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ALICJA PIENIĄŻEK

ON A CONSTRUCTION OF HYPOPARABOLIC POLYNOMIALS OF TWO SPATIAL VARIABLES AND ITS APPLICATION

ABSTRACT: The aim of the paper is to construct hypoparabolic polynomials of two spatial variables satisfying an hypoparabolic equation and to apply the theory of those polynomials for a finding of a solution of a initial-boundary value problem.

KEY WORDS: hypoparabolic polynomials, hypoparabolic equation, initial-boundary value problem, existence of a solution.

1. INTRODUCTION

In this paper we construct hypoparabolic polynomials of two space variables satisfying the equation

(1) $$Lu(x,t) = 0, \quad x = (x_1, x_2) \in R^2, \quad t \in R,$$

where

$$L := D_{x_2}^2 - D_{x_1}^2 - D_t.$$

For this purpose we prove two theorems on the forms of k-hyperbolic polynomials and we prove a theorem on properties of hypoparabolic functions.

In the paper we also give a theorem on an application of the hypoparabolic functions for finding of a solution of a initial-boundary value problem. Moreover, we present six constructions of hyperbolic polynomials in particular cases.

Some methods of parabolic and polyharmonic polynomials were also used in [1-4].

2. CONSTRUCTIONS OF K-HYPERBOLIC POLYNOMIALS

Let us consider the homogeneous polynomial W_m, given by the formula

(2) $$W_m(x) = \sum_{j=0}^{m} a_{m-j,j}\, x_2^{m-j} x_1^j,$$

where $a_{m-j,j}$ $(j = 0,1,\ldots,m)$ are the constant real coefficients.

Let $H := D_{x_2}^2 - D_{x_1}^2$ and $H^n := H(H^{n-1})$ $(n \in N)$. Suppose that the polynomial W_m satisfies the conditions

(3) $$H^n W_m(x) \neq 0, \qquad H^{n+1} W_m(x) = 0.$$

It follows, by (3), that $m = 2n$ or $m = 2n+1$.

We call the polynomials W^i_{2k+r} $(i = 0,1,\ldots,2k-1; r = 0,1)$ satisfying the equations $H^k W^i_{2k+r}(x) = 0$, where

$$W^i_{2k+r}(x) = \sum_{j=0}^{2k+r} a^i_{2k+r-j,j}\, x_2^{2k-j}\, x_1^j$$

$(i = 0,1,\ldots,2k-1;\ r = 0,1)$, *k*-hyperbolic polynomials.

At first, we shall construct the polynomials W^0_0, W^i_j $(i = 0,1;\ j = 1,2,3)$ and W^i_j $(i = 0,1,2,3;\ j = 4,5)$.

By C_i $(i = 1,2,\ldots,2k;\ k \in N)$, we denote real constants.

Construction 1. There exists one linearly independent homogeneous hyperbolic polynomial W^0_0, given by the formula $W^0_0(x) = 1$.

Indeed, let $W_0(x) = a_{0,0} \cdot 1$, where $a_{0,0} \neq 0$. We have that $HW_0(x) = 0$. Consequently, the function W^0_0 is a solution of equation (1). Moreover, this solution is linearly independent.

Construction 2. The polynomials W^0_1 and W^1_1, given by the formulae $W^0_1(x) = x_2$ and $W^1_1(x) = x_1$, are two linearly independent homogeneous hyperbolic polynomials of the first degree.

Since $W_1(x) = a_{1,0}\, x_2 + a_{0,1}\, x_1$ then $HW_1(x) = 0$. Applying the Kronecker matrix $[K_{ij}]$ $(i, j = 0,1)$ we obtain that $W^0_1(x) = K_{0,0}\, x_2$ and $W^1_1(x) = K_{1,1}\, x_1$.

The polynomials W^i_1 $(i = 0,1)$ are linearly independent. Indeed, differentiating the identity $C_1 x_2 + C_2 x_1 = 0$ with respect to x_2, we obtain that $C_1 = 0$. Consequently, $C_2 x_1 = 0$ and, therefore, $C_2 = 0$.

Construction 3. The polynomials W^0_2 and W^1_2, given by the formulae $W^0_2(x) = x_2^2 + x_1^2$, and $W^1_2(x) = x_1 x_2$, are two linearly independent homogeneous hyperbolic polynomials of the second degree.

