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THE RADIAL SOLUTION OF THE FACTORISED POLYPARABOLIC EQUATION IN SPHERICAL SHALL

ABSTRACT: The subject of the paper is a construction of an explicit radial solution of the factorised polyparabolic equation in spherical shall. In this paper the theorems on uniqueness and existence of solution of considered of (1) – (4) problem is given.

KEY WORDS: polyparabolic factorised equation, Green function, iterated Green potentials.

1. INTRODUCTION

The subject of the paper is the existence and uniqueness of the radial solution $U(r,t)=u(x,t)$, $r=|x|$, of the factorised polyparabolic equation

(1) $$P_m P_{m-1} \dots P_2 P_1 u(x,t) = f(x,t), \quad x=(x_1,x_2,x_3), \quad P_i = \Delta - c_i^2 D_t,$$

$$\Delta = \sum_{i=1}^{3} D_{x_i}^2, \quad c_i \neq c_j \text{ for } i \neq j, \quad c_i > 0, \quad i=1,2,\dots,m, \quad m \in N - \{1\}, \text{ (for fixed } m),$$

in the spherical shall

$$D = \{(x,t) : a < |x| < b, \ 0 < a < b, \ t \in (0,T), \ |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}\}.$$

The radial solution $u(x,t)=U(r,t)$, $r=|x|$, satisfies the initial conditions

(2) $$P_{j-1} P_{j-2} \dots P_2 P_1 P_0 U(r,0) = f_j(r), \quad j=1,2,\dots,m, \quad a<r<b, \quad P_0 U = U,$$

and the boundary conditions

(3) $$P_{j-1} P_{j-2} \dots P_2 P_1 P_0 U(a,t) = h_j(t), \quad j=1,2,\dots,m, \quad 0<t<T,$$

(4) $$P_{j-1} P_{j-2} \dots P_2 P_1 P_0 U(b,t) = k_j(t), \quad j=1,2,\dots,m, \quad 0<t<T.$$

In [1] the similar problem for factorised polyparabolic equation for the strip is treated.

In the paper [4] a limit problem for the equation $P^m u = f$ for spherical shall D with the initial conditions $D_t^i U(r,0) = f_i(r)$, $r \in (a,b)$, $i=0,1,\dots,m-1$, and the boundary conditions $P^i U(a,t) = H_i(t)$, $P^i U(b,t) = K_i(t)$, $t \in (0,T)$, $i=0,1,\dots,m-1$ was solved.

In the paper [2] similar problem for the polyparabolic equation $P^m u = f$ for the spherical shall D with the initial conditions $P^{j-1}U(r,0) = f_j(r)$, $j = 1,2,...,m$, $r \in (a,b)$ and the boundary conditions $P^{j-1}U(a,t) = h_j(t)$, $P^{j-1}U(b,t) = k_j(t)$, $j = 1,2,...,m$, $t \in (0,T)$ is considered.

2. SOME DENOTATIONS AND LEMMAS

Definition 1. Denote by (K_1) *the class of all functions* $F: D_1 \to \Re$, *such that* $F \in C^{1,0}(D_1) \cap C^{0,0}(\overline{D_1})$, $D_1 = \{(r,t) : r \in (a,b), t \in (0,T)\}$.

Definition 2. Denote by (K_2) *the class of all functions* $L:[a,b] \to \Re$, *such that* $L \in C^1([a,b])$ *and* $D_r^{i-1}F(a) = 0$, $i = 1,2,...,m$.

Definition 3. Denote by (K_3) *the class of all functions* $H:[0,T] \to \Re$, *such that* $H \in C^1([0,T])$ *and* $D_t^{i-1}H(0) = 0$, $i = 1,2,....,m$.

Definition 4. Denote by (K_4) *the class of all functions* $v: D_1 \to \Re$, *such that* $v \in C^{2m,m}(D_1) \cap C^{m,m-1}(\overline{D_1})$.

Let

(5) $Q_j = D_r^2 - c_j^2 D_t$, $c_i \neq c_k$ for $i \neq k$, $c_i > 0$, $i,k = 1,2,...,m$

Lemma 1. If $u \in C^{2m,m}(D)$, *then* $U \in C^{2m,m}((a,b) \times (0,T))$ *and*

$$P_m P_{m-1} ... P_2 P_1 U(r,t) = r^{-1} Q_m Q_{m-1} ... Q_2 Q_1 (rU(r,t)).$$

Proof. We have

$$\Delta u(x,t) = \Delta U(r,t) = D_r^2 U(r,t) + 2r^{-1} D_r U(r,t) = r^{-1} D_r^2 (rU(r,t)),$$

$$P_1 U(r,t) = \Delta U(r,t) - c_1^2 D_t U(r,t) = r^{-1} (D_r^2 (rU(r,t) - c_1^2 D_t (rU(r,t)))) =$$
$$= r^{-1} Q_1 (rU(r,t)).$$

