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INTERPOLATION INEQUALITIES IN ORLICZ-SOBOLEV SPACE

ABSTRACT: We consider the problem of determining upper bounds for norms of functions from Orlicz-Sobolev space $W^{j,M}(\Omega)$, $j=0,1,2,...,m$ in terms of norms of the space $W^{m,M}(\Omega)$ and Orlicz space $L^M(\Omega)$.

The interpolation inequalities of this type are well-known for classical Sobolev spaces $W^{m,p}(\Omega)$, $p \geq 1$, (see e.g. [6], [7] and also [1]).

KEY WORDS: Orlicz-Sobolev space, Orlicz space, modular, modular space.

Let $\Re$ be a real vector space. A functional $\rho:\Re \to \overline{R}_+$, where $\overline{R}_+ = [0,+\infty]$, is called a convex pseudomodular in $\Re$, if $\rho(0)=0$, $\rho(-x)=\rho(x)$ and $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ for $x,y \in \Re$, $\alpha,\beta \geq 0$, $\alpha+\beta=1$. If, additionally, $\rho(x)=0$ only for $x=0$, then ρ is called a convex modular on $\Re$, (see [5]).

By a function $M:[0,+\infty) \to [0,+\infty)$ we mean a map which is convex, vanishing and continuous at zero and not vanishing everywhere. A function M satisfies the condition Δ_2 if there exists a constant $K>0$ such that $M(2u) \leq K\,M(u)$ for every $u>0$ (for more details see e.g. [5]).

Let Ω be a nonempty open set in the N-dimensional real Euclidean space R^N. By X we shall denote the vector space of all locally integrable functions on Ω with equality almost everywhere on Ω. Let m be a fixed non-negative integer number. The convex modular J on X we define in the following manner

$$J(f)=\sum_{|\alpha|\leq m} \int_\Omega M(|D^\alpha f(x)|)dx \qquad \text{for } f \in X,$$

where $D^\alpha f$ is the distributional derivative of f. By the Orlicz-Sobolev space $W^{m,M}(\Omega)$ we mean the set of all functions $f \in X$, possessing distributional derivatives $D^\alpha f$ up to order m, for which there exists a constant $a>0$, depending of f, such that $J(af)<\infty$. The space $W^{m,M}(\Omega)$ equipped with the Luxemburg norm $\|\ \|_{W^{m,M}}$ generated by the modular J, is a Banach space, (see [2]). In the sequel we shall use $\|\ \|_m$ in place $\|\ \|_{W^{m,M}}$. Let $W_0^{m,M}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the space $W^{m,M}(\Omega)$.

In the Orlicz-Sobolev space $W^{m,M}(\Omega)$ we define the family functionals $(I_j)_{0\le j\le m}$ as follows

$$I_j(f)=\sum_{|\alpha|=j}\int_\Omega M(|D^\alpha f(x)|)dx \qquad \text{for } f\in X.$$

For $j=0$ the above functional is a convex modular and for $1\le j\le m$ the functionals I_j are convex pseudomodulars. Moreover we define the second family of functionals $(J_j)_{0\le j\le m}$ by

$$J_j(f)=\sum_{|\alpha|\le j}\int_\Omega M(|D^\alpha f(x)|)dx \qquad \text{for } f\in X.$$

The functionals J_j are convex modulars. In particulary, for $j=m$, we obtain the modular J generating $W^{m,M}(\Omega)$.

The following theorem is proveded in [4]:

Theorem 1. Let ρ *be a convex pseudomodular in a real vector space* $\mathfrak{R}$ *and let* $\rho(cu)<\infty$ *for some* $c>0$. *If* $\rho(a(u_n-u))\to 0$ *as* $n\to\infty$ *for every* $a>0$, *then there exists* $b>0$ *such that* $\rho(bu_n)\to\rho(bu)$ *as* $n\to\infty$.