Since $W_2(x) = a_{2,0}\, x_2^2 + a_{1,1}\, x_2 x_1 + a_{0,2}\, x_2^2$ then $HW_2(x) = 2a_{2,0} - 2a_{0,2} = 0$. Applying the Kronecker matrix, we obtain that

$$W_2^0(x) = K_{0,0}\, x_2^2 + K_{0,1}\, x_2\, x_1 + K_{0,2}\, x_1^2 = x_2^2 + x_1^2$$

and

$$W_2^1(x) = K_{1,0}\, x_2^2 + K_{1,1}\, x_2\, x_1 + K_{1,0}\, x_2^2 = x_2\, x_1.$$

The polynomials W_2^i $(i = 0,1)$ are linearly independent. Indeed, differentiating twice the identity $C_1(x_2^1 + x_1^2) + C_2\, x_2\, x_1 = 0$, with respect to x_2, we get $2C_1 = 0$. Hence, we have that $C_2 = 0$

Let

(4) $$c_{p,q,r,s} := (c_{p,q})^{-1} c_{r,s},$$

where $c_{p,q}$ and $c_{r,s}$ are coefficients.

Construction 4. The polynomials W_3^0 and W_3^1, given by the formulae $W_3^0(x) = x_2^3 + 3x_2^2\, x_1$, and $W_3^1(x) = x_2^2\, x_1 + 1/3\, x_1^3$, are two linearly independent homogeneous hyperbolic polynomials of the third degree.

From the equation $HW_3(x) = 0$, where

$$W_3(x) = \sum_{j=0}^{3} a_{3-j,j}\, x_2^{3-j}\, x_1^j,$$

we obtain the system of the equations

$$(3!)\, a_{3,0} - (2!)\, a_{1,2} = 0$$

and

$$(2!)\, a_{2,1} - (3!)\, a_{0,3} = 0.$$

Applying (4), we have that $c_{3,0} = 3!$, $c_{1,2} = 2!$, $c_{2,1} = 2!$, $c_{0,3} = 3!$, $c_{1,2,3,0} = 3$ and $c_{0,3,2,1} = 1/3$. By Kronecker matrix, we obtain the formulae

$$W_3^0(x) = \sum_{j=0}^{1} K_{0,j}\, x_2^{3-j} x_1^j + c_{1,2,3,0}\, K_{0,0}\, x_2\, x_1^2 + c_{0,3,2,1}\, K_{0,1}\, x_1^3$$

and

$$W_3^1(x) = \sum_{j=0}^{1} K_{1,j}\, x_2^{3-j} x_1^j + c_{1,2,3,0}\, K_{1,0}\, x_2\, x_1^2 + c_{0,3,2,1}\, K_{1,1}\, x_1^3.$$

Similarly to the argument given for W_2^i $(i = 0,1)$, differentiating three times, with respect to x_2, the both sides of the following identity:

$$C_1(x_2^3 + 3x_2\, x_1^2) + C_2(x_2^2\, x_1 + 1/3\, x_1^3) = 0,$$

we obtain that $C_1 = C_2 = 0$.

Construction 5. The polynomials W_4^i $(i = 0,1,2,3)$, given by the formulae $W_4^0(x) = x_2^4 - x_1^4$, $W_4^1(x) = x_2^2 x_1^2 + 1/3\, x_2^4$, $W_4^2(x) = x_2^3 x_1$ and $W_4^3(x) = x_1^3 x_2$, are four linearly independent homogeneous bihyperbolic polynomials of the fourth degree.

From the equation $H^2 W_4(x) = 0$, where

$$W_4(x) = \sum_{j=0}^{4} a_{4-j,j}\, x_2^{4-j} x_1^j,$$

we obtain that $(4!)a_{4,0} - 2(2!)(2!)a_{2,2} + (4!)a_{0,4} = 0$. By Kronecker matrix, we have the formulae

$$W_4^i(x) = \sum_{j=0}^{3} K_{i,j}\, x_2^{4-j} x_1^j + (c_{0,4,4,0} K_{i,0} + c_{0,4,2,2} K_{i,2}) x_1^4 \qquad (i = 0,1,2,3),$$

where $c_{0,4,4,0} = -1$ and $c_{0,4,2,2} = 1/3$.

