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A MULTIVARIATE OSCILLATION THEOREM

ABSTRACT: An oscillation criterion is derived for a multivariate partial difference equation.

KEY WORDS: partial difference equation, oscillation criterion.

Qualitative theory of discrete time processes has drawn some attention in recent years. In particular, oscillation properties of difference equations have been studied recently by a number of authors (see e.g. Györi and Ladas [1, Chapter 7]). In [1, Theorem 7.5.1], Ladas et al. established the following oscillation criterion.

Theorem 1. Assume that σ is a positive integer, that $p(i) \geq 0$ for $i \geq 0$ and that

$$\liminf_{m\to\infty} \frac{1}{\sigma} \sum_{i=m-\sigma}^{m-1} p(i) > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}.$$

Then every solution of the following delay difference equation

$$u_{i+1} - u_i + p(i) u_{i-\sigma} = 0, \qquad i = 0,1,2,...$$

is oscillatory.

In [2], Zhang et al. established the following oscillation criterion for a partial difference equation of the form

$$u_{i+1,j} + u_{i,j+1} - u_{ij} + p(i,j) u_{i-\sigma,j-\tau} = 0, \qquad i,j = 0,1,2,... \tag{1}$$

where σ, τ are non-negative integers.

Theorem 2. Assume that σ, τ are positive integers, that $p(i,j) \geq 0$ for $i,j \geq 0$ and that

$$\liminf_{m,n\to\infty} \frac{1}{\sigma\tau} \sum_{i=m-\sigma}^{m-1} \sum_{j=n-\tau}^{n-1} p(i,j) > \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}},$$

where $\lambda = 2\sigma\tau/(\sigma+\tau)$. Then every solution of (1) is oscillatory.

In this note, we will establish a multivariate generalization of Theorem 1 which is also an improvement of Theorem 2 when the dimension in concern is two. A few preparatory notations will be needed to state this extension. We write Z^n to denote the subset of R^n consisting of all vectors $x=(x_1,...,x_n)$ where x_i is an integer for each $1\le i\le n$. The vector (1,1,...,1) will be denoted by ℓ, and the vector $\delta^j=(\delta_1^j,...,\delta_n^j)$ will denote the vector defined by $\delta_i^j=0$ if $j\ne i$ and $\delta_i^i=1$. Let $x,y\in R^n$. The usual quantitative concepts related to vectors will be defined componentwise. In particular, we write $x\le y$ if their corresponding components x_i and y_i satisfy $x_i\le y_i$ for $1\le i\le n$. The meaning of $x<y$ is similarly imposed.

By a rectangle in Z^n, we mean a subset of the form

$$[a,b]=\{x\in Z^n|a\le x\le b\}$$

or of the form $\{x\in Z^n|x\ge a\}$. A multiple sequence can be written in the form $\{u(x)\}$ where x belongs to some rectangle of Z^n. The translation operator E_k is defined by $E_k u(x)=u(x+\delta^k)$.

For any $x\in R^n$, the vector in R^{n-1} obtained by removing the $k-$component of x will be denoted by $\partial_k x$. Also for any $S\subseteq R^n$, the notation $\partial_k S$ is used to denote the set $\{\partial_k x|x\in S\}$. Finally, for any $x=(x_1,...,x_n)\in R^n$, we will adopt the following notations:

$$\mu(x)=x_1...x_n,$$

and

$$\pi(x)=x_2x_3...x_n+x_1x_3x_4...x_n+...+x_1x_2...x_{n-1},$$

where it is understood that $\pi(x)=1$ if $x\in R$.

We will obtain an oscillation criterion for the following multivariate recurrence relation

$$\sum_{k=1}^{n} E_k u(x)-u(x)+p(x)u(x-\sigma)\le 0, \qquad x\in Z^n,\ x\ge 0, \tag{2}$$

where $\sigma=(\sigma_1,...,\sigma_n)\in Z^n$ and $\sigma\ge 0$. As in [1], it is easy to state an existence and uniqueness theorem for the solutions of the recurrence relation (2). Furthermore, a solution $\{u(x)\}_{x\ge-\sigma}$ is said to be eventually positive (or eventually negative) if $u(x)>0$ (respectively $u(x)<0$) for all large x, and oscillatory if it is neither eventually positive nor eventually negative.