Let the mapping $f\to f^*$, for $f\in X$, denote zero extension of f outside a set Ω:

$$f^*(x)=\begin{cases} f(x) & \text{if} \quad x\in\Omega,\\ 0 & \text{if} \quad x\in R^N-\Omega.\end{cases} \tag{1}$$

Lemma 1. Let $f\in W_0^{m,M}(\Omega)$. *The mapping* $f\to f^*$ *specified by* (1) *is an isometric isomorphism of* $W_0^{m,M}(\Omega)$ *into* $W^{m,M}(R^N)$.

Proof. Let $f\in W_0^{m,M}(\Omega)$ and let $(\phi_n)_{n=1}^\infty\subset C_0^\infty(\Omega)$ be a sequence converning to f in the space $W_0^{m,M}(\Omega)$. For any $\varphi\in C_0^\infty(R^N)$ we have for $|\alpha|\le m$

$$\left|\int_\Omega f(x)D^\alpha\varphi(x)dx-\int_\Omega \phi_n f(x)D^\alpha\varphi(x)dx\right|\le c\left\|f-\phi_n\right\|_{W^{m,M}(\Omega)}.$$

Since

$$\int_{R^N} f^*(x)D^\alpha\varphi(x)dx=\lim_{n\to\infty}\int_\Omega \phi_n(x)D^\alpha\varphi(x)dx=$$

$$=(-1)^{|\alpha|}\lim_{n\to\infty}\int_{\Omega} D^{\alpha}\phi_n(x)\varphi(x)dx=(-1)^{|\alpha|}\int_{\Omega} D^{\alpha}f(x)\varphi(x)dx=$$

$$=(-1)^{|\alpha|}\int_{R^N}(D^{\alpha}f)^*(x)\varphi(x)dx,$$

so $D^{\alpha}f^*=(D^{\alpha}f)^*$ in the distributional sense on R^N. Then we have

$$\int_{\Omega} M(|D^{\alpha}f(x)|)dx=\int_{R^N} M(|(D^{\alpha}f)^*(x)|)dx=\int_{R^N} M(|D^{\alpha}f^*(x)|)dx$$

for $|\alpha|\leq m$. Hence $\|f\|_{W^{m,M}(\Omega)}=\|f^*\|_{W^{m,M}(R^N)}$.

This lemma, for the case $M(u)=u^p$, $1\leq p<\infty$ can be found in [1].

Lemma 2. Let M satisfy the condition Δ_2. *Let* $-\infty\leq a<b\leq\infty$, *and let* $0<\varepsilon_0<\infty$. *There exists a constant* $C=C(\varepsilon_0,M,b-a)$ *for* $0<b-a<\infty$, *such that for every function* $f\in C^2(a,b)$ *and for every* $0<\varepsilon\leq\varepsilon_0$

$$\int_a^b M(|f'(t)|)dt\leq C\int_a^b M(\varepsilon|(f''(t)|)dt+C\int_a^b M(\varepsilon^{-1}|(f(t)|)dt. \tag{2}$$

If $b-a=\infty$, *then* (2) *holds with a constant* $C=C(M)$ *for every* $\varepsilon>0$.

Proof. We assume, that $\varepsilon_0=1$ and $0<b-a<\infty$. If $\xi\in(a,a+(1/3)(b-a))$ and $\eta\in(a+(2/3)(b-a),b)$, then there exists $\lambda\in(\xi,\eta)$ such that

$$|f'(\lambda)|=\left|\frac{f(\eta)-f(\xi)}{\eta-\xi}\right|\leq\frac{3}{b-a}(|f(\eta)|+|f(\xi)|).$$

It follows, by convexity of M, that for any $x\in(a,b)$

$$M(|f'(x)|)\leq\frac{1}{3}M\left(\frac{9}{b-a}|f(\eta)|\right)+\frac{1}{3}M\left(\frac{9}{b-a}|f(\xi)|\right)+\frac{1}{3}M\left(3\int_a^b|f''(t)|dt\right).$$

Applying Jensen's inequality and integrating the above inequality with respect to ξ over $\xi\in(a,a+(1/3)(b-a))$ and with respect to η over $\eta\in(a+(2/3)(b-a),b)$, we obtain

$$M(|f'(x)|)\leq\frac{1}{b-a}\int_a^b M\left(\frac{9}{b-a}|f(t)|\right)+\frac{1}{b-a}\int_a^b M(3(b-a)|f''(t)|)dt.$$

