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B.G. PACHPATTE

INTEGRAL INEQUALITIES ASSOCIATED TO THE ZEROS OF SOLUTIONS OF CERTAIN HIGHER ORDER DIFFERENTIAL EQUATIONS

ABSTRACT: The aim of the present paper os to establish some new integral inequalities associated to the zeros of the solutions of certain higher order differential equations by using elementary analysis. The inequalities obtained here can be effectively used in the study of qualitative behavior of solutions of the corresponding equations.

KEY WORDS: Integral inequalities, zeros of solutions, Higher order differential equations, qualitative behavior, Hölder's inequality.

1. INTRODUCTION

In this paper we consider the differential equations of the form:

(A) $$(r_{n-1}(t)(r_{n-2}(t)(\dots(r_2(t)(r_1(t)|y'(t)|^{\alpha-1}y'(t))')'\dots)')'+ \\ +q(t)|y(t)|^{\beta-1}y(t)=0,$$

(B) $$(r_{n-1}(t)(r_{n-2}(t)(\dots(r_2(t)(r_1(t)|y(t)|^{p}|y'(t)|^{k-2}y'(t))')'\dots)')')'+ \\ +q(t)|y(t)|^{p+k-2}y(t)=0,$$

where $n\geq 2$, and also the differential equations of the forms:

(C) $$(m(t)(r(t)|y'(t)|^{\alpha-1}y'(t))^{(n-1)})^{(n)}+q(t)|y(t)|^{\beta-1}y(t)=0,$$

(D) $$(m(t)(r(t)|y(t)|^{p}|y'(t)|^{(k-2)}y'(t))^{(n-1)})^{(n)}+q(t)|y(t)|^{p+k-2}y(t)=0,$$

where $n\geq 1$. In equations (A) – (D) we assume that $t\in I=[t_0,\infty)$, $t_0\geq 0$, and I contains the points a and b $(a<b)$, $\alpha\geq 1$, $\beta\geq 1$, $p\geq 0$, $k\geq 2$ are real constants and $k>b$, the functions $r_i,r,m:I\to R$, $R=(-\infty,\infty)$, $i=1,2,\dots,n-1$ are sufficiently smooth and $r_i(t)>0$, $r(t)>0$, $m(t)>0$, the function $q:I\to R$ is continuous.

A large number of papers have treated the various special versions and variants of equations (A) – (D) and a number of results have been obtained, see[1-16] and the references cited therein. For the case $n=2$ and $\alpha=1$, the equation (A) reduces to the well-known Emden-Fowler equation which has been extensively investigated from various viewpoints. For the case $n=2,\ p=0$, the equation (B) reduces to the equation recently studied by the authors in [5], see also [6]. The special versions of equations (C) and (D) with (i) $\alpha=1,\ \beta=1,\ r(t)=1$, (ii) $p=0,\ k=2,\ r(t)=1$ are recently studied by the authors in [7], see also [8]. Concerning the existence of solutions of equations of the type (A) – (D), we refer the papers [2, 5 – 8, 10] and the references given therein. Our objective here is to establish some new integral inequalities which not only relates points a and b in I at which the solutions of equations (A) – (D) have zeros but also any point $c\in(a,b)$ where the solutions of (A) – (D) are maximized. The inequalities that we propose here can be effectively used in the study of the qualitative nature of the solutions of equations (A) – (D). Here we give some such applications to convey the importance of our results to the literature.

2. STATEMENT OF RESULTS

We shall use the following notations for simplification of details of presentation:

(*) $$E[t,\bar r,h(s)]=E[t,r_2,r_3,r_4,\dots,r_{n-1},h(s)]=$$

$$=\frac{1}{r_2(t)}\int_t^{\alpha_1}\frac{1}{r_3(s_2)}\int_{s_2}^{\alpha_2}\frac{1}{r_4(s_3)}\cdots\int_{s_{n-3}}^{\alpha_{n-3}}\frac{1}{r_{n-1}(s_{n-2})}\int_{s_{n-2}}^{\alpha_{n-2}}h(s)ds\,ds_{n-2}\dots ds_3\,ds_2\,,$$