Let

$$Y_1(r,t) = r^{-1} Q_1 (rU(r,t)) \quad \text{or} \quad Q_1(rU(r,t)) = rY_1(r,t).$$

From $m = 2$, we have

$$P_2 P_1 U(r,t) = P_2(Y_1(r,t)) = r^{-1} Q_2 (rY_1(r,t)) = r^{-1} Q_2 Q_1 (rU(r,t)).$$

Assuming that

$$P_{m-1}P_{m-2}\dots P_2P_1U(r,t)=r^{-1}Q_{m-1}Q_{m-2}\dots Q_2Q_1(rY_{m-1}))$$

we obtain

$$P_mP_{m-1}..P_2P_1U(r,t)=r^{-1}Q_{m-1}Q_{m-2}\dots Q_2Q_1(rU(r,t)).$$

By Lemma 1, (1) – (4) and by (5) we obtain follow lemma

Lemma 2. If the function $u\in C^{2m,m}(D)$ *is a solution of the* (1) – (4) *problem, then the function* $U(r,t)\in C^{2m,m}(D_1)$ *and satisfies the conditions:*

(1a) $Q_mQ_{m-1}\dots Q_2Q_1v(r,t)=\overline{F}(r,t),\ v=rU,\ \overline{F}(r,t)=rf(r,t),\ (r,t)\in D_1,$

(2a) $Q_{j-1}Q_{j-2}\dots Q_2Q_1Q_0v(r,0)=F(r,t),\ Q_0v=v,\ j=1,2,\dots,m,\ r\in(a,b),$

(3a) $Q_{j-1}Q_{j-2}\dots Q_2Q_1Q_0v(a,t)=H_j(t),\ H_j(t)=ah_j(t),\ j=1,2,\dots,m,\ t\in(0,T),$

(4a) $Q_{j-1}Q_{j-2}\dots Q_2Q_1Q_0v(b,t)=K_j(t),\ K_j(t)=bk_j(t),\ j=1,2,\dots,m,\ t\in(0,T).$

Conversely. If the function v satisfies conditions (1a) – (4a), then the function $u(x,t)=U(r,t)=r^{-1}v(r,t)$, $r=|x|$, is the solution of the problem (1) – (4).

3. Uniqueness Theorem

Theorem 1. If $v_1(r,t),v_2(r,t)\in(K_4)$ *are solutions of the problem* (1a) – (4a), *then* $v_1(r,t)=v_2(r,t)$ *for* $(r,t)\in D_1$.

Proof. Let

$$L_i=Q_iQ_{i-1}\dots Q_2Q_1,\quad i=1,2,\dots,m,\quad m\in N.$$

Let

$$D_t=\{(r,t):\ r\in(a,b),s\in(0,t)\}.$$

Let

$$V(r,t)=v_1(r,t)-v_2(r,t)\quad\text{for}\quad(r,t)\in D_1.$$

The function $V(r,t)$ satisfying the conditions:

(6) $L_mV(r,s)=0,\quad (r,t)\in D_1,$

(7) $L_{i-1}V(r,0)=0,\quad r\in(a,b),\ i=1,2,\dots,m,\quad L_0V=V,$

(8) $L_{i-1}V(a,t)=0,\quad t\in(0,T),\ i=1,2,\dots,m,$

(9) $L_{i-1}V(b,t)=0,\quad t\in(0,T),\ i=1,2,\dots,m.$

Multiplying both sides of the equation (6) by $L_{m-1}V(r,t)$, integrating over D_t and by (7) – (9) we follow obtain

$$(L_{m-1}V(r,s))L_{m-1}(r,s)) = 0, \quad (r,s) \in D_1$$

i.e.

$$(L_{m-1}V(r,s))(D_r^2 - c_m^2 D_s)(L_{m-1}V(r,s)) = 0$$

and

$$(L_{m-1}V(r,s))D_x^2 L_{m-1}V(r,s) - (L_{m-1}V(r,s))(c_m^2 D_s L_{m-1}V(r,s)) = 0. \tag{10}$$

Integrating both sides of (10) over D_t,

$$\int_0^t \int_a^b (L_{m-1}V(r,s))D_x^2 L_{m-1}V(r,s)drds -$$

$$- c_m^2 \int_0^t \int_a^b (L_{m-1}V(r,s))(D_s L_{m-1}V(r,s))drds = 0.$$

Let

$$W_{m-1}(r,s) = L_{m-1}V(r,s)$$

We have

$$\int_0^t\int_a^b W_{m-1}(r,s)D_x^2 W(r,s))drds - c_m^2 \int_0^t\int_a^b (W_{m-1}(r,s))D_s W_{m-1}(r,s)ds = 0.$$