Integrating with respect to x over (a,b), we are led to

$$(3)\qquad \int_a^b M(|f'(t)|)dt \le \int_a^b M\left(\frac{9}{b-a}|f(t)|\right)dt + \int_a^b M(3(b-a)|f''(t)|)dt.$$

Since $0<\varepsilon\le 1$ there exists a positive integer n such that $(1/2)\varepsilon \le (1/n) \le \varepsilon$. Setting $a_j = a+(b-a)(j/n)$ for $j=0,1,\dots,n$, we obtain from (3), noting that $a_{j+1}-a_j=(b-a)/n$,

$$\int_a^b M(|f'(t)|)dt \le \sum_{j=0}^{n-1}\left\{\int_{a_j}^{a_{j+1}} M\left(\frac{18}{b-a}\varepsilon^{-1}|f(t)|\right)dt + \int_{a_j}^{a_{j+1}} M(3(b-a)\varepsilon|f''(t)|)dt\right\}.$$

Let p be a positive integer number such that

$$\max\left\{\frac{18}{b-a}, 3(b-a)\right\} \le 2^p.$$

Then, applying the condition Δ_2, we have

$$(4)\qquad \int_a^b M(|f'(t)|)dt \le K^p \int_a^b M(\varepsilon^{-1}|f(t)|)dt + K^p \int_a^b M(\varepsilon|f''(t)|)dt.$$

Let now ε_0 be an arbitrary positive number. For $0<\varepsilon\le\varepsilon_0$ we have $0<(\varepsilon/\varepsilon_0)\le 1$. Thus, from (4), we obtain

$$\int_a^b M(|f'(t)|)dt \le C\int_a^b M\left(\frac{\varepsilon_0}{\varepsilon}|f(t)|\right)dt + C\int_a^b M\left(\frac{\varepsilon}{\varepsilon_0}|f''(t)|\right)dt.$$

Suppose, that $b-a=\infty$. To be specific we assume $a<\infty$ and $b=\infty$. For given $\varepsilon>0$ let $a_j=a+j\varepsilon$, $j=0,1,2,\dots$. Using (3) we have

$$\int_a^\infty M(|f'(t)|)dt = \sum_{j=0}^{\infty}\int_{a_j}^{a_{j+1}} M(|f'(t)|)dt \le$$

$$\le \int_a^\infty M(9\varepsilon^{-1}|f(t)|)dt + \int_a^\infty M(3\varepsilon|f''(t)|)dt.$$

By the condition Δ_2

$$\int_a^\infty M(|f'(t)|)dt \le C\int_a^\infty M(\varepsilon^{-1}|f(t)|)dt + C\int_a^\infty M(\varepsilon|f''(t)|)dt,$$

which is the desired inequality, where the constant C depends only on M.

The other possibilities are similar.

Lemma 3. Let M satisfy the condition Δ_2. *Let* $0<\delta_0<\infty$, *let* $m\geq 2$ *and let* $\varepsilon_0=\min\{\delta_0,\delta_0^2,\dots,\delta_0^{m-1}\}$. *Suppose that there exists a constant* $K=K(\delta_0,M,\Omega)$ *such that for every* δ, $0<\delta\leq\delta_0$ *and for every* $u\in W^{2,M}(\Omega)$ *we have*

$$I_1(u)\leq KI_2(\delta u)+KI_0(\delta^{-1}u). \tag{5}$$

Then there exists a constant $C=C(\varepsilon_0,m,M,\Omega)$ *such that for every* $0<\varepsilon\leq\varepsilon_0$, *every integer* j, $0\leq j\leq m-1$, *and every* $u\in W^{m,M}(\Omega)$, *we have*

$$I_j(u)\leq CI_m(\varepsilon u)+KI_0\left(\varepsilon^{-\frac{j}{m-j}}u\right). \tag{6}$$