(**) $$H[t,m,h(s)]=\int_t^{\alpha_2}\int_{s_3}^{\alpha_3}\dots\int_{s_{n-1}}^{\alpha_{n-1}}\frac{1}{m(s_n)}\int_{s_n}^{\alpha_n}\int_{s_{n+1}}^{\alpha_{n+1}}\dots\times$$

$$\times\int_{s_{2n-1}}^{\alpha_{2n-1}}h(s)ds\,ds_{2n-1}\dots ds_{n+1}ds_n ds_{n-1}\dots ds_3,$$

where $t\in I$, $h(t)$, $r_i(t)$, $m(t)$ are real-valued continuous functions defined on I, $r_i(t)>0$, $m(t)>0$, and α_i are some suitable points in I. We denote respectively by $\overline{E}[t,\bar r,h(s)]$, $\overline{H}[t,m,h(s)]$ the integrales on the right sides of (*), (**) when the upper limits α_i are all replaced by the largest of α_i. For example, if α_1 is the largest of all α_i in (*), then $\overline{E}[t,\bar r,h(s)]$ is given by the integral on the right side of (*) when all the upper limits of integrals are replaced by α_1.

Our main results are given in the following theorems.

Theorem 1. (i) *Let* $\alpha_1 > \alpha_2 > \ldots > \alpha_{n-2}$ *be respectively zeros of*

$$(r_1(t)|y'(t)|^{\alpha-1}y'(t))',\ (r_2(t)(r_1(t)|y'(t)|^{\alpha-1}y'(t))')',\ \ldots,(r_{n-2}(t)(r_{n-3}(t)\times$$
$$\times(\ldots(r_2(t)(r_1(t)|y'(t)|^{\alpha-1}y'(t))')'\ldots)')')',$$

where $y(t)$ *is a solution of* (A)*, let* $a<\alpha_{n-2}$ *and* $b>\alpha_1$ *be zeros of* $y(t)$ *and* $|y(t)|$ *is maximized in* $c\in(a,b)$. *Then*

$$1\le M^{\beta-\alpha}\left(\int_a^b r_1^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_a^b \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right), \tag{1}$$

$$1\le 2^{\alpha+1}M^{\beta-\alpha}\left(\int_a^c r_1^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_a^c \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right), \tag{2}$$

$$1\le 2^{\alpha+1}M^{\beta-\alpha}\left(\int_c^b r_1^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_c^b \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right), \tag{3}$$

where $M=\max|y(t)|=|y(c)|,\ c\in(a,b)$.

(ii) *Let* $\alpha_1 > \alpha_2 > \ldots > \alpha_{n-2}$ *be respectively zeros of*

$$(r_1(t)|y(t)|^p|y'(t)|^{k-2}y'(t))',\ (r_2(t)(r_1(t)|y(t)|^p|y'(t)|^{k-2}y'(t))')',$$
$$\ldots,(r_{n-2}(t)(r_{n-1}(t)(\ldots(r_2(t)(r_1(t)|y(t)|^p|y'(t)|^{k-2}y'(t))')'\ldots)')')'$$

where $y(t)$ *is a solution of (B), let* $a<\alpha_{n-2}$ *and* $b>\alpha_1$ *be zeros of* $y(t)$ *and* $|y(t)|$ *is maximized in* $c\in(a,b)$. *Then*

$$1\le\left(\int_a^b r_1^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_a^b \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right), \tag{4}$$

$$1\le 2^{k}\left(\int_a^c r_1^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_a^c \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right), \tag{5}$$

$$1\le 2^{k}\left(\int_c^b r_1^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_c^b \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right). \tag{6}$$

Theorem 2. (i) *Let* $\alpha_2>\alpha_3>\ldots>\alpha_{n-1}>\alpha_n>\alpha_{n+1}>\ldots>\alpha_{2n-1}$ *be respectively zeros of*

$$(r(t)|y'(t)|^{\alpha-1}y'(t))',\ (r(t)|y'(t)|^{\alpha-1}y'(t))'',\ \ldots,(r(t)|y'(t)|^{\alpha-1}y'(t))^{(n-2)},$$
$$(r(t)|y'(t)|^{\alpha-1}y'(t))^{(n-1)},\ (m(t)(r(t)|y'(t)|^{\alpha-1}y'(t))^{(n-1)})',\ldots,\ (m(t)(r(t)\times$$
$$\times|y'(t)|^{\alpha-1}y'(t))^{(n-1)})^{(n-1)},$$