We first state two useful lemmas whose proofs employ essentially the same ideas as those of Lemma 2.1 and Lemma 2.5 in [2].

Lemma 1. Assume that $p(x) \geq 0$ *for* $x \geq 0$. *If* $\{u(x)\}$ *is an eventually positive solution of* (2), *then* $u(x)$ *is non-increasing in each independent variable for all large* x.

The proof follows from the fact that

$$E_k u(x) - u(x) \leq \sum_{k=1}^{n} E_k u(x) - u(x) + p(x)u(x-\sigma) \leq 0.$$

Lemma 2. Assume that $\sigma > 0$, *that* $p(x) \geq 0$ *for* $x \geq 0$ *and that*

$$\sum_{x\in[m-\sigma,m-\ell]} p(x) \geq \delta > 0 \tag{3}$$

for all large m. *If* $\{u(x)\}$ *is an eventually positive solution of* (2), *then the quotient* $u(x-\sigma)/u(x)$ *is bounded for all large* x.

Proof. By means of (3), for any sufficiently large $t = (t_1,\ldots,t_n)$, there exists (see e.g. [3]) an integer m_1 such that $m_1 - \sigma_1 \leq t_1 \leq m_1$ and

$$\sum_{x_1=m_1-\sigma_1}^{t_1} \sum_{x\in\partial_1[t,t+\sigma]} p(x) \geq \frac{\delta}{2}, \qquad \sum_{x_1=t_1+1}^{m_1} \sum_{x\in\partial_1[t,t+\sigma]} p(x) \geq \frac{\delta}{2}.$$

In view of (3) and the decreasing nature of $u(x)$, we have

$$\sum_{x_1=m_1-\sigma_1}^{t_1} (E_1 u(x) - u(x)) + \sum_{x_1=m_1-\sigma_1}^{t_1} \sum_{x\in\partial_1[t,t+\sigma]} p(x)u(x-\sigma) \leq 0,$$

which implies

$$\sum_{x_1=m_1-\sigma_1}^{t_1} (E_1 u(x) - u(x)) + u(t_1-\sigma_1,t_2,\ldots,t_n) \sum_{x_1=m_1-\sigma_1}^{t_1} \sum_{x\in\partial_1[t,t+\sigma]} p(x) \leq 0.$$

Thus,

$$u(t_1+1,t_2,\ldots,t_n) - u(m_1-\sigma_1,t_2,\ldots,t_n) + \frac{\delta}{2} u(t_1-\sigma_1,t_2,\ldots,t_n) \leq 0,$$

which implies that

$$u(m_1-\sigma_1,t_2,\ldots,t_n) \geq \frac{\delta}{2} u(t_1-\sigma_1,t_2,\ldots,t_n).$$

By similar arguments, we see form

$$\sum_{x_1=t_1+1}^{m_1}(E_1u(x)-u(x))+\sum_{x_1=t_1+1}^{m_1}\sum_{x\in\partial_1[t,t+\sigma]}p(x)u(x-\sigma)\le 0$$

that

$$u(t)\ge u(t_1+1,t_2,\dots,t_n)\ge\frac{\delta}{2}u(m_1-\sigma_1,t_2,\dots,t_n).$$

Combining the above two inequalities, we have

$$u(t)\ge\frac{\delta^2}{4}u(t_1-\sigma_1,t_2,\dots,t_n)$$

for any large $t=(t_1,\dots,t_n)$. Similarly, we have

$$u(s)\ge\frac{\delta^2}{4}u(s_1,\dots,s_{j-1},s-\sigma_j,s_{j+1},\dots,s_n)$$

for any large $s=(s_1,\dots,s_n)$.

Finally,

$$\frac{u(x-\sigma)}{u(x)}=\frac{u(x_1-\sigma_1,x_2-\sigma_2,\dots,x_n-\sigma_n)}{u(x_1,x_2-\sigma_2,\dots,x_n-\sigma_n)}\times\dots\times$$

$$\times\frac{u(x_1,\dots,x_{n-1},x_n-\sigma_n)}{u(x_1,\dots,x_n)}\le\left(\frac{\delta^2}{4}\right)^n$$

for all large x. The proof is complete.