Proof. For $j=0$ the inequality (6) is obvious. We consider only the case $1\leq j\leq m-1$. We first prove (6) for $j=m-1$ by induction on *m*. Then for $m=2$ the inequality (6) is exactly (5). Assume, that (6) holds for some *k*, $2\leq k\leq m-1$,

$$I_{k-1}(u)\leq K_1I_k(\delta\ u)+K_1I_0(\delta^{-(k-1)}u) \tag{7}$$

for every $0<\delta\leq\delta_0$ and $u\in W^{k,M}(\Omega)$. Let $u\in W^{k+1,M}(\Omega)$ and let $|\alpha|=k-1$. Then for $u\in W^{k+1,M}(\Omega)$ we have $D^\alpha u\in W^{2,M}(\Omega)$. Thus, from (5) we obtain

$$I_1(D^\alpha u)\leq K_2I_2(\delta\ D^\alpha u)+K_2I_0(\delta^{-1}D^\alpha u). \tag{8}$$

Then, by (7), we have for $0<\eta\leq\delta_0$

$$I_k(u)\leq\sum_{|\alpha|=k-1}I_1(D^\alpha u)\leq K_3I_{k+1}(\delta u)+K_3I_{k-1}(\delta^{-1}u)\leq$$

$$\leq K_3I_{k+1}(\delta\ u)+K_1K_3I_k(\delta^{-1}\eta u)+K_1K_3I_0(\delta^{-1}\eta^{1-k}u).$$

We may assume without loss of generality, that $2K_1K_3\geq 1$. Taking $\eta=\delta/2K_1K_3$ and applying Δ_2, we obtain

$$I_k(u)\leq K_4I_{k+1}(\delta\ u)+K_4I_0(\delta^{-k}u).$$

This completes the induction establishing (6) for $j=m-1$ with $\varepsilon=\delta$.

By induction on j we prove

$$(9) \qquad I_j(u) \le K_5 I_m(\delta^{m-j} u) + K_5 I_0(\delta^{-j} u)$$

for $1 \le j \le m-1$ and $0 < \delta \le \delta_0$. Setting $k = m$ in (7) we obtain (9) in the special case $j = m-1$. Thus for $j = m-1$ (9) holds. Assume, therefore, that (9) holds for some j, $2 \le j \le m-1$. We prove that it also holds for $j-1$. From (7) and (8) we obtain

$$(10) \qquad I_{j-1}(u) \le K_6 I_m(\delta^{m-(j-1)} u) + K_6 I_0(\delta^{1-j} u).$$

Thus (9) holds. Let $0 < \varepsilon \le \min\{\delta_0, \delta_0^2, \dots, \delta_0^{m-1}\}$ be arbitrary. Then $\varepsilon \le \delta_0^{m-j}$ for every $j = 1, 2, \dots, m-1$. Hence $\varepsilon^{1/(m-j)} \le \delta_0$. Now (6) follows by setting $\delta = \varepsilon^{1/(m-j)}$ in (9).

Theorem 2. Let M satisfy the condition Δ_2. There exists a constant $K = K(m, M, N)$ such that for any $\varepsilon > 0$, any integer j, $0 \le j \le m-1$, and any $u \in W_0^{m,M}(\Omega)$

$$I_j(u) \le K I_m(\varepsilon u) + K I_0\left(\varepsilon^{-\frac{j}{m-j}} u\right).$$

Proof. The operator $Tu = u^*$, $u \in W_0^{m,M}(\Omega)$, specified by (1) is, by Lemma 1, an isometric isomorphism of $W_0^{m,M}(\Omega)$ into $W_0^{m,M}(R^N)$. Thus it is sufficient to prove the theorem for $\Omega = R^N$. By Lemma 3 we need consider only the case $j = 1$, $m = 2$. For $j = 0$, $m = 1$ the desired thesis is obvious.