Let

$$I_{m-1}^1 = \int_0^t\int_a^b W(r,s)D_x^2 W_{m-1}(r,s))drds,$$

$$I_{m-1}^2 = -c_m^2 \int_0^t\int_a^b (W_{m-1}(r,s))(W_{m-1}(r,s))ds,$$

Integrating by parts I_{m-1}^1 and I_{m-1}^2, we obtaine

$$I_{m-1}^1 = \int_0^t (W_{m-1}(r,s)D_x W_{m-1}(r,s)\Big|_{r=a}^{r=b} ds - \tag{11}$$

$$- \int_0^t\int_a^b (D_r W_{m-1}(r,s)D_r W_{m-1}(r,s))drds =$$

$$= (-1/2)\int_0^t\int_a^b (D_r W_{m-1}(r,s))^2 drds \le 0$$

and

(12) $$I_{m-1}^2 = -c_m^2 \int_0^t \int_a^b (1/2)\, D_s W_{m-1}^2(r,s)\, dr\, ds = (-1/2)c_m^2 \int_a^b \int_0^t D_s W_{m-1}^2(r,s) dr ds =$$

$$= (-c_m^2/2) \int_a^b W_{m-1}^2(r,s)\Big|_{s=0}^{s=t} dr = (-c_m^2/2) \int_a^b (W_{m-1}^2(r,t) - W_{m-1}^2(r,0))\, dr =$$

$$= (-c_m^2/2) \int_a^b W_{m-1}^2(r,t) dr \le 0.$$

Since $I_{m-1}^1 + I_{m-1}^2 = 0$ thus by (11) and (12) follows that $I_{m-1}^1 = I_{m-1}^2 = 0$. Consequently

$$W(r,t) = 0 \quad \text{for} \quad (r,s) \in \overline{D_1}.$$

and

(13) $$L_{m-1}V(r,s) = 0 \quad \text{for} \quad (r,s) \in \overline{D_1}.$$

Multiplying (13) by $W_{m-2}(r,s) = L_{m-2}V(r,s)$, we have

$$L_{m-2}V(r,s) L_{m-1}V(r,s) = 0,$$

i.e.

$$(W_{m-2}(r,s))(D_r^2 - c_{m-1}^2 D_s W_{m-2}(r,s)) = 0,$$

i.e.

(14) $$(W_{m-2}(r,s)) D_r^2 W_{m-2}(r,s) - W_{m-2}(r,s)(c_{m-1}^2 D_s W_{m-2}(r,s)) = 0.$$

Integrating (14) over the set D_t, we get

$$\int_0^t \int_a^b W_{m-2}(x,s) D_r^2 W(r,s) dr ds -$$

$$- c_{m-1}^2 \int_0^t \int_a^b W_{m-2}(r,s)(D_s W_{m-2}(r,s))\, dr ds = I_{m-2}^1 + I_{m-2}^2,$$

where

$$I_{m-2}^1 = \int_0^t \int_a^b W_{m-2}(r,s) D_r^2 W_{m-2}(r,s) dr ds,$$

$$I_{m-2}^2 = -c_{m-1}^2 \int_0^t \int_a^b W_{m-2}(r,s)(D_s W_{m-2}(r,s)) dr ds.$$

Integrating I^1_{m-2} and I^2_{m-2} by parts we get

$$I^1_{m-2} = (-1/2)\int_0^t\int_a^b D_r W^2_{m-2}(r,s)drds \le 0, \tag{15}$$

$$I^2_{m-2} = (-c^2_m/2)\int_a^b W^2_{m-2}(r,t)dr \le 0. \tag{16}$$

Since $I^1_{m-2} + I^2_{m-2} = 0$ and $I^1_{m-2} \le 0,\ I^2_{m-2} \le 0$, thus (see (14) – (16))

$$W_{m-2}(r,s) = 0 \quad \text{for } (r,s) \in \overline{D_1}$$

and

$$L_{m-2}V(r,s) = 0 \qquad \text{for} \quad (r,s) \in \overline{D_1}. \tag{17}$$

Similarly we get

$$L_{m-2}V(r,s) = 0 \quad \text{for } (r,s) \in \overline{D_1}, \quad i = 3,4,...,m. \tag{18}$$

Multiplying (18) both sides by $L_{m-i-1}V(r,s) = W_{m-i-1}(r,s)$, we obtain

$$W_{m-i-1}(r,s)L_{m-i}V(r,s) = 0, \quad (r,s) \in \overline{D_1}, \quad i = 3,4,...,m,$$

i.e.

$$W_{m-i-1}(r,s)D^2_r W_{m-i-1}(r,s) - c^2_{m-i}D_s W_{m-i-1}(r,s) = 0, \tag{19}$$
$$(r,s) \in \overline{D_1}, \quad i = 3,4,...,m.$$