Lemma 3. Let $\{w(x)\}$ be a multiple sequence and let $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)\in Z^n$ such that $w(x)>0$ for $a\le x\le b+\ell$, then

$$\prod_{x\in[a,b]}\left\{\frac{1}{w(x)}\left(\prod_{k=1}^{n}E_kw(x)\right)^{1/n}\right\}=\left\{\prod_{\partial_1x\in\partial_1[a,b]}\left(\frac{w(b_1+1,x_2,\dots,x_n)}{w(a_1,x_2,\dots,x_n)}\right)^{1/n}\right\}\times$$

$$\times\left\{\prod_{\partial_2x\in\partial_2[a,b]}\left(\frac{w(x_1,b_2+1,x_3,\dots,x_n)}{w(x_1,a_2,x_3,\dots,x_n)}\right)^{1/n}\right\}\times\dots\times$$

$$\times\left\{\prod_{\partial_nx\in\partial_n[a,b]}\left(\frac{w(x_1,x_2,\dots,x_{n-1},b_n+1)}{w(x_1,x_2,\dots,x_{n-1},a_n)}\right)^{1/n}\right\}.$$

Proof. This follows from

$$\prod_{x\in[a,b]}\left\{\frac{1}{w(x)}\left(\prod_{k=1}^{n}E_k w(x)\right)^{1/n}\right\}=\exp\left\{\frac{1}{n}\sum_{x\in[a,b]}\sum_{k=1}^{n}\log\frac{E_k w(x)}{w(x)}\right\}=$$

$$=\exp\left\{\frac{1}{n}\sum_{k=1}^{n}\sum_{x\in[a,b]}\log\frac{E_k w(x)}{w(x)}\right\}=$$

$$=\exp\left\{\frac{1}{n}\sum_{k=1}^{n}\left[\sum_{\partial_k x\in\partial_k[a,b]}\sum_{k=a_k}^{b_k}\log\frac{E_k w(x)}{w(x)}\right]\right\}=$$

$$=\exp\left\{\frac{1}{n}\left[\sum_{\partial_1 x\in\partial_1[a,b]}\log\frac{w(b_1+1,x_2,\ldots,x_n)}{w(a_1,x_2,\ldots,x_n)}+\right.\right.$$

$$+\sum_{\partial_2 x\in\partial_2[a,b]}\log\frac{w(x_1,b_2+1,x_3,\ldots,x_n)}{w(x_1,a_2,x_3,\ldots,x_n)}+\ldots+$$

$$\left.\left.+\sum_{\partial_n x\in\partial_n[a,b]}\log\frac{w(x_1,\ldots,x_{n-1},b_n+1)}{w(x_1,\ldots,x_{n-1},a_n)}\right]\right\}.$$

We now state our main result as follows.

Theorem 3. Assume that $\sigma=(\sigma_1,\ldots,\sigma_n)\in Z^n$ *and* $\sigma>0$, *that* $p(x)>0$ *for* $x\geq 0$, *and that*

$$\liminf_{m\to\infty}\frac{1}{\mu(\sigma)}\sum_{x\in[m-\sigma,m-\ell]}p(x)>\frac{1}{n^{\lambda}}\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}},$$

where $\lambda=n\mu(\sigma)/\pi(\sigma)$. *Then every solution of* (2) *is oscillatory.*

Proof. Assume to the contrary that $\{u(x)\}$ is an eventually positive solution of (2). Then in view of (2) and Lemma 1, we see that

$$\sum_{k=1}^{n}E_k u(x)\leq u(x)-p(x)u(x-\sigma)\leq u(x)-p(x)u(x)$$

for all large x. Hence for all large $m=(m_1,\ldots,m_n)\in Z^n$, we have

$$\prod_{x\in[m-\sigma,m-\ell]}\{1-p(x)\}\geq\prod_{x\in[m-\sigma,m-\ell]}\left\{\frac{1}{u(x)}\sum_{k=1}^{n}E_k u(x)\right\}$$

which implies, by means of the arithmetic-geometric mean inequality, that

$$\left\{1-\frac{1}{\mu(\sigma)}\sum_{x\in[m-\sigma,m-\ell]}p(x)\right\}^{\mu(\sigma)} \geq n^{\mu(\sigma)}\prod_{x\in[m-\sigma,m-\ell]}\left\{\frac{1}{u(x)}\left(\sum_{k=1}^{n}E_k u(x)\right)^{1/n}\right\}=$$

