FASCICULI MATHEMATICI

Nr 30 1999

B.G. PACHPATTE

A NOTE ON MULTIVARIATE VARIANTS OF HARDY'S INEQUALITY

ABSTRACT: In the present note we establish multivariate generalizations of the certain variants of the well known Hardy's inequality. Our results are obtained by using the well known Fubini's theorem.

KEY WORDS: multivariate variants, Hardy's inequality Fubini's theorem, logarithmic, factor, Hölder's inequality.

1. INTRODUCTION

In 1928 G.H. Hardy [4, Theorem 330] proved the following inequality.

Theorem A. If $p>1$, $m\neq 1$, $f(x)\geq 0$ *and* $F(x)$ *is defined by*

$$F(x)=\begin{cases}\int_0^x f(t)dt, & m>1,\\ \int_x^\infty f(t)dt, & m<1,\end{cases}$$

then

$$\int_0^\infty x^{-m}F^p(x)dx<\left(\frac{p}{|m-1|}\right)^p\int_0^\infty x^{-m}(xf(x))^p dx, \tag{1}$$

unless $f\equiv 0$. *The constant is the best possible.*

In the years thereafter, many alternative proofs of Hardy's inequality were found, and more general results were obtained (see [1-10]). In 1979, the case of special interest of Theorem A when $m=1$ was discussed by L.Y. Chan [3] by decomposing $[0,\infty)$, the interval of integration, into $[0,1)$ and $[1,\infty)$. The following variants of Theorem A were obtained in [3].

Theorem B. Let $1<p<\infty$, $f(x)$ *is a nonnegative measurable function such that* $f\in L(x,\infty)$ *for every* $x\in(1,\infty)$, *and* $F(x)=\int_x^\infty f(t)dt$. *Then we have*

(2) $$\int_1^\infty x^{-1}F^p(x)dx \le p^p \int_1^\infty x^{-1}[(x\log x)f(x)]^p dx.$$

Theorem C. Let $1<p<\infty$, $f(x)$ *is a nonnegative measurable function such that* $f\in L(0,x)$ *for every* $x\in(0,1)$, *and* $F(x)=\int_x^\infty f(t)dt$. *Then we have*

(3) $$\int_0^1 x^{-1}F^p(x)dx \le p^p \int_0^1 x^{-1}[(x\,|\log x|\,f(x)]^p dx.$$

Theorems B and C give inequalities which depart somewhat from the structure of the Hardy's inequality in Theorem A, since (2) and (3) contains an extra logarithmic factor in the integrals on the right side. In the present note we establish the multivariate generalizations of Theorems B and C, which we believe are of interest in their own right. Our proofs are elementary and based on the applications of the well known Fubini's theorem.

2. Main results

In what follows, we let $B_{u,v}$ be a subset of the n-dimensional Euclidean space R^n defined by $B_{u,v}=\{z\in R^n : u<z<v\}$ where $u,v\in R^n$. We denote by $\int_{B_{x,y}} f(z)dz$ the n-fold integral

$$\int_{x_1}^{y_1}\cdots\int_{x_n}^{y_n} f(z_1,\dots,z_n)dz_n\dots dz_1,$$

where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ are in R^n such that $x_i<y_i$ for $i=1,\dots,n$. Throught we assume that all inequalities between vectors are componentwise and all the integrals exists on the respective domains of their definitions.

A generalization of Theorem B to the case of multivariate functions is contained in the following

Theorem 1. Let $p>1$ *be a constant and* $f(x)$ *be a nonnegative and integrable function on* $B_{1,a}$. *If*

(4) $$F(x)=\int_{B_{x,a}} f(y)dy, \quad x\in B_{1,a},$$

then

$$\int_{B_{1,a}} \prod_{i=1}^{n} x_i^{-1} F^p(x)dx \le p^{np} \int_{B_{1,a}} \prod_{i=1}^{n} x_i^{-1} \left[\left(\prod_{i=1}^{n} x_i \log x_i \right) f(x) \right]^p dx. \tag{5}$$