Integrating (19) over the set D_t, we have

$$\int_0^t\int_a^b W_{m-i-1}(r,s)D^2_r W_{m-i-1}(r,s)drds -$$
$$- c^2_{m-i}\int_0^t\int_a^b W_{m-i-1}(r,s)(D_s W_{m-i-1}(r,s))drds = I^1_{m-i-1} + I^2_{m-i-1} = 0,$$

where

$$I^1_{m-i-1} = \int_a^b\int_0^t W_{m-i-1}(r,s)D^2_r W_{m-i-1}(r,s)drds, \qquad i = 3,4,...,m,$$

$$I^2_{m-i-1} = -c^2_{m-1}\int_0^t\int_a^b W_{m-i-1}(r,s)(D_s W_{m-i-1}(r,s))drds, \qquad i = 3,4,...,m.$$

Integrating by parts I^1_{m-i-1}, I^2_{m-i-1}, we obtain

$$I^1_{m-i-1} = \int_0^t (W_{m-i-1}(r,s)D_r W_{m-i-1}(r,s))\Big|_{r=a}^{r=b} ds -$$

$$-\int_0^t\int_a^b (D_r W_{m-i-1}(r,s)D_r W_{m-i-1}(r,s))drds =$$

$$=(-1/2)\int_0^t\int_a^b D_r W^2_{m-i-1}(r,s)drds \le 0$$

and

$$I^2_{m-i-1} = -c^2_{m-i}\int_0^t\int_a^b W_{m-i-1}(r,s)(D_s W_{m-i-1}(r,s))drds =$$

$$=(-c^2_{m-i}/2)\int_0^t\int_a^b (D_s W^2_{m-i-1}(r,s)drds =$$

$$=(-c^2_{m-i}/2)\int_a^b\int_0^t (D_s W^2_{m-i-1}(r,s))dsdr =$$

$$=(-c^2_{m-i}/2)\int_a^b W^2_{m-i-1}(r,s)\Big|_{s=0}^{s=t} dr = (-c^2_{m-i}/2)\int_a^b W^2_{m-i-1}(r,t)dr \le 0.$$

Consequently

$$W_{m-i-1}(r,s) = 0, \qquad i = 1,2,\dots,m-1.$$

For $i = m-1$, we obtain

$$V(r,s) = 0 \qquad \text{for} \quad (r,s) \in \overline{D_1}.$$

Finally we have

$$v_1(r,s) = v_2(r,s) \qquad \text{for} \quad (r,s) \in \overline{D_1}.$$

4. Green functions

By [3] p. 476 the Green function for the equations $Q_i(G_i(r,t;p,s,c_i)) = 0$, $i = 1,2,\dots,m$ for the domain D_1 and boundary conditions of Dirichlet type $G_i(a,t;p,s,c_i) = G_i(b,t;p,s,c_i) = 0$, $i = 1,2,\dots,m$, are of the form

$$G_i(r,t;p,s,c_i) = U_0(r,t;p,s,c_i) + H(r,t;p,s,c_i),$$

where

$$H(r,t;p,s,c_i)=\sum_{n=1}^{\infty}(-1)^{n-1}(U_n^1(r,t;p,s,c_i)+U_n^2(r,t;p,s,c_i)),$$

$$U_0(r,t;p,s,c_i)=(t-s)^{-1/2}\exp(B(t,s)(r-p)^2c_i^{-1}),$$

$$U_n^1(r,t;p,s,c_i)=(t-s)^{-1/2}\exp(B(t,s)(r_n^1-p)^2c_i^{-1}),\quad n=1,2,3,...,$$

$$U_n^2(r,t;p,s,c_i)=(t-s)^{-1/2}\exp(B(t,s)(r_n^2-p)^2c_i^{-1}),\quad n=1,2,3,...,$$

$$B(t,s)=(-4(t-s))^{-1},$$

$$r=r_0^1=r_0^2,$$

$$r_{2n}^1=r+2n(b-a),\quad n=1,2,3,...,$$

$$r_{2n+1}^1=-r-2n(b-a)+2a,\quad n=1,2,3,...,$$

$$r_{2n}^2=r-2n(b-a),\quad n=1,2,3,...,$$

$$r_{2n+1}^2=-r+2n(b-a)+2b,\quad n=1,2,3,....$$

5. POTENTIALE COMPATIBLE WITH INITIAL CONDITIONS

Consider the following potentials:

$$Z_1(r,t)=A\int_a^b F_1(p)G_1(r,t;p,0,c_1)S_0(p,s)dp,\qquad S_0(p,s)=1,$$

$$Z_2(r,t)=A\int_0^t\int_a^b G_1(r,t;p,s,c_1)S_1(p,s)dpds,$$

$$S_1(r,t)=\int_a^b G_2(r,t;p_1,0,c_1)F_2(p_1)dp_1$$

....

