1° $f: N(n_0)\times R^k \to R^k, \quad |f(n,y)| \le g(n)|y|$

for all $n\in N(n_0),\ y\in R^k$, *where* $g(n)$ *is a positive function on* $N(n_0)$,

2° $\sum_{j=n_0}^{\infty} |\Phi(j,n_0)| \left|\Phi^{-1}(j+1,n_0)\right| g(j) < \infty$

where $\Phi(n,n_0)$ *is the fundamental matrix of the equation (7),* $\Phi(n_0,n_0)=I$, *then for every solution of the equation (6) there is a constant vector b such that*

(17) $$x(n)=\Phi(n,n_0)b+o(|\Phi(n,n_0)|) \quad as \quad n\to\infty.$$

Proof. The solution $x(n)$ of (6) can now be written as

(18) $$x(n)=\Phi(n,n_0)\left[x(n_0)+\sum_{j=n_0}^{n-1}\Phi^{-1}(j+1,n_0)f(j,x(j))\right].$$

From 1° we obtain

$$|x(n)| \le |\Phi(n,n_0)| \left[|x_0| + \sum_{j=n_0}^{n-1} g(j)\left|\Phi^{-1}(j+1,n_0)\right| |x(j)|\right].$$

By Lemma 1 we have

$$|x(n)| \le |x_0| \, |\Phi(n,n_0)| \exp\left(\sum_{j=n_0}^{n-1} |\Phi(j,n_0)| \left|\Phi^{-1}(j+1,n_0)\right| g(j)\right)$$

for $n\in N(n_0)$.

From the above and 2^o it follows that

$$\sum_{j=n_0}^{\infty}\Phi^{-1}(j+1,n_0)f(j,x(j)) < \infty.$$

Indeed

$$\sum_{j=n_0}^{n-1}\left|\Phi^{-1}(j+1,n_0)\right| |f(j,x(j))| \le \sum_{j=n_0}^{n-1}\left|\Phi^{-1}(j+1,n_0)\right| g(j)\,|x(j)| \le$$

$$\le K|x_0| \sum_{j=n_0}^{n-1}\left|\Phi^{-1}(j+1,n_0)\right| |\Phi(j,n_0)| g(j),$$

where $K=\exp\left(\sum_{j=n_0}^{\infty}|\Phi(j,n_0)| \left|\Phi^{-1}(j+1,n_0)\right| g(j)\right)$.

By setting

(19) $$b = x_0 + \sum_{j=n_0}^{\infty} \Phi^{-1}(j+1,n_0) f(j,x(j)),$$

$$v(n) = -\sum_{k=n}^{\infty} \Phi^{-1}(k+1,n_0) f(k,x(k))$$

from (18) we obtain $x(n) = \Phi(n,n_0)b + \Phi(n,n_0)v(n)$.
Since $b = \text{const}$ and $v(n)$ approach zero as $n \to \infty$, then $x(n) = \Phi(n,n_0)b + o(|\Phi(n,n_0)|)$.

Remark. Condition 2^o in Theorem 3 can be replaced by

$$\sum_{n=n_0}^{\infty} g(n) < \infty \text{ and } |\Phi(n,n_0)|\left|\Phi^{-1}(j+1,n_0)\right| \le N = \text{const}$$

for $n_0 \le j+1 \le n < \infty$.

Corollary. If the condition

$$\sum_{j=n_0}^{\infty} |\Phi(j,n_0)|\left|\Phi^{-1}(j+1,n_0)\right|\left|B(j) - A(j)\right| < \infty$$

holds, then for every solution $x(n)$ *of the system* $x(n+1) = B(n)x(n)$, *where B is* $k \times k$ *matrix there is a constant vector b such that*

$$x(n) = \Phi(n,n_0)b + o(|\Phi(n,n_0)|) \text{ as } n \to \infty.$$

3. In this section we consider the following system of difference equation

(20) $$\Delta x(n) = F(n,x(n)).$$

For this equation we give the conditions under which only part components of the solutions of (20) approches finite limit.

Let R^k be the direct sum

$$R^k = E_1 + E_2.$$

We denote by P_i $(i = 1,2)$ the corresponding projection, i.e.

$$P_i R^k = E_i \quad (i = 1,2),$$

$$P_1 + P_2 = I, \quad P_i^2 = P_i, \quad P_1 P_2 = 0.$$

Simultaneously with equation (20) we consider the equation

(21) $$\Delta P_1 x(n) = P_1 F(n, P_1 x(n)).$$

Theorem 4. Suppose that

(i) $f : N(n_0) \times D \to D^k$ *is continuous in the last argument for any* $n \in N(n_0)$,

(ii) $|P_1 F(n,x) - P_1 F(n, P_1 x)| \le \varphi(n) |P_1 x|$, $\varphi(n)$ *is a nonnegative function on* $N(n_0)$, $x \in D$,

(iii) $|P_1 F(n, P_1 x) - P_1 F(n, P_1 y)| \le \psi(n) |P_1 x - P_1 y|$, $\psi(n)$ *is a nonnegative function on* $N(n_0)$, $x, y \in D$,

(iv) $\sum_{n=n_0}^{\infty} \varphi(n) \le \varphi_1 < \infty$, $\sum_{n=n_0}^{\infty} \psi(n) \le \psi_1 < \infty$, $\sum_{n=n_0}^{\infty} |P_1 F(n_0, 0)| \le F_1 < \infty$.

Then the part components of the solution of (20) tends to finite limit as $n \to \infty$.