Proof. If f is null, the inequality (5) is trivially true. We assume that f is not null. Denote by I the integral on the left side in (5). By Fubini's theorem [1, p. 18] we observe that

$$I = \int_1^{a_1} \dots \int_1^{a_{n-1}} \prod_{i=1}^{n-1} x_i^{-1} I_n \, dx_{n-1} \dots dx_1, \tag{6}$$

where

$$I_n = \int_1^{a_n} x_n^{-1} \left(\int_{x_n}^{a_n} \left(\int_{x_1}^{a_n} \dots \int_{x_{n-1}}^{n-1} f(y_1, \dots, y_{n-1}, y_n \right) \times \right.$$
$$\left. \times \, dy_{n-1} \dots dy_1 \right) dy_n \Bigg)^p dx_n. \tag{7}$$

By keeping $x_1, \dots x_{n-1}$ fixed in (7) and integrating by parts, we have

$$I_n = \Bigg[(\log x_n) \left(\int_{x_n}^{a_n} \left(\int_{x_1}^{a_1} \dots \int_{x_{n-1}}^{a_{n-1}} f(y_1, \dots, y_{n-1}, y_n) \times \right. \right.$$
$$\left. \left. \times \, dy_{n-1} \dots dy_1 \right) dy_n \right)^p \Bigg]_1^{a_n} -$$
$$- \int_1^{a_n} (\log x_n) p \left[\int_{x_n}^{a_n} \left(\int_{x_1}^{a_1} \dots \int_{x_{n-1}}^{a_{n-1}} f(y_1, \dots, y_{n-1}, y_n) \times \right. \right.$$
$$\left. \left. \times \, dy_{n-1} \dots dy_1 \right) dy_n \right]^{p-1} \times$$
$$\times \left(- \int_{x_1}^{a_1} \dots \int_{x_{n-1}}^{a_{n-1}} f(y_1, \dots, y_{n-1}, x_n) dy_{n-1} \dots dy_1 \right) dy_n \Bigg) dx_n. \tag{8}$$

From (8) we observe that

$$(9)\qquad I_n = p\int_1^{a_n}\left[x_n^{(p-1)/p}(\log x_n)\left(\int_{x_1}^{a_1}\cdots\int_{x_{n-1}}^{a_{n-1}} f(y_1,\dots,y_{n-1},x_n)\times \times dy_{n-1}\dots dy_1\right)\right]\times$$

$$\times\left[x_n^{-(p-1)/p}\int_{x_n}^{a_n}\left(\int_{x_1}^{a_1}\cdots\int_{x_{n-1}}^{a_{n-1}} f(y_1,\dots,y_{n-1},y_n)\times \times dy_{n-1}\dots dy_1\right)dy_n\right]^{p-1} dx_n .$$

Now applying Hölder's inequality with indices *p, p/*(*p*-1) on the right hand side of (9) we obtain

$$(10)\qquad I_n \le p\left[\int_1^{a_n}\left(x_n^{(p-1)}(\log x_n)^p\left(\int_{x_1}^{a_1}\cdots\int_{x_{n-1}}^{a_{n-1}} f(y_1,\dots,y_{n-1},x_n)\times \times dy_{n-1}\dots dy_1\right)\right)^p dx_n\right]^{1/p} I_n^{(p-1)/p} .$$

Dividing both sides of (10) by the second factor on right-hand side of (10) since $I_n > 0$, and then raising both sides to the *p*-th power we obtain

$$(11)\qquad I_n \le p^p\int_1^{a_n} x_n^{(p-1)}(\log x_n)^p\left(\int_{x_1}^{a_1}\cdots\int_{x_{n-1}}^{a_{n-1}} f(y_1,\dots,y_{n-1},x_n)\times \times dy_{n-1}\dots dy_1\right)^p dx_n .$$

Substituting (11) in (6) and using Fubini's theorem we observe that

$$(12)\qquad I \le p^p\int_1^{a_1}\cdots\int_1^{a_{n-2}}\prod_{i=1}^{n-2} x_i^{-1}\int_1^{a_n} x_n^{(p-1)}(\log x_n)^p I_{n-1}\times \times dx_n dx_{n-2}\dots dx_1 ,$$

where

$$(13)\qquad I_{n-1} = \int_1^{a_{n-1}} x_{n-1}^{-1} \left(\int_{x_{n-1}}^{a_{n-1}} \left(\int_{x_1}^{a_1} \cdots \int_{x_{n-2}}^{a_{n-2}} f(y_1, \ldots, y_{n-2}, y_{n-1}, x_n) \times \right.\right.$$

$$\left.\left. \times\, dy_{n-2} \ldots dy_1 \right) dy_{n-1} \right)^p dx_{n-1}.$$