and so one

$$Z_i(r,t)=A\int_0^t\int_a^b G_1(r,t;p_{i-1},s_{i-1},c_1)S_{i-1}(p_{i-1},s_{i-1})dp_{i-1}dp_{i-s},\qquad i=1,2,...,m,$$

$$S_{i-1}(r,t)=\int_0^t\int_a^b G_{i-2}(r,t;p_{i-2},s_{i-2},c_{i-2})S_{i-2}(p_{i-2},s_{i-2})dp_{i-2}ds_i,$$

$$S_{i-2}(r,t)=\int_0^t\int_a^b G_{i-3}(r,t;p_{i-3},s_{i-3},c_{i-3})S_{i-3}(p_{i-3},s_{i-3})dp_{i-3}ds_i,$$

$$S_1(r,t)=\int_a^b G_1(r,t;p_1,0,c_1)F_2(p_1)dp_1,$$

$$S_k(r,t)=\int_0^t\int_a^b G_k(r,t;p_{k-1},s_{k-1},c_k)\left(\int_a^b F_{k+1}(p_{k-1},s_{k-1};\right.$$

$$\left. p_k,0,c_{k+1})dp_k\right)dp_{k-1}\,ds_{k-1},\quad k=1,2,...,m-1.$$

6. PROPERTIES OF THE POTENTIALS Z_i

Lemma 3. If $F_i(r)\in(K_1)$, *then*

1° $Q_i...Q_2Q_1Z_i(r,t)=0,\quad (r,t)\in D_1,\ i=1,2,...,m,\ m\in N$

2° $Q_{i-1}...Q_2Q_1Z_i(r,0)=F_i(r),\quad r\in(a,b),\ i=1,2,...,m,$

3° $Q_{i-1}...Q_2Q_1Z_i(a,t)=Q_{i-1}...Q_2Q_1Z_i(b,t)=0,\quad t\in(0,T),\ i=1,2,...,m.$

Proof. Ad 1°. We have

$$Q_1Z_i(r,t)=Q_1\int_0^t\int_a^b G_1(r,t;p_1,s_1,c_1)S_{i-1}(y_{i-1},s_{i-1})dp_{i-1}\,ds_{i-1}.$$

Applying Poisson's theorem

$$Q_1Z_i(r,t)=S_{i-1}(r,t),\quad (r,t)\in D_1,\ i=1,2,3,...,m.$$

Similarly

$$Q_2Q_1Z_i(r,t)=Q_2(S_{i-1}(r,t))=$$

$$=Q_2\int_0^t\int_a^b G_2(r,t;y_2,s_2,c_2)S_{i-2}(F_{i-2},s_{i-2})dp_{i-2}\,ds_{i-2}$$

on applying Poisson's theorem, we have

$$Q_2Q_1Z_i(r,t)=S_{i-2}(r,t),\quad (r,t)\in D_1,\ i=1,2,...,m,$$

and so one

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(r,t)=$$

$$=Q_{i-1}\int_0^t\int_a^b G_{i-1}(r,t;p_{i-1},s_{i-1},c_{i-1})S_1(p_1,s_1)dp_{i-1}\,ds_{i-1}$$

and applying Poisson's theorem we obtain

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(r,t) = S_1(r,t), \quad (r,t)\in D_1, \ i=1,2,\ldots,m.$$

Further

$$Q_i S_1(p,s) = Q_i \int_a^b G_i(r,t;p_i,0,c_i)F_i(p_i)dp_i =$$

$$= \int_a^b F_i(p_i)Q_i G_i(r,t;p_i,0,c_i)dp_i = 0.$$

Consequently

$$Q_iQ_{i-1}\cdots Q_2Q_1Z_i(r,t) = 0, \quad (r,t)\in D_1, \ i=1,2,\ldots,m.$$

Ad 2°. By 1°, we have

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(r,t) = S_1(r,t), \quad (r,t)\in D_1, \ i=1,2,\ldots,m.$$

Consequently

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(r,t) = \int_a^b F_i(p_i)G_i(r,t;p_i,0,c_i)dp_i \to F_i(x_0)$$

as $(r,t)\to(r_0,0)$, $(r,t)\in D_1$, $i=1,2,\ldots,m$, $r\in(a,b)$.

Ad 3°. By 1°, we have

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(r,t)=S_i(r,t)=\int_a^b F_i(p_i)G_i(r,t;y_i,0,c_i)dp_i=J_i(r,t)$$

because by the astimation

$$\int_a^b F_i(p_i)G_i(r,t;p_i,0,c_i)dp_i \le Ct^{1/2} \le CT^{1/2}$$

and by of the properties of the Green function [3], p. 476, we obtain that the integral $J_i(r,t)$ is locally uniformly convergent at the points $r=a$ and $r=b$. Consequently

$$\lim_{x\to a} J_i(r,t) = \lim_{x\to b} J_i(r,t), \qquad \lim_{x\to a} J_i(r,t) = \lim_{x\to b} J_i(r,t)$$

and

$$Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(a,t) = Q_{i-1}Q_{i-2}\cdots Q_2Q_1Z_i(b,t) = 0,$$

$t\in(0,T)$, $i=1,2,\ldots,m$.