Similarly as in Construction 4, we can prove that W_4^i $(i = 0,1,2,3)$ are linearly independent.

Construction 6. The polynomials W_5^i $(i = 0,1,2,3)$, given by the formulae $W_5^0(x) = x_2^5 - 5x_1^4 x_2$, $W_5^1(x) = x_2^4 x_1 - 1/5\, x_1^5$, $W_5^2(x) = x_2^3 x_1^2 + x_2 x_1^4$ and $W_5^3(x) = x_2^2 x_1^3 + 1/5\, x_1^5$, are four linearly independent homogeneous bihyperbolic polynomials of the fifth degree.

From the equation $H^2 W_5(x) = 0$, where

$$W_5(x) = \sum_{j=0}^{5} a_{5-j,j}\, x_2^{5-j} x_1^j,$$

we obtain the system

$$a_{5,0}(5!)(0!) - 2(3!)(2!)a_{3,2} + (1!)(4!)a_{1,4} = 0$$

and

$$a_{4,1}(4!)(1!) - 2(2!)(3!)a_{2,3} + (0!)(5!)a_{0,5} = 0.$$

Consequently, we have the formulae

$$W_5^i(x) = \sum_{j=0}^{3} K_{i,j}\, x_2^{5-j} x_1^j + (c_{1,4,5,0} K_{i,0} + c_{1,4,3,2} K_{i,2}) x_2 x_1^4 +$$

$$+ (c_{0,5,4,1} K_{i,1} + c_{0,5,2,3} K_{i,3}) x_1^5 \qquad (i = 0,1,2,3),$$

where $c_{1,4,5,0} = -5$, $c_{1,4,3,2} = 1$, $c_{0,5,4,1} = -1/5$ and $c_{0,5,2,3} = 1/5$.

Similarly, as in Construction 4, we can prove that W_5^i $(i = 0,1,2,3)$ are linearly independent.

Theorem 1. If $m = 2k$ *then there exist* $2k$ *linearly independent homogeneous k-hyperbolic polynomials* W_{2k}^j $(j = 0,1,\ldots,2k-1)$ *of degree* $2k$, *given by the formulae*

$$W_{2k}^{2i}(x) = x_2^{2k-2i} x_1^{2i} - c_{0,2k,2k-2i,2i} x_1^{2k} \qquad (i = 0,1,\ldots,k-1)$$

and

$$W_{2k}^{2i+1}(x) = x_2^{2k-(2i+1)} x_1^{2i+1} \qquad (i = 0,1,\ldots,k-1),$$

where

$$c_{0,2k,2k-2i,2i} = [(-1)^k (2k)!]^{-1} (2k-2i)!(2i)!.$$

Proof. Since $W_{2k}(x) = \sum_{j=0}^{2k} a_{2k-j,j} x_2^{2k-j} x_1^j$ and $H^k W_{2k}(x) = 0$ then we get that

$$\begin{aligned} H^k W_{2k}(x) = &\sum_{j=0}^{2k} a_{2k-j,j} D_{x_2}^{2k} (x_2^{2k-j} x_1^j) + \\ &-\binom{k}{1} \sum_{j=0}^{2k} a_{2k-j,j} D_{x_2}^{2k-2} D_{x_1}^2 (x_2^{2k-j} x_1^j) + \ldots + \\ &+(-1)^p \binom{k}{p} \sum_{j=0}^{2k} a_{2k-j,j} D_{x_2}^{2k-2p} D_{x_1}^{2p} (x_2^{2k-j} x_1^j) + \ldots + \\ &+(-1)^{k-1} \binom{k}{k-1} \sum_{j=0}^{2k} a_{2k-j,j} D_{x_2}^2 D_{x_1}^{2k-2} (x_2^{2k-j} x_1^j) + \\ &+(-1)^k \binom{k}{k} \sum_{j=0}^{2k} a_{2k-j,j} D_{x_1}^{2k} (x_2^{2k-j} x_1^j) = 0. \end{aligned}$$