Proof. Equation (20) and (21) can be written in the form

(22) $$x(n) = x_0 + \sum_{s=n_0}^{n-1} F(s, x(s))$$

and

(23) $$P_1 y(n) = P_1 y_0 + \sum_{s=n_0}^{n-1} P_1 F(s, P_1 y(s)),$$

where $P_1 y(n)$ denote the solution of (21).

Let $x = x(n)$ be the solution of (20) such that $P_1 y_0 = P_1 x_0$. Then using the assumptions of Theorem, it is easy to obtain

(24) $$|P_1 x(n) - P_1 y(n)| \le \sum_{s=n_0}^{n-1} |P_1 F(s, x(s)) - P_1 F(s, P_1 y(s))| \le$$
$$\le \sum_{s=n_0}^{n-1} (|P_1 F(s, x(s)) - P_1 F(s, P_1 x(s))| +$$
$$+ |P_1 F(s, P_1 x(s)) - P_1 F(s, P_1 y(s))|) \le$$
$$\le \sum_{s=n_0}^{n-1} (\varphi(s) |P_1 x(s)| + \psi(s) |P_1 x(s) - P_1 y(s)|);$$

(25) $$|P_1 x(n)| \le |P_1 x(n) - P_1 y(n)| + |P_1 y(n)|;$$

$$|P_1y(n)|\le|P_1y_0|+\sum_{s=n_0}^{n-1}|P_1F(s,P_1y(s))|\le$$
$$\le|P_1y_0|+\sum_{s=n_0}^{n-1}|P_1F(s,P_1y(s))-P_1F(s,0)|+\sum_{s=n_0}^{n-1}|P_1F(s,0)|\le$$
$$\le|P_1y_0|+\sum_{s=n_0}^{n-1}|P_1F(s,0)|+\sum_{s=n_0}^{n-1}\psi(s)|P_1y(s)|.$$

Applying the discrete version of Gronwall inequality [4] we have

$$(26)\qquad |P_1y(n)|\le|P_1y_0|\exp\left(\sum_{s=n_0}^{n-1}\psi(s)\right)+\sum_{s=n_0}^{n-1}|P_1F(s,0)|\exp\left(\sum_{r=s+1}^{n-1}\psi(r)\right)\le$$
$$\le|P_1y_0|e^{\psi_1}+e^{\psi_1}F_1.$$

Employing inequalities (25) and discrete version of Gronwall inequality [4] from (23) we get

$$(27)\quad |P_1x(n)-P_1y(n)|\le\sum_{s=n_0}^{n-1}(\varphi(s)|P_1x(s)|+\psi(s)|P_1x(s)-P_1y(s)|)\le$$
$$\le\sum_{s=n_0}^{n-1}\{(\varphi(s)|P_1y(s)|+(\varphi(s)+\psi(s))|P_1x(s)-P_1y(s)|)\le$$
$$\le\varphi_1(|P_1y_0|+F_1)e^{\psi_1}+\sum_{s=n_0}^{n-1}(\varphi(s)+\psi(s))|P_1x(s)-P_1y(s)|\le$$
$$\le\varphi_1(|P_1y_0|+F_1)e^{\psi_1}\exp\left(\sum_{s=n_0}^{n-1}(\varphi(s)+\psi(s))\right).$$

It now follows that

$$|P_1x(n)|\le|P_1x_0|+\sum_{s=n_0}^{n-1}|P_1F(s,x(s))|\le$$
$$\le|P_1x_0|+\sum_{s=n_0}^{n-1}(|P_1F(s,x(s))-P_1F(s,P_1x(s))|+P_1F(s,P_1x(s))|)\le$$
$$\le|P_1x_0|+\sum_{s=n_0}^{n-1}(\varphi(s)|P_1x(s)|+|P_1F(s,P_1x(s))-P_1F(s,P_1y(s))|+$$
$$+|P_1F(s,P_1y(s))|)\le$$

$$\leq |P_1 x_0| + \sum_{s=n_0}^{n-1} |P_1 F(s,0)| + \sum_{s=n_0}^{n-1} \psi(s) |P_1 y(s)| +$$

$$+ \sum_{s=n_0}^{n-1} \psi(s) |P_1 x(s) - P_1 y(s)| + \sum_{s=n_0}^{n-1} \varphi(s) |P_1 x(s)| \leq$$

$$\leq |P_1 x_0| + F_1 + \psi_1 (|P_1 y_0| + F_1) e^{\psi_1} +$$

$$+ \varphi_1 (|P_1 y_0| + F_1) e^{\psi_1} e^{(\varphi_1 + \psi_1)} + \sum_{s=n_0}^{n-1} \varphi(s) |P_1 x(s)| =$$

$$= (|P_1 x_0| + F_1) \left[1 + \psi_1 e^{\psi_1} + \varphi_1 e^{\psi_1} e^{(\varphi_1 + \psi_1)} \right] + \sum_{s=n_0}^{n-1} \varphi(s) |P_1 x(s)| =$$

$$= c + \sum_{s=n_0}^{n-1} \varphi(s) |P_1 x(s)|.$$

From which we conclude that

$$|P_1 x(n)| \leq c \exp \sum_{s=n_0}^{n-1} \varphi(s).$$

Hence $P_1 x(n)$ is bounded on $N(n_0)$ and $\sum_{n=n_0}^{\infty} |P_1 F(n, x(n))| < \infty$.

So that from (22) $P_1 x(n)$ tends to finite limit as $n \to \infty$, i.e.

$$\lim_{n \to \infty} P_1 x(n) = P_1 x_0 + \sum_{n=n_0}^{\infty} P_1 F(n, x(n)).$$

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