Lemma 4. If $F_i(r)\in(K_1)$, $i=1,2,...,m$, *and if* $Z(r,t)=\sum_{i=1}^{m} Z_i(r,t)$, *then:*

1° $Q_m Q_{m-1}...Q_2 Q_1 Z(r,t)=0, \quad (r,t)\in D_1,$

2° $Q_{i-1}Q_{i-2}...Q_2 Q_1 Z(r,0) = F_i(r), \quad r\in(a,b), \quad i=1,2,...,m,$

3° $Q_{i-1}Q_{i-2}...Q_2 Q_1 Z(a,t) = Q_{i-1}Q_{i-2}...Q_2 Q_1 Z(b,t) = 0, \quad t\in(0,T),\ i=1,2,...,m.$

Proof. Ad 1°. By 1° of the Lemma 3, we get 1° of Lemma 4.

Ad 2°. $Z(r,0)=\sum_{i=1}^{m} Z_i(r,0)=Z_1(r,t)=F_1(r)$

because

$$Z_i(r,0) = 0 \quad \text{for} \quad i=2,3,...,m,$$

since

$$|Z_i(r,t)| \le A\|S_{i-1}\| \int_0^t\int_a^b G_{i-1}(r,t;p,s,c_{i-1})dpds \le$$

$$\le A \sup_{[a,b]}|F_i| \int_0^t\int_a^b G_{i-1}(r,t;p,s,c_{i-1})dpds \le C_i \int_0^t (t-s)^{-1/2} ds \le$$

$$\le C_{i+1}t^{1/2} \le C_{i+1}T^{1/2} \quad \text{and} \quad Z(r,t)\to 0 \quad \text{as} \quad t\to 0.$$

Proof. For $Q_1 Z(r,t)$, we have

$$L_1 Z(r,t) = \sum_{n=1}^{m} Q_1 Z_i(r,t) = S_1(r,t)\to F_2(r_0) \quad \text{as} \quad (r,t)\to(r,0),$$

because by Lemma 3 and by following estimation

$$|Q_1 Z_i(r,t)| \le C_i\, t^{1/2} \le C_i\, T^{1/2},$$

$$Q_1 Z_i(r,t)\to 0 \quad \text{as} \quad t\to 0, \quad \text{for} \quad i=2,3,...,m.$$

Similarly for $L_i Z(r,t) = Q_i Q_{i-1}...Q_2 Q_1 Z(r,t) = \sum_{n=1}^{m} Z_n(r,t)$, we have

$$L_i Z(r,t) = L_i \sum_{n=1}^{m} Z_i(r,t) = \sum_{n=1}^{m} \int_a^b F_i(p) G_i(r,t;p,0,c_i)ds \to F_i(r) \quad \text{as} \quad t\to 0$$

by Lemma 3, point 2° and by the estimation

$$|L_i Z_p(r,t)| \le C_p t^{1/2} \le C_p T^{1/2} \quad \text{for} \quad p>i, \quad i=2,3,...,m,$$

we have

$$\left|L_i Z_p(r,t)\right| \to 0 \text{ as } t \to 0, \ p > i, \ i = 2,3,\ldots,m.$$

Ad 3°. For $Z(r,t)$, we have

$$Z(a,t) = \sum_{i=1}^{m} Z_i(a,t) = 0$$

because

$$Z_i(a,t) = \int_0^t \int_a^b G_1(a,t;p_1,s_1,c_1) S_{i-1}(p_{i-1},s_{i-1}) dp_{i-1}\, ds_{i-1} = 0 \text{ for } i = 1,2,\ldots,m.$$

Similarly we obtain

$$Z(b,t) = 0.$$

For $L_i Z(r,t) = Q_i Q_{i-1} \ldots Q_2 Q_1 Z(r,t)$, we have

$$L_i Z(a,t) = L_i \sum_{p=1}^{m} Z_p(a,t) = \int_a^b F_i(p_i) G_i(a,t;p_i,0,c_i)\, dp +$$

$$+ \sum_{p=2}^{m} \int_0^t \int_a^b G_1(a,t;p_1,s_1,c_1) S_{i-1}(p_1,s_1) dp_1\, ds_1 = 0$$

for $t \in (0,T)$, $i = 1,2,\ldots,m$.

Similarly we obtain

$$L_i Z(b,t) = 0.$$

7. POTENTIALE COMPATIBLE WITH THE BOUNDARY CONDITIONS

Let us consider the potentials

$$V_{1,i}(r,t) = \int_0^t \int_a^b G_1(r,t;p,s,c_1) R^1_{i-1}(p,s) dp ds, \qquad i = 1,2,\ldots,m,$$

where

$$R^1_{i-1}(r,t) = \int_0^t \int_a^b G_2(r,t;p,s,c_2) R^1_{i-2}(p,s) dp ds,$$

$$R^1_{i-2}(r,t) = \int_0^t \int_a^b G_3(r,t;p,s,c_3) R^1_{i-3}(p,s) dp ds,$$

....