Calculating in the above equality the suitable derivatives, we obtain

$$\begin{aligned} &\sum_{j=0}^{2k} a_{2k-j,j} (2k-j) \cdot \ldots \cdot (1-j)\, x_2^{-j} x_1^j - \binom{k}{1} \sum_{j=0}^{2k} a_{2k-j,j} (2k-j) \cdot \ldots \times \\ &\quad \times (3-j) j (j-1) x_2^{2-j} x_1^{j-2} + \ldots + (-1)^p \binom{k}{p} \sum_{j=0}^{2k} a_{2k-j,j} (2k-j) \cdot \ldots \times \\ &\quad \times (2p+1-j) j (j-1) \cdot \ldots \cdot (j-2p+1) x_2^{2p-j} x_1^{j-2p} + \ldots + \end{aligned}$$

$$+(-1)^{k-1}k\sum_{j=0}^{2k}a_{2k-j,j}(2k-j)(2k-j-1)j(j-1)\cdot\ldots\times$$
$$\times(j-2k+3)x_2^{2k-j-2}x_1^{j-(2k-2)}+$$
$$+(-1)^k\sum_{j=0}^{2k}a_{2k-j,j}(j-1)\cdot\ldots\cdot(j-2k+1)x_2^{2k-j}x_1^{j-2k}=$$
$$=\sum_{j=0}^{0}a_{2k-j,j}(2k-j)\cdot\ldots\cdot(1-j)x_2^{-j}x_1^j+$$
$$-\binom{k}{1}\sum_{j=0}^{0}a_{2k-j-2,j+2}(2k-j-2)\cdot\ldots\cdot(1-j)(j+2)(j+1)x_2^{-j}x_1^j+$$
$$+\ldots+(-1)^p\binom{k}{p}\sum_{j=0}^{0}a_{2k-j-2p,j+2p}(2k-j-2p)\cdot\ldots\times$$
$$\times(1-j)(j+2p)(j+2p-1)\cdot\ldots\cdot(j+1)x_2^{-j}x_1^j+\ldots+$$
$$+(-1)^{k-1}k\sum_{j=0}^{0}a_{2-j,j+2k-2}(2-j)(1-j)(j+2k-2)(j+2k-3)\cdot\ldots\times$$
$$\times(j+1)x_2^{-j}x_1^j+(-1)^k\sum_{j=0}^{0}a_{2k-(j+2k),j+2k}(j+2k)\times$$
$$\times(j+2k-1)\cdot\ldots\cdot(j+1)x_2^{-j}x_1^j=0.$$

By the last formula, we get

$$\binom{k}{0}(2k)!a_{2k,0}-\binom{k}{1}(2k-2)!(2)!a_{2k-2,2}+\ldots+$$
$$+(-1)^p\binom{k}{p}(2k-2p)!(2p)!a_{2k-2p,2p}+\ldots+$$
$$+(-1)^{k-1}\binom{k}{k-1}(2)!(2k-2)!a_{2,2k-2}+(-1)^k\binom{k}{k}(2k)!a_{0,2k}=0. \tag{5}$$

From equation (5), we obtain that

$$a_{0,2k}=-\sum_{r=0}^{k-1}c_{0,2k,2k-2r,2r}\,a_{2k-2r,2r},$$

where

$$c_{0,2k,2k-2r,2r} = [(-1)^k (2k)!]^{-1} \binom{k}{r} (2k-2r)!(2r)!$$

$(r = 0,1,\dots,p,\dots,k-1)$.
The coefficient $a_{2k-j,j}$ $(j = 0,1,\dots,2k-1)$ are arbitrary.

Let

$$W_{2k}^i(x) = \sum_{j=0}^{2k} a_{2k-j,j}^i \, x_2^{2k-j} \, x_1^j \quad (i = 0,1,\dots,2k-1),$$

where

$$a_{2k-j,j}^i = K_{i,j} \qquad (i, j = 0,1,\dots,2k-1)$$

and

$$a_{0,2k}^i = -c_{0,2k,2k,0} K_{i,0} - c_{0,2k,2k-2,2} K_{i,2} - c_{0,2k,2k-2p,2p} K_{i,2p} + \\ + \dots - c_{0,2k,2,2k-2} K_{i,2k-2} \qquad (i = 0,1,\dots,2k-1).$$