where $y(t)$ *is a solution of* (C), *let* $a<\alpha_{2n-1}$ *and* $b>\alpha_2$ *be zeros of* $y(t)$ *and* $|y(t)|$ *is maximized in* $c\in(a,b)$. *Then*

$$1\le M^{\beta-\alpha}\left(\int_a^b r^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_a^b \overline{H}[s_1,m,|q(s)|]ds_1\right), \tag{7}$$

$$1\le 2^{\alpha+1}M^{\beta-\alpha}\left(\int_a^c r^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_a^c \overline{H}[s_1,m,|q(s)|]ds_1\right), \tag{8}$$

$$1\le 2^{\alpha+1}M^{\beta-\alpha}\left(\int_c^b r^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha}\left(\int_c^b \overline{H}[s_1,m,|q(s)|]ds_1\right), \tag{9}$$

where $M=\max|y(t)=|y(c)|,\ c\in(a,b)$.

(ii) *Let* $\alpha_2>\alpha_3>\ldots>\alpha_{n-1}>\alpha_n>\alpha_{n+1}>\ldots>\alpha_{2n-1}$ *be respectively zeros of*

$$(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t))',\ (r(t)|y(t)|^p|y'(t)|^{k-2}y'(t))'',\ldots,$$
$$(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t))^{(n-2)},\ (r(t)|y(t)|^p|y'(t)|^{k-2}y'(t))^{(n-1)},(m(t)(r(t)\times$$
$$\times|y(t)|^p|y'(t)|^{k-2}y'(t))^{(n-1)})',\ldots,\ (m(t)(r(t)|y(t)|^p|y'(t)|^{k-2}y'(t))^{(n-1)})^{(n-1)},$$

where $y(t)$ *is a solution of* (D), *let* $a<\alpha_{2n-1}$ *and* $b>\alpha_2$ *be zeros of* $y(t)$ *and* $|y(t)|$ *is maximized in* $c\in(a,b)$. *Then*

$$1\le\left(\int_a^b r^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_a^b \overline{H}[s_1,m,|q(s)|]ds_1,\right), \tag{10}$$

$$1\le 2^{k}\left(\int_a^c r^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_a^c \overline{H}[s_1,m,|q(s)|]ds_1\right), \tag{11}$$

$$1\le 2^{k}\left(\int_c^b r^{-(1/(k-1))}(s_1)ds_1\right)^{k-1}\left(\int_c^b \overline{H}[s_1,m,|q(s)|]ds_1\right). \tag{12}$$

Our next result deals with the application of the inequality (1) given in Theorem 1.

Theorem 3. If

$$(13)\qquad \int_t^{\infty} r_1^{-(1/\alpha)}(s_1)ds_1 < \infty, \qquad \int_t^{\infty} \overline{E}\left[s_1, \overline{r}, | q(s) |\right] ds_1 < \infty,$$

then every oscillatory of (A) *converges to zero at* $t \to \infty$.

Remark 1. We note that the results analogous to that of given in our Theorem 3 can very easily be extended to the equations (B) – (D). The precise formulations of such results is very close to that of Theorem 3 with suitable modifications. Here we do not discuss the details.

3. Proof of Theorem 1

(i) Integrating $n-2$ times Eq (A), by hypotheses, we get

$$(14)\qquad (-1)^{n-2}(r_1(t)\,|\,y'(t)\,|^{\alpha-1}\,y'(t))' + E\left[t, \overline{r}, q(s)\,|\,y(s)\,|^{\beta-1}\,y(s)\right] = 0.$$

Let $M = |\,y(c)\,|$, $c \in (a,b)$. Since $y(a) = y(b) = 0$, it is easy to observe that

$$(15)\qquad M^2 = y^2(c) = 2\int_a^c y(s_1)y'(s_1)ds_1,$$

$$(16)\qquad M^2 = y^2(c) = -2\int_c^b y(s_1)y'(s_1)ds_1.$$

From (15) and (16) we observe that

$$(17)\qquad M^2 \le \int_a^b |\,y(s_1)\,|\,|\,y'(s_1)\,|ds_1 =$$

$$= \int_a^b \left(r_1^{-(1/(\alpha+1))}(s_1)\,|\,y(s_1)\,|\right)\left(r_1^{(1/(\alpha+1))}(s_1)\,|\,y'(s_1)\,|\right)ds_1.$$