$$R_2^1(r,t) = \int_0^t\int_a^b G_{i-1}(r,t;p,s,c_{i-1})\, R_1^1(p,s)\, dp\, ds,$$

$$R_1^1(r,t) = \int_0^t H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds, \quad i=j \quad \text{for } V_{1,j}, \quad j=1,2,\ldots,m.$$

Similarly we define

$$V_{2,i}(r,t) = \int_0^t\int_a^b G_1(r,t;p,s,c_1) R_{i-1}^2(p,s) dp ds, \qquad i=1,2,\ldots,m,$$

where

$$R_{i-1}^2(r,t) = \int_0^t\int_a^b G_2(r,t;p,s,c_1)\, R_{i-1}^2(p,s)\, dp\, ds,$$

....

$$R_2^2(r,t) = \int_0^t\int_a^b G_{i-1}(r,t;p,s,c_{i-1})\, R_1^2(p,s)\, dp\, ds,$$

$$R_1^2(r,t) = \int_0^t K_i(s) D_p\, G_i(r,t;b,s,c_i)\, ds, \quad i=j \quad \text{for } V_{2,j}, \quad j=1,2,\ldots,m.$$

8. PROPERTIES OF THE POTENTIALS $V_{1,i}$ AND $V_{2,i}$

Lemma 5. If H_i, $K_i \in (K_2)$, then

1° $Q_m Q_{m-1} \cdots Q_2 Q_1 V_{1,i}(r,t) = 0, \quad i=1,2,\ldots,m, \quad (r,t)\in D_1,$

2° $Q_i Q_{i-1} \cdots Q_2 Q_1 V_{1,i}(r,0) = 0, \quad i=1,2,\ldots,m-1, \quad r\in(a,b),$

3° $Q_k Q_{k-1} \cdots Q_2 Q_1 V_{1,i}(a,t) = \begin{cases} H_i(t), & k=i,\ i=1,2,\ldots,m-1,\ t\in(0,T), \\ 0 & k\neq i,\ i=1,2,\ldots,m-1,\ t\in(0,T), \end{cases}$

4° $Q_m Q_{m-1} \cdots Q_2 Q_1 V_{2,i}(r,t) = 0, \quad i=1,2,\ldots,m, \quad (r,t)\in D_1,$

5° $Q_i Q_{i-1} \cdots Q_2 Q_1 V_{2,i}(r,0) = 0, \quad i=1,2,\ldots,m-1, \quad r\in(a,b),$

6° $Q_k Q_{k-1} \cdots Q_2 Q_1 V_{2,i}(b,t) = \begin{cases} K_i(t), & k=i,\ i=1,2,\ldots,m-1,\ t\in(0,T), \\ 0 & k\neq i,\ i=1,2,\ldots,m-1,\ t\in(0,T). \end{cases}$

Proof. Ad 1°. By Poisson's theorem we obtain

$Q_1 V_{1,i}(r,t) = R_{i-1}^1(r,t),$

$$Q_2 Q_1 V_{1,i}(r,t) = Q_2 R^1_{i-1}(r,t) = R^1_{i-2}(r,t),$$

....

$$Q_{i-1}\, Q_{i-2} \dots Q_2\, Q_1\, V_{1,i}(r,t) = \int_0^t H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds,$$

$$Q_i\, Q_{i-1} \dots Q_2\, Q_1\, V_{1,i}(r,t) = Q_i \int_0^t H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds =$$

$$= (D_r^2 - c_i^2\, D_t) \int_0^t H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds =$$

$$= \int_0^t D_r^2 (H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds -$$

$$- c_i^2 (\lim_{s \to t} H_i(s)\, D_p\, G_i(r,t;a,s,c_i)) -$$

$$- c_i^2 \int_0^t H_i(s)\, D_t\, D_p\, G_i(r,t;a,s,c_i) ds =$$

$$= \int_0^t Q_m\, H_i(s)\, D_p\, G_i(r,t;a,s,c_i)\, ds = 0$$

therefore

$$Q_k\, Q_{k-1} \dots Q_2 Q_1 V_{1,i}(r,t) = 0 \quad \text{for} \quad k > 1, \quad (r,t) \in D_1.$$

Ad 2°. For $V_{1,i}(r,t)$, we have

$$\left| V_{1,1}(r,t) \right| \le \sup_{[0,t]} |H_1| \int_0^t D_p\, G_1(r,t;a,s,c_1) ds \le$$

$$\le \sup_{[0,t]} |H_1|\, C_{1,1} \to H_1(0)\, C_{1,1} = 0.$$

Similarly, we have

$$\left| V_{1,i}(r,t) \right| \le C_{1,i} t^{1/2} \to 0 \quad \text{as} \quad t \to 0, \quad t = 2,3,\dots,m, \quad (r,t) \in D_1 \tag{20}$$

Consequently

$$Q_i Q_{i-1} \dots Q_2 Q_1 V_{1,i}(r,0) = 0.$$

Ad 3°. We have

$$Q_i\, Q_{i-1} \cdots Q_2\, Q_1\, V_{1,k}(r,t) = \begin{cases} 0 \quad \text{for} \quad i>k, \;\text{by}\quad 1^\circ, \\ \int_0^t H_i(s) D_p\, G_i(r,t;a,s,c_i)\, ds \to H_i(t), \\ \qquad\qquad\qquad \text{for} \quad i=k, \;\text{as}\; x \to a, \\ 0 \quad \text{for} \quad i<k \quad \text{by estimation (20).} \end{cases}$$

The proof 4°, 5°, 6° is similar to these 1°, 2° and 3°.