Observe that

(6) $$W_{2k}^i(x) = \sum_{j=0}^{2k-1} K_{i,j} x_2^{2k-j} x_1^j + (-c_{0,2k,2k,0} K_{i,0} - c_{0,2k,2k-2,2} K_{i,2} - \dots + \\ - c_{0,2k,2k-2p,2p} K_{i,2p} - \dots - c_{0,2k,2,2k-2} K_{i,2k-2}) x_1^{2k} \\ (i = 0,1,\dots,2k-1).$$

By (6), we obtain that

$$W_{2k}^{2i}(x) = x_2^{2k-2i} \, x_1^{2i} - c_{0,2k,2k-2i,2i} \, x_1^{2k} \quad (i = 0,1,\dots,k-1)$$

and

$$W_{2k}^{2i+1}(x) = x_2^{2k-(2i+1)} x_1^{2i+1} \qquad (i = 0,1,\dots,k-1).$$

The polynomials W_{2k}^i $(i = 0,1,\dots,2k-1)$ are linearly independent. Indeed, if

(7) $$\sum_{i=0}^{2k-1} C_{i+1} W_{2k}^i(x) = 0$$

then differentiating $2k$-times, with respect to x_2, the both sides of equation (7), we obtain that $C_{i+1} = 0 \quad (i = 0,1,\dots,2k-1)$.

Theorem 2. If $m=2k+1$ *then there exist* $2k$ *linearly independent homogeneous k-hyperbolic polynomials* W_{2k}^{j} $(j=0,1,\dots,2k-1)$ *of degree* $2k+1$, *given by the formulae*

$$W_{2k+1}^{2i}(x)=x_2^{2k+1-2i}x_1^{2i}-c_{1,2k,2k-2i+1,2i}\,x_2x_1^{2k}\qquad (i=0,1,\dots,k-1)$$

and

$$W_{2k+1}^{2i+1}(x)=x_2^{2k+1-(2i+1)}x_1^{2i+1}-c_{0,2k+1,2k-2i,2i+1}\,x_1^{2k+1}\qquad (i=0,1,\dots,k-1),$$

where

$$c_{1,2k,2k-2i+1,2i}=[(-1)^k(2k)!]^{-1}(2k-2i+1)!(2i)!\qquad (i=0,1,\dots,k-1)$$

and

$$c_{0,2k+1,2k-2i,2i+1}=[(-1)^k(2k+1)!]^{-1}(2k-2i)!(2i+1)!\qquad (i=0,1,\dots,k-1)$$

Proof. Since $W_{2k+1}(x)=\sum_{j=0}^{2k+1}a_{2k+1-j,j}\,x_2^{2k+1-j}x_1^j$ satisfies the equation $H^kW_{2k+1}(x)=0$ then we have

$$H^kW_{2k+1}(x)=\sum_{j=0}^{2k+1}a_{2k+1-j,j}(D_{x_2}^2-D_{x_1}^2)^k x_2^{2k+1-j}x_1^j=$$

$$=\sum_{j=0}^{2k+1}a_{2k+1-j,j}D_{x_2}^{2k}(x_2^{2k+1-j}x_1^j)+$$

$$-\binom{k}{1}\sum_{j=0}^{2k+1}a_{2k+1-j,j}D_{x_2}^{2k-2}D_{x_1}^2(x_2^{2k+1-j}x_1^j)+$$

$$+\dots+(-1)^p\binom{k}{p}\sum_{j=0}^{2k+1}a_{2k-1-j,j}D_{x_1}^{2k-2p}D_{x_2}^{2p}(x_2^{2k+1-j}x_1^j)+\dots+$$

$$+(-1)^{k-1}\binom{k}{k-1}\sum_{j=0}^{2k+1}a_{2k+1-j,j}D_{x_2}^{2k-2}D_{x_1}^2(x_2^{2k+1-j}x_1^j)+$$

$$+(-1)^k\binom{k}{k}\sum_{j=0}^{2k+1}a_{2k+1-j,j}D_{x_1}^{2k}(x_2^{2k+1-j}x_1^j)=$$

$$=\sum_{j=0}^{2k+1}a_{2k+1-j,j}(2k+1-j)\cdot\ldots\cdot(2-j)(x_2^{1-j}x_1^j)+$$

$$-\binom{k}{1}\sum_{j=0}^{2k+1} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\cdot(4-j)j(j-1)\times$$

$$\times(x_2^{3-j}x_1^{j-2})+\ldots+(-1)^p\binom{k}{p}\sum_{j=0}^{2k+1} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\times$$