Now by using the Hölder's inequality on the right side of (18) with indices $(\alpha+1)/\alpha$, $(\alpha+1)$; performing integration by parts and using the facts that $y(t)$ is a solution of (A) and satisfies the equivalent integral equation (14), $y(a) = y(b) = 0$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_{n-1}$, we observe that

$$(18)\qquad M^2 \le \left(\int_a^b (r_1^{-(1/\alpha)}(s_1)\,|\,y(s_1)\,|^{(\alpha+1)/\alpha}ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(\int_a^b r_1(s_1)\,|\,y'(s_1)\,|^{\alpha+1}ds_1\right)^{1/(\alpha+1)} =$$

$$=\left(\int_a^b r_1^{-(1/\alpha)}(s_1)\,|\,y(s_1)\,|^{(\alpha+1)/\alpha}ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(\int_a^b (r_1(s_1)\,|\,y'(s_1)\,|^{\alpha-1}y'(s_1))y'(s_1)ds_1\right)^{1/(\alpha+1)} =$$

$$=\left(\int_a^b r_1^{-(1/\alpha)}(s_1)\,|\,y(s_1)\,|^{(\alpha+1)/\alpha}ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(-\int_a^b (r_1(s_1)\,|\,y'(s_1)\,|^{\alpha-1}y'(s_1))'y(s_1)ds_1\right)^{1/(\alpha+1)} \leq$$

$$\leq\left(\int_a^b r_1^{-(1/\alpha)}(s_1)\,|\,y(s_1)\,|^{(\alpha+1)/\alpha}ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(\int_a^b |(r_1(s_1)\,|\,y'(s_1)\,|^{(\alpha-1)}y'(s_1))'||\,y(s_1)\,|ds_1\right)^{1/(\alpha+1)} \leq$$

$$\leq\left(\int_a^b r_1^{-(1/\alpha)}(s_1)\,|\,y(s_1)\,|^{(\alpha+1)/\alpha}ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(\int_a^b |\,y(s_1)\,|\overline{E}\left[s_1,\overline{r},|\,q(s)\,||\,y(s)\,|^{\beta-1}|\,y(s)\,|\right]ds_1\right)^{1/(\alpha+1)} \leq$$

$$\leq M(M)^{(\beta+1)/(\alpha+1)}\left(\int_a^b (r_1^{-(1/\alpha)}(s_1)ds_1\right)^{\alpha/(\alpha+1)} \times$$

$$\times\left(\int_a^b \overline{E}\left[s_1,\overline{r},|\,q(s)\,|\right]ds_1\right)^{1/(\alpha+1)}$$

Dividing both sides of (18) by M^2 and then raising the power $\alpha+1$ to both sides of the resulting inequality we get the required inequality in (1).

From (15) and (16) we observe that

(19) $$M^2 \le 2\int_a^c |y(s_1)||y'(s_1)|ds_1 =$$

$$= 2\int_a^c (r_1^{-(1/(\alpha+1))}(s_1)|y(s_1)|)(r_1^{(1/(\alpha+1))}(s_1)|y'(s_1)|)ds_1,$$

(20) $$M^2 \le 2\int_c^b |y(s_1)||y'(s_1)|ds_1 =$$

$$= 2\int_c^b (r_1^{-(1/(\alpha+1))}(s_1)|y(s_1)|)(r_1^{(1/(\alpha+1))}(s_1)|y'(s_1)|)ds_1.$$

Now the inequalities (2) and (3) follows by using the Hölder's inequality on the right sides of (19) and (20) with indices $(\alpha+1)/\alpha,\ \alpha+1$; performing the integration by parts; the facts that $y(t)$ is a solution of (A) and hence it also satisfies the equivalent integral equation (14), $y(a)=y(b)=0,\ \ y'(c)=0$; $\alpha_1>\alpha_2>\ldots>\alpha_{n-2}$ and following the last argument as given above in the proof of inequality (1).