9. Volume potentials

Let us consider

$$N_m(r,t) = A \int_0^t \int_a^b G_1(r,t;p,s,c_1)\, M_1(p,s)\, dp\, ds, \qquad i = 1,2,\ldots,m,$$

$$M_1(r,t) = \int_0^t \int_a^b G_2(r,t;p,s,c_2)\, M_2(p,s)\, dp\, ds,$$

$$M_2(r,t) = \int_0^t \int_a^b G_3(r,t;p,s,c_3)\, M_3(p,s)\, dp\, ds,$$

....,

$$M_{m-1}(r,t) = \int_0^t \int_a^b G_m(r,t;p,s,c_m)\, F(p,s)\, dp\, ds,$$

$$M_m(r,t) = F(r,f).$$

10. Properties of the volume potentials

Lemma 6. If $F \in (K_4)$, *then*

1° $Q_m Q_{m-1} \cdots Q_2 Q_1 N_m(r,t) = F(r,t), \qquad (r,t) \in D_1,$

2° $Q_i Q_{i-1} \cdots Q_2 Q_1 N_m(r,0) = 0, \qquad i = 0,1,2,\ldots,m-1, \quad r \in (a,b),$

3° $Q_i Q_{i-1} \cdots Q_2 Q_1 N_m(a,t) = Q_i Q_{i-1} \cdots Q_2 Q_1 N_m(b,t) = 0,$

$$i = 0,1,2,\ldots,m-1, \quad t \in (0,T).$$

Proof. Ad 1°. By Poisson's theorem we have

$$Q_1 N_m(r,t) = M_1(r,t), \quad (r,t) \in D_1,$$

$$Q_2 Q_1 N_m(r,t) = M_2(r,t), \quad (r,t) \in D_1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$Q_m Q_{m-1} \dots Q_2 Q_1 N_m(r,t) = F(r,t), \quad (r,t) \in D_1.$$

Ad 2°. Similarly to these 2° of the Lemma 3 we get

$$|N_m(r,t)| \le C_{m,1} t^{1/2} \to 0 \quad \text{as} \quad t \to 0,$$

$$|Q_1 N_m(r,t)| \le C_{m,2} t^{1/2} \to 0 \quad \text{as} \quad t \to 0,$$

$$|Q_2 Q_1 N_m(r,t)| \le C_{m,3} t^{1/2} \to 0 \quad \text{as} \quad t \to 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$|Q_{m-1} Q_{m-2} \dots Q_2 Q_1 N_m(r,t)| \le C_{m,m-1} t^{1/2} \to 0 \quad \text{as} \quad t \to 0.$$

Ad 3°. We have

$$N_m(a,t) = A \int_0^t \int_a^b G_1(a,t;p,s,c_1) M_1(p,s)\, dp\, ds = 0,$$

$$Q_1 N_m(a,t) = A \int_0^t \int_a^b G_2(r,t;p,s,c_2) M_2(p,s) dp ds = 0,$$

$$Q_2\, Q_1\, N_m(a,t) = A \int_0^t \int_a^b G_3(a,t;p,s,c_3) M_3(p,s)\, dp\, ds = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$Q_m\, Q_{m-1} \dots Q_2\, Q_1\, N_m(a,t) = A \int_0^t \int_a^b G_m(a,t;p,s,c_m) F(p,s)\, dp\, ds = 0.$$

Similarly we obtain

$$N_m(b,t) = 0,\ Q_1(b,t) = 0,\ Q_2 Q_1(b,t) = 0,\ \dots,\ Q_{m-1} Q_{m-2} \dots Q_2 Q_1 N_m(b,t) = 0.$$

11. THEOREM ON EXISTENCE OF THE SOLUTION

By Lemmas 1 – 6 we get

Theorem 2. If the assumptions of Lemmas 1 – 6 are satisfies, then the function

$$u(x,t) = U(r,t)\Big|_{r=|x|} = |x|^{-1} \left\{ \sum_{i=1}^{m} \Big[Z_i(r,t)\Big|_{r=|x|} + V_{1,i}(r,t)\Big|_{r=|x|} + \right.$$

$$\left. + V_{2,i}(r,t)\Big|_{r=|x|} \Big] + N_m(r,t)\Big|_{r=|x|} \right\}$$

is solution of the problem (1) – (4).

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