$$\times(2p+2-j)j(j-1)\cdot\ldots\cdot(j-2p+1)x_2^{2p+1-j}x_1^{j-2p}+\ldots+$$

$$+(-1)^{k-1}\binom{k}{k-1}\sum_{j=0}^{2k+1} a_{2k+1-j,j}(2k+1-j)(2k-j)j\times$$

$$\times(j-1)\cdot\ldots\cdot(j-2k+3)x_2^{2k-1-j}x_1^{j-(2k-2)}+$$

$$+(-1)^k\binom{k}{k}\sum_{j=0}^{2k+1} a_{2k+1-j,j}j(j-1)\cdot\ldots\cdot(j-2k+1)x_2^{2k+1-j}x_1^{j-2k}=$$

$$=\sum_{j=0}^{1} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\cdot(2-j)x_2^{1-j}x_1^{j}+$$

$$-\binom{k}{1}\sum_{j=2}^{3} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\cdot(4-j)j(j-1)(x_2^{3-j}x_1^{j-2})+$$

$$+\ldots+(-1)^p\binom{k}{p}\sum_{j=2p}^{2p+1} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\times$$

$$\times(2p+2-j)j(j-1)\cdot\ldots\cdot(j-2p+1)x_2^{2p+1-j}x_1^{j-2p}+\ldots+$$

$$+(-1)^{k-1}\binom{k}{k-1}\sum_{j=2k-2}^{2k-1} a_{2k+1-j,j}(2k+1-j)(2k-j)j\times$$

$$\times(j-1)\cdot\ldots\cdot(j-2k+3)x_2^{2k-1-j}x_1^{j-(2k-2)}+$$

$$+(-1)^k\binom{k}{k}\sum_{j=2k}^{2k+1} a_{2k+1-j,j}j(j-1)\cdot\ldots\cdot(j-2k+1)x_2^{2k+1-j}x_1^{j-2k}=$$

$$=\sum_{j=0}^{1} a_{2k+1-j,j}(2k+1-j)\cdot\ldots\cdot(2-j)x_2^{1-j}x_1^{j}+$$

$$-\binom{k}{1}\sum_{j=0}^{1} a_{2k-j-1,j+2}(2k-1-j)\cdot\ldots\cdot(2-j)(j+2)(j+1)x_2^{1-j}x_1^{j}+$$

$$+\ldots+(-1)^p\binom{k}{p}\sum_{j=0}^{1} a_{2k+1-(j+2p),j+2p}(2k+1-j-2p)\cdot\ldots\times$$

$$\times (2-j)(j+2p)(j+2p-1)\cdot\ldots\cdot(j+1)x_2^{1-j}x_1^j+\ldots+$$

$$+(-1)^{k-1}\binom{k}{k-1}\sum_{j=0}^{1}a_{3-j,j+2k-2}(3-j)(2-j)(j+2k-2)\times$$

$$\times (j+2k-3)\cdot\ldots\cdot(j+1)x_2^{1-j}x_1^j+$$

$$+(-1)^k\binom{k}{k}\sum_{j=0}^{1}a_{1-j,j+2k}(j+2k)(j+2k-1)\cdot\ldots\cdot(j+1)x_2^{1-j}x_1^j=$$

$$=[a_{2k+1,0}(2k+1)!-\binom{k}{1}a_{2k-1,2}(2k-1)!(2)!+\ldots+$$

$$+(-1)^p\binom{k}{p}a_{2k+1-2p,2p}(2k+1-2p)!(2p)!+\ldots+$$

$$+(-1)^{k-1}\binom{k}{k-1}a_{3,2k-2}(2k-2)!(3)!+(-1)^k\binom{k}{k}a_{1,2k}(2k)!]x_2+$$

$$+[a_{2k,1}(2k)!-\binom{k}{1}a_{2k-2,3}(2k-2)!(3)!+\ldots+$$

$$+(-1)^p\binom{k}{p}a_{2k-2p,1+2p}(2k-2p)!(2p+1)+\ldots+$$

$$+(-1)^{k-1}\binom{k}{k-1}a_{2,2k-1}(2)!(2k-1)+$$

$$+(-1)^k\binom{k}{k}a_{0,1+2k}(2k+1)!]x_1=0.$$

Hence, we get the system of two equations

(8) $$a_{2k+1,0}(2k+1)!+a_{2k-1,2}(-k)(2k-1)!(2)!+\ldots+$$

$$+a_{2k+1-2p,2p}(-1)^p\binom{k}{p}(2k+1-2p)!(2p)!+\ldots+$$

$$+a_{3,2k-2}(-1)^{k-1}\binom{k}{k-1}(3)!(2k-2)!+a_{1,2k}(-1)^k\binom{k}{k}(2k)!=0,$$