(ii) Integrating $n-2$ times Eq (B), by hypotheses, we get

(21) $$(-1)^{n-2}(r_1(t)|y(t)|^p|y'(t)|^{k-2}\,y'(t))' + E\left[t,\bar{r},q(s)|y(s)|^{p+k-2}\,y(t)\right] = 0.$$

By following the proof of part (i), from the hypotheses we have (15) and (16). From (15) and (16) we observe that

(22) $$M^2 \le \int_a^b |y(s_1)||y'(s_1)|\,ds_1 =$$

$$= \int_a^b (r_1^{-(1/k)}(s_1)|y(s_1)|^{1-p/k})(r_1^{(1/k)}(s_1)|y(s_1)|^{p/k}|y'(s_1)|)\,ds_1.$$

Now by using the Hölder's inequality on the right side of (22) with indices $k/(k-1),\ k$; performing the integration by parts and using the facts that $y(t)$ is a solution of (B) and it satisfies the equivalent integral equation (21), $y(a)=y(b)=0$ and $\alpha_1>\alpha_2>\ldots>\alpha_{n-2}$, we observe that

(23) $$M^2 \le \left(\int_a^b r_1^{-(1/(k-1))}(s_1)|y(s_1)|^{(k-p)/(k-1)}ds_1\right)^{(k-1)/k} \times$$

$$\times\left(\int_a^b r_1(s_1)|y(s_1)|^p|y'(s_1)|^k ds_1\right)^{1/k} =$$

$$=\left(\int_a^b r_1^{-(1/(k-1))}(s_1)|y(s_1)|^{(k-p)/(k-1)} ds_1\right)^{(k-1)/k} \times$$

$$\times\left(\int_a^b (r_1(s_1)|y(s_1)|^p|y'(s_1)|^{k-2} y'(s_1))y'(s_1) ds_1\right)^{1/k} =$$

$$=\left(\int_a^b r_1^{-(1/(k-1))}(s_1)|y(s_1)|^{(k-p)/(k-1)} ds_1\right)^{(k-1)/k} \times$$

$$\times\left(-\int_a^b (r_1(s_1)|y(s_1)|^p|y'(s_1)|^{k-2} y'(s_1))' y(s_1) ds_1\right)^{1/k} \leq$$

$$\leq\left(\int_a^b r_1^{-(1/(k-1))}(s_1)|y(s_1)|^{(k-p)/(k-1)} ds_1\right)^{(k-1)/k} \times$$

$$\times\left(\int_a^b |r_1(s_1)|y(s_1)|^p|y'(s_1)|^{k-2} y'(s_1))'||y(s_1)| ds_1\right)^{1/k} \leq$$

$$\leq\left(\int_a^b r_1^{-(1/(k-1))}(s_1)|y(s_1)|^{(k-p)/(k-1)} ds_1\right)^{(k-1)/k} \times$$

$$\times\left(\int_a^b |y(s_1)|\overline{E}\left[s_1, \overline{r}, |q(s)||y(s)|^{p+k-2}|y(s)|\right] ds_1\right)^{1/k} \leq$$

$$\leq (M)^{(k-p)/k}(M)^{(p+k)/k}\left(\int_a^b r_1^{-(1/(k-1))}(s_1) ds_1\right)^{(k-1)/k} \times$$

$$\times\left(\int_a^b \overline{E}\left[s_1, \overline{r}, |q(s)|\right] ds_1\right)^{1/k}.$$

Dividing both sides of (23) by M^2 and then raising the power k to both sides of the resulting inequality we get the required inequality in (4).

The inequalities (5) and (6) follow in similar fashion as indicated in the proofs of inequalities (2) and (3) and the proof of the inequality (4) given above with suitable modifications. The proof is complete.

4. Proof of Theorem 2

Integrating $2n-2$ times Eq (C) and (D), by corresponding hypotheses, we get

(24) $(-1)^{2n-2}(r(t)|y'(t)|^{\alpha-1}y'(t))' + H[t,m,q(s)|y(s)|^{\beta-1}y(s)] = 0,$

(25) $(-1)^{2n-2}(r(t)|y(t)|^{p}|y'(t)|^{k-2}y'(t))' + H[t,m,q(s)|y(s)|^{p+k-2}y(s)] = 0.$

Now by following the proof of Theorem 1 with suitable modifications, we get the desired inequalities in (7) – (9) and (10) – (12). This completes the proof.