Now, by following exactly the same arguments as above with suitable modifications, we obtain

$$(14)\qquad I_{n-1} \le p^p \int_1^{a_{n-1}} x_{n-1}^{(p-1)} (\log x_{n-1})^p \left(\int_{x_1}^{a_1} \cdots \int_{x_{n-2}}^{a_{n-2}} f(y_1, \ldots, y_{n-2}, x_{n-1}, x_n) \times \right.$$

$$\left. \times\, dy_{n-2} \ldots dy_1 \right)^p dx_{n-1}.$$

Substituting (14) in (12) and again using Fubini's theorem we have

$$I \le p^{2p} \int_1^{a_1} \cdots \int_1^{a_{n-3}} \prod_{i=1}^{n-3} x_i^{-1} \int_1^{a_{n-1}} x_{n-1}^{(p-1)} (\log x_{n-1})^p \times$$

$$\times \int_1^{a_n} x_n^{(p-1)} (\log x_n)^p I_{n-2}\, dx_n dx_{n-1} dx_{n-3} \ldots dx_1$$

where

$$I_{n-2} = \int_1^{a_{n-2}} x_{n-2}^{-1} \left(\int_{x_{n-2}}^{a_{n-2}} \left(\int_{x_1}^{a_1} \cdots \int_{x_{n-3}}^{a_{n-3}} \times \right.\right.$$

$$\left.\left. \times\, f(y_1, \ldots, y_{n-3}, y_{n-2}, x_{n-1}, x_n) dy_{n-3} \ldots dy_1 \right) dy_{n-2} \right)^p dx_{n-2}.$$

Continuing in this way, we finally get

$$I \le p^{np} \int_{B_{1,a}} \prod_{i=1}^{n} x_i^{-1} \left[\left(\prod_{i=1}^{n} x_i \ \log x_i \right) f(x) \right]^p dx.$$

This is the required inequality in (5) and the proof is complete.

A multivariate analog of Theorem C is embodied in the following

Theorem 2. *Let* $p>1$ *be a constant and* $f(x)$ *be a nonnegative and integrable function on* $B_{0,1}$. *If*

$$F(x)=\int_{B_{0,x}} f(y)dy, \qquad x\in B_{0,1}, \tag{15}$$

then

$$\int_{B_{0,1}} \prod_{i=1}^{n} x_i^{-1} F^p(x)dx \le p^{np} \int_{B_{0,1}} \prod_{i=1}^{n} x_i^{-1}\left[\left(\prod_{i=1}^{n} x_i \,|\log x_i|\right) f(x)\right]^p dx. \tag{16}$$

The proof of this theorem follows by the same arguments as in the proof of Theorem 1 given above with suitable modifications. We omit the details.

We note that in [3] Chan has obtained inequalities (2) and (3) by using the method due to Benson [2]. Here our proofs are more direct and elementary than those contained in [3]. In the special case when $n=1$, the inequalities established in Theorems 1 and 2 reduces respectively to the inequalities given by Chan [3] in Theorems B and C.

REFERENCES

[1] R.A. Adams, *Sobolev spaces,* Academic Press, New York , 1975.

[2] D.C. Benson, Inequalities involving integrals of functions and their derivatives, *J. Math. Anal. Appl.* 17(1967), 292-308.

[3] L.Y. Chan, Some extensions of Hardy's inequality, *Canadian Math. Bull.* 22(1979), 165-169.

[4] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1934.

[5] B.G. Pachpatte, On a new class of Hardy type inequalities, *Proc. Royal Soc. Edinburgh* 105(A)(1987), 265-274.

[6] B.G. Pachpatte, A many variable generalization of Hardy's inequality concerning a series of terms, *Tamkang J. Math.* 23(1992), 349-354.

[7] B.G. Pachpatte, On certain two dimensional Hardy type integral inequalities, *J. Ramanujan Math. Soc.* 7(1992), 93-102.

[8] B.G. Pachpatte, On generalizations of Hardy's integral inequality for function of several variables, *Demonstratio Mathematica* 27(1994), 43-51.

[9] B.G. Pachpatte, On Hardy type integral inequalities for functions of two variables, *Demonstratio Mathematica* 28(1995), 239-244.

[10] A. Wanebo, Hardy inequalities, *Proc. Amer. Math. Soc.* 109(1990), 85-95.

(Department of Mathematics, Marthwada University, Aurangabad 431 004, Maharashtra, India)
Received on 10.04.1996 and, in revised form, on 31.12.1996.