$$a_{2k,1}(2k)! + a_{2k-2,3}(-k)(2k-2)!(3)! + \ldots +$$
$$+ a_{2k-2p,2p+1}(-1)^p \binom{k}{p}(2k-2p)!(2p+1)! + \ldots +$$
$$+ a_{2,2k-1}(-1)^{k-1}\binom{k}{k-1}(2)!(2k-1)! +$$
$$+ a_{0,2k+1}(-1)^k \binom{k}{k}(2k+1)! = 0.$$

By (8), we obtain the formulae

$$a_{1,2k} = -a_{2k+1,0}\, c_{1,2k,2k+1,0} - a_{2k-1,2}\, c_{1,2k,2k-1,2} +$$
$$- \ldots - a_{2k+1-2p,2p}\, c_{1,2k,2k+1-2p,2p} - \ldots - a_{3,2k-2}\, c_{1,2k,3,2k-2}$$

and

$$a_{0,2k+1} = -a_{2k,1}\, c_{0,2k+1,2k,1} - a_{2k-2,3}\, c_{0,2k+1,2k-2,3} +$$
$$- \ldots - a_{2k-2p,2p+1}\, c_{0,2k+1,2k-2p,2p+1} - \ldots - a_{2,2k-1}\, c_{0,2k+1,2,2k-1},$$

where

$$c_{1,2k,2k+1-2r,2r} = [(-1)^k (2k)!]^{-1}(2k+1-2r)!(2r)!$$

and

$$c_{0,2k+1,2k-2r,2r+1} = [(-1)^k (2k+1)!]^{-1}(2k-2r)!(2r+1)!$$
$$(r = 0,1,\ldots,p,\ldots,k-1).$$

The cofficients $a_{2k+1-2r,2r}$, $a_{2k-2r,2r+1}$ $(r = 0,1,\ldots,p,\ldots,k-1)$ are arbitrary .

Let

$$W^i_{2k+1}(x) = \sum_{j=0}^{2k+1} a^i_{2k+1-j,j}\, x_2^{2k+1-j}\, x_1^j,$$

where

$$a^i_{2k+1-j,j} = K_{i,j} \qquad (i,j = 0,1,\ldots,2k-1),$$

$$a^i_{1,2k} = -c_{1,2k,2k+1,0}\, K_{i,0} - c_{1,2k,2k-1,2}\, K_{i,2} +$$
$$- \ldots - c_{1,2k,2k+1-2p,2p}\, K_{i,2p} - \ldots - c_{1,2k,3,2k-2}\, K_{i,2k-2}$$
$$(i = 0,1,\ldots,2k-1)$$

and

$$a^i_{0,2k+1} = -c_{0,2k+1,2k,1}\, K_{i,1} - c_{0,2k+1,2k-2,3}\, K_{i,3} +$$

$$-\ldots-c_{0,2k+1,2k-2p,2p+1}K_{i,2p+1}-c_{0,2k+1,2,2k-1}K_{i,2k-1}$$

$$(i=0,1,\ldots,2k-1).$$

Observe that

$$W_{2k+1}^{i}(x)=\sum_{j=0}^{2k-1}K_{i,j}\,x_2^{2k+1-j}x_1^{j}-(c_{1,2k,2k+1,0}K_{i,0}+$$

$$+c_{1,2k,2k-1,2}K_{i,2}+\ldots+c_{1,2k,2k+1-2p,2p}K_{i,2p}+$$

$$+c_{1,2k,3,2k-2}K_{i,2k-2})x_2x_1^{2k}-(c_{0,2k+1,2k,1}K_{i,1}+c_{0,2k+1,2k-2,3}K_{i,3}+\ldots+$$