$$=n^{\mu(n)}\left\{\prod_{x\in[m-\sigma,m-\ell]}\left(\frac{u(m_1,x_2,\dots,x_n)}{u(m_1-\sigma_1,x_2,\dots,x_n)}\right)^{1/n}\right\}\times\dots\times$$

$$\times\left\{\prod_{\partial_n x\in\partial_n[m-\sigma,m-\ell]}\left(\frac{u(x_1,x_2,\dots,x_{n-1},m_n)}{u(x_1,x_2,\dots,x_{n-1},m_n-\sigma_n)}\right)^{1/n}\right\}\geq$$

$$\geq n^{\mu(n)}\left\{\prod_{\partial_1 x\in\partial_1[m-\sigma,m-\ell]}\left(\frac{u(m)}{u(m-\sigma)}\right)^{1/n}\right\}\times\dots\times$$

$$\times\left\{\prod_{\partial_n x\in\partial_n[m-\sigma,m-\ell]}\left(\frac{u(m)}{u(m-\sigma)}\right)^{1/n}\right\}=$$

$$=n^{\mu(\sigma)}\left(\frac{u(m)}{u(m-\sigma)}\right)^{\pi(\sigma)/n},$$

where the first equality follows from Lemma 3 and the second inequality follows from Lemma 1. Thus

$$1-\frac{1}{\mu(\sigma)}\sum_{x\in[m-\sigma,m-\ell]}p(x)\geq n\left(\frac{u(m)}{u(m-\sigma)}\right)^{1/\lambda}$$

for all large $m\in Z^n$. Let

$$\Gamma=\frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}},\qquad \Gamma'=\frac{\Gamma}{n^\lambda},$$

and let Ψ be a constant such that for sufficiently large $m\in Z^n$,

$$\Gamma'<\Psi\leq\frac{1}{\sigma\tau}\sum_{x\in[m-\sigma,m-1]}p(x).$$

Then we have

$$1-\Psi\geq n\left(\frac{u(m)}{u(m-\sigma)}\right)^{1/\lambda},$$

so that $\Psi<1$. In addition, in view of the inequality [1, p. 181]

$$1-\Psi \le \left\{ \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}} \right\}^{1/\lambda} \Psi^{-1/\lambda} = \left(\frac{\Gamma}{\Psi} \right)^{1/\lambda}, \qquad 0<\Psi\le 1,$$

we see further that

$$\frac{u(m)}{u(m-\sigma)} \le \frac{\Gamma}{n^{\lambda}\Psi} = \frac{\Gamma'}{\Psi}$$

say for $m \ge M_1$.

If we apply the above inequality to (2), we obtain

$$\sum_{k=1}^{n} E_k u(x) \le u(x) - p(x)\frac{\Psi}{\Gamma'}u(x), \qquad x \ge M_1.$$

A similar procedure then leads to

$$\frac{u(x)}{u(x-\sigma)} \le \left(\frac{\Gamma'}{\Psi} \right)^2,$$

say, for $x \ge M_2 \ge M_1$. Inductively, we see that for any positive integer t, there are $M_t \in Z^n$ such that

$$\frac{u(x)}{u(x-\sigma)} \le \left(\frac{\Gamma'}{\Psi} \right)^t, \qquad x \ge M_t.$$

Thus, $u(x-\sigma)/u(x)$ diverges as x tends to infinity, which contradicts Lemma 2.

We remark that when $n=2$, Theorem 3 reduces to the following result: Assume that σ, τ are positive integers, that $p(i,j) \ge 0$ for $i,j \ge 0$ and that

$$\liminf_{m,n\to\infty} \frac{1}{\sigma\tau} \sum_{i=m-\sigma}^{m-1} \sum_{j=n-\tau}^{n-1} p(i,j) > \frac{1}{2^{\lambda}} \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}},$$

where $\lambda = 2\sigma\tau/(\sigma+\tau)$. Then every solution of (1) is oscillatory.

Since $\lambda \ge 1$ when $\sigma, \tau > 0$, we have indeed an improvement of Theorem 2 as claimed.

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