5. Proof of Theorem 3

Let $y(t)$ be an arbitrary solution of (A) and suppose to the contrary that $\lim_{t\to\infty} y(t) \neq 0$. Then

$$\liminf_{t\to\infty} |y(t)| \neq 0,$$

and for some positive d,

$$\limsup_{t\to\infty} |y(t)| > 2d.$$

Due to the oscillatory nature of $y(t)$; $(r_1(t)|y'(t)|^{\alpha-1}y'(t))'$, $(r_2(t)(r_1(t)\times \times|y'(t)|^{\alpha-1}y'(t))')', \ldots, (r_{n-2}(t)(r_{n-1}(t)(\ldots(r_2(t)(r_1(t)|y'(t)|^{\alpha-1}y'(t))')'\ldots)')')'$, must be oscillatory, see [3, p. 165]. Let T be large enough and $T < t_1 < \alpha_{n-2} < \ldots < \alpha_2 < \alpha_1 < T_0$ be points such that

$$y(t_1) = 0,$$

$$(r_{n-2}(\alpha_{n-2})(r_{n-3}(\alpha_{n-2})(\ldots(r_2(\alpha_{n-2})(r_1(\alpha_{n-2})|y'(\alpha_{n-2})|^{\alpha-1}y'(\alpha_{n-2}))')'\ldots)')')' = 0,$$

$$\vdots$$

$$(r_2(\alpha_2)(r_1(\alpha_2)|y'(\alpha_2)|^{\alpha-1}y'(\alpha_2))')' = 0,$$

$$(r_1(\alpha_1)|y'(\alpha_1)|^{\alpha-1}y'(\alpha_1))' = 0,$$

and

$$M = \sup_{t_1\le t\le T_0} |y(t)| > d.$$

Let $t_2 > T_0$ be another zero of $y(t)$. Let

$$M_0 = \sup_{t_1\le t\le t_2} |y(t)|.$$

Clearly $M_0 > d$. Becouse of (13) and since T is large enough, we have for every $t \geq T$.

$$(26) \qquad \int_t^\infty r_1^{-(1/\alpha)}(s_1)ds_1 < M_0^{-(\beta-\alpha)/\alpha}, \qquad \int_t^\infty \overline{E}[s_1,\overline{r},|q(s)|]ds_1 < 1.$$

Clearly, the inequality (1) in Theorem 1 is true on the interval (t_1,t_2) and we have

$$(27) \qquad 1 \leq M_0^{\beta-\alpha}\left(\int_{t_1}^{t_2} r_1^{-(1/\alpha)}(s_1)ds_1\right)^\alpha \left(\int_{t_1}^{t_2} \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right).$$

From (27) and (26), we have

$$(28) \qquad 1 \leq M_0^{\beta-\alpha}\left(\int_{t_1}^{\infty} r_1^{-(1/\alpha)}(s_1)ds_1\right)^\alpha \left(\int_{t_1}^{\infty} \overline{E}[s_1,\overline{r},|q(s)|]ds_1\right) < 1.$$

This contradiction proves the theorem.

Remark 2. We note that the results given in Theorems 1–3 can be very easily extended to the following more general equations of the forms:

$$(\text{A}') \quad (r_{n-1}(t)(r_{n-2}(t)(\ldots(r_2(t)(r_1(t)|y'(t)|^{\alpha-1}y'(t))')'\ldots)')')' + \\ + q(t)|y(t)|^{\beta-1}y(t)f(t,y(t)) = 0,$$

$$(\text{B}') \quad (r_{n-1}(t)(r_{n-2}(t)(\ldots(r_2(t)(r_1(t)|y(t)|^p|y'(t)|^{k-2}y'(t))')'\ldots)')')' + \\ + q(t)|y(t)|^{p+k-2}y(t)f(t,y(t)) = 0,$$

$$(\text{C}') \quad (m(t)(r(t)|y'(t)|^{\alpha-1}y'(t))^{(n-1)})^{(n)} + q(t)|y(t)|^{\beta-1}y(t)f(t,y(t)) = 0,$$

$$(\text{D}') \quad (m(t)(r(t)|y(t)|^p|y'(t)|^{(k-2)}y'(t))^{(n-1)})^{(n)} + \\ + q(t)|y(t)|^{p+k-2}y(t)f(t,y(t)) = 0,$$

where α, β, p, k, r_i, m, r are as defined in equations (A) – (D) and the function $f: I \times R \to R$ is continuous and satisfies the condition $|f(t,y)| \leq w(t,|y|)$, where the function $w: I \times R_+ \to R_+$, $R_+ = [0,\infty)$, is continuous and satisfies $w(t,u) \leq w(t,v)$ for $0 \leq u \leq v$. For the use of such a condition on *f*, see [13-15].

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(Department of Mathematics, Marathwada University, Aurangabad 431 004, (Mahaarashtra) India)

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