$$+c_{0,2k+1,2k-2p,2p+1}K_{i,2p+1}+\ldots+c_{0,2k+1,2,2k-1}K_{i,2k-1})x_1^{2k+1}$$

$$(i=0,1,\ldots,2k-1).$$

By the above formula, we obtain that

$$W_{2k+1}^{2i}=x_2^{2k+1-2i}\,x_1^{2i}-c_{1,2k,2k-2i+1,2i}\,x_2\,x_1^{2k}\qquad(i=0,1,\ldots,k-1)$$

and

$$W_{2k+1}^{2i+1}=x_2^{2k+1-(2i+1)}\,x_1^{2i+1}-c_{0,2k+1,2k-2i,2i+1}\,x_1^{2k+1}\qquad(i=0,1,\ldots,k-1).$$

The polynomials W_{2k+1}^{i} $(i=0,1,\ldots,k-1)$ are linearly independent. Indeed, let

$$\sum_{i=0}^{2k-1}C_{i+1}W_{2k+1}^{i}(x)=0. \tag{9}$$

Differentiating $2k+1-$times, with respect to x_2, the both sides of equation (9), we obtain that $C_{i+1}=0$ $(i=0,1,\ldots,2k-1)$.

3. THE HYPOPARABOLIC POLYNOMIALS

Theorem 3. If the polynomial W_m, *defined by formula* (2), *satisfies condition* (3) *then the hypoparabolic function* S_n, *given by the formula*

$$S_n(x,t)=\sum_{j=0}^{n}\frac{t^j}{j!}H^j\,W_m(x), \tag{10}$$

satisfies equation (1).

Proof. Observe that

$$HS_n(x,t)=\sum_{j=0}^{n}\frac{t^j}{j!}H^{j+1}W_m(x)=\sum_{j=0}^{n-1}\frac{t^j}{j!}H^{j+1}W_m(x)=\sum_{j=0}^{n}\frac{t^{j-1}}{(j-1)!}H^jW_m(x)$$

and

$$D_tS_n(x,t)=\sum_{j=1}^{n}\frac{t^{j-1}}{(j-1)!}H^jW_m(x).$$

Consequently, we obtain (1).

Therefore, the proof is complete.

Remark. The hypoparabolic polynomials are of the forms:

$$S_{2k}^p(x,t)=W_{2k}^p(x)+tHW_{2k}^p(x)+\frac{t^2}{2}H^2W_{2k}^p(x)+\ldots+$$
$$+\frac{t^{2k-2}}{(2k-2)!}H^{2k-2}W_{2k}^p(x)\qquad (p=0,1,\ldots,2k-1)$$

and

$$S_{2k+1}^p(x,t)=W_{2k+1}^p(x)+tHW_{2k+1}^p(x)+\frac{t^2}{2}W_{2k+1}^p(x)+\ldots+$$
$$+\frac{t^{2k-2}}{(2k-2)!}H^{2k-2}W_{2k+1}^p(x)\quad (p=0,1,\ldots,2k-1).$$

4. AN EXAMPLE OF A SOLUTION TO A INITIAL-BOUNDARY VALUE PROBLEM

Let us consider the problem

(11) $$(D_{x_2}^2-D_{x_1}^2-D_t)u(x,t)=(ax_1+bx_2+c)V(t),\qquad (x,t)\in D,$$

where

$$D=\{(x,t):x=(x_1,x_2),\ \ x\in[0,1]^2,\ \ t\in[0,T]\},$$

(12) $$u(x,0)=S_n(x,0),\quad x\in[0,1]^2,$$

(13) $$u(0,x_2,t)=S_n(0,x_2,t)-(bx_2+c)\int_0^t V(s)ds,$$

$$x_2\in[0,1],\quad t\in[0,T],$$

$$u(x_1,0,t) = S_n(x_1,0,t) - (ax_1 + c)\int_0^t V(s)ds, \tag{14}$$

$$x_1 \in [0,1], \quad t \in [0,T],$$

where V is a polynomial of the variable t, S_n is given by formula (10) and a, b, c are constant coefficients.

It is possible to prove the following:

Theorem 4. The function u given by the formula

$$u(x,t) := S_n(x,t) - (ax_1 + bx_2 + c)\int_0^t V(s)ds, \quad (x,t) \in D,$$

is a solution of problem (11)-(14).

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(Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Cracow, Poland)

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