Let $\varepsilon > 0$ be arbitarary and let $\phi \in C_0^\infty(R^N)$. By Lemma 2 we have

$$\int_{-\infty}^{\infty} M(|D_j \phi(x)|) dx_j \le K \int_{-\infty}^{\infty} M(\varepsilon |D_j^2 \phi(x)|) dx_j + K \int_{-\infty}^{\infty} M(\varepsilon^{-1} |\phi(x)|) dx_j.$$

Integrating the above inequality with respect to the remaining components of x, we obtain

$$\int_{R^N} M(|D_j \phi(x)|) dx \le K \int_{R^N} M(\varepsilon |D_j^2 \phi(x)|) dx + K \int_{R^N} M(\varepsilon^{-1} |\phi(x)|) dx.$$

Hence

$$(11) \qquad I_1(\phi) = \sum_{j=1}^{N} \int_{R^N} M(|D_j \phi(x)|) dx \le K \sum_{|\alpha|=2} \int_{R^N} M(\varepsilon |D^\alpha \phi(x)|) dx +$$

$$+KN\int_{R^N} M(\varepsilon^{-1}|\phi(x)|)dx \le K_1 I_2(\varepsilon\phi) + K_1 I_0(\varepsilon^{-1}\phi).$$

Let $u \in W_0^{m,M}(\Omega)$. Since $C_0^\infty(R^N)$ is dense in $W^{m,M}(R^N)$, (see [3]), it follows that there exists a sequence $(\phi_n) \in C_0^\infty(R^N)$ such that $J(a(u-\phi_n)) \to 0$ as $n \to \infty$ for every $a > 0$. Hence $I_i(a(u-\phi_n)) \to 0$ as $n \to \infty$ for every $a > 0$ and $i = 0,1,2$. By Theorem 1 and the condition Δ_2 we have

$$I_0(\varepsilon^{-1}\phi_n) \le I_0(\varepsilon^{-1}u), \quad I_1(\phi_n) \to I_1(u) \text{ and } I_2(\varepsilon\phi_n) \le I_2(\varepsilon u) \text{ as } n \to \infty.$$

Using (10) we have

$$I_1(\phi_n) \le K_1 I_2(\varepsilon\phi_n) + K_1 I_0(\varepsilon^{-1}\phi_n) \quad \text{for } n = 1,2,\dots. \tag{12}$$

Let $n \to \infty$ in (12). Then, we obtain

$$I_1(u) \le K_1 I_2(\varepsilon u) + K_1 I_0(\varepsilon^{-1}u).$$

This completes the proof.

Theorem 3 Let Ω be arbitrary open set in R^N and let M satisfy the condition Δ_2. Then there exists a constant $K = K(m,M,N)$ such that for $0 \le j \le m$ and any $u \in W_0^{m,M}(\Omega)$

$$\|u\|_j \le K\, \|u\|_m^{j/m}\, \|u\|_0^{(m-j)/m}.$$

Proof. For $j = 0$ and $j = m$ the desired inequality is obvious. Let $0 < j < m$. By means of Theorem 2 for $0 < \varepsilon \le 1$ we have

$$J_j(u) = \sum_{i=0}^{j} I_i(u) \le \sum_{i=0}^{j} \left\{ K I_m(\varepsilon u) + K I_0\left(\varepsilon^{-\frac{i}{m-i}} u\right) \right\} \le \tag{13}$$

$$\le K_1 J_m(\varepsilon u) + K_1 J_0\left(\varepsilon^{-\frac{i}{m-i}} u\right)$$

for any $u \in W_0^{m,M}(\Omega)$. It follows by continuity of M that

$$J_m\left(\frac{u}{\|u\|_m}\right) \le 1 \quad \text{and} \quad J_0\left(\frac{u}{\|u\|_0}\right) \le 1$$

for any $u \in W_0^{m,M}(\Omega)$, $u \neq 0$. Now we set $\varepsilon = (\|u\|_0 / \|u\|_m)^{((m-j)/m)}$ and denote $B = \|u\|_m^{j/m} \|u\|_0^{(m-j)/m}$. Then, by (13), we obtain for fixed j, $J_j(u/B) \leq 2K_1$. We may assume that $2K_1 \geq 1$. Then, by convexity of J_j we have

$$\|u\|_j \leq 2K_1 \|u\|_m^{j/m} \|u\|_0^{(m-j)/m}.$$

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