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A LIMIT PROBLEM FOR A FACTORISED PARABOLIC EQUATION OF THE FOURTH ORDER

ABSTRACT: The aim of this paper is the construction of the classical solution to a nonlinear factorised differential biparabolic equation of the fourth order, in a rectangular domain, with boundary-value conditions of the Dirichlet and the Neumann types.

KEY WORDS: a limit problem, a factorised parabolic equation, Green functions, an integral equation, the Banach fixed point method.

1. INTRODUCTION

In the present paper, for given functions F, f_i, h_i and k_i $(i=1,2)$, we construct the classical solution u to the nonlinear factorised biparabolic equation

(1) $$P_2 P_1 u(x,t) = F(x,t,u(x,t)), \qquad (x,t)\in D_1,$$

where $P_i = D_x^2 - C_i^{-1} D_t$ $(i=1,2)$, C_i $(i=1,2)$ are positive constants and

$$D_1 := \{(x,t) : x\in \mathrm{J} := (0,1),\ t\in(0,T]\},$$

satisfying the initial conditions

(2) $$u(x,0) = f_1(x), \qquad x\in \mathbf{J},$$

(3) $$P_1 u(x,0) = f_2(x), \qquad x\in \mathbf{J}$$

and the boundary-value conditions

(4) $$u(0,t) = h_1(t), \quad t\in(0,T],$$

(5) $$u(1,t) = h_2(t), \quad t\in(0,T],$$

(6) $$D_x P_1 u(0,t) = k_1(t), \quad t\in(0,T],$$

(7) $$D_x P_1 u(1,t) = k_2(t), \quad t\in(0,T].$$

Iterated and factorised problems were also studied in papers [2], [4-6] and [8].

2. SOME DEFINITIONS

Let $\mathcal{K}_1$ denote the class of all functions $w: D_1 \to \mathbf{R} := (-\infty,+\infty)$ such that $w\in C^{4,2}(D_1)\cap C^{3,1}(\overline{D_1})$ and $|w|\le R$ in D_1, where $R>0$ is a constant.

By $\mathcal{K}_2$ we denote the class of all functions $\xi : \bar{\mathbf{J}} \to \mathbf{R}$ such that $\xi \in C(\bar{\mathbf{J}})$ and $\xi(0) = \xi(1) = 0$.

Let $\mathcal{K}_3$ denote the class of all functions $\eta : [0,T] \to \mathbf{R}$ such that $\eta \in C([0,T])$ and $\eta(0) = \eta(T) = 0$.

By $\mathcal{K}_4$ we denote the class of all functions γ satisfying the conditions:

(i) function γ is defined and continuous in the set

$$D_2 := \{(x,t,z) : (x,t) \in \overline{D}_1,\ z \in \mathrm{R}\},$$

(ii) function γ is uniformly bounded in D_2, i.e.

$$|\gamma(x,t,z)| \le M \quad \text{for } (x,t,z) \in D_2,$$

where M is a positive constant,

(iii) function γ satisfies the following Lipschitz condition with a positive constant L:

$$|\gamma(x,t,z_1) - \gamma(x,t,z_2)| \le L|z_1 - z_2|,$$
$$(x,t) \in \overline{D}_1, \qquad z_i \in \mathbf{R} \quad (i = 1,2).$$

In the sequel we will need the notation

$$\widetilde{F}(y,s,z) = \begin{cases} 0 & \text{for } (y,s,z) \in [\mathbf{R} \setminus \bar{\mathbf{J}}] \times [0,T] \times \mathbf{R}, \\ F(y,s,z) & \text{for } (y,s,z) \in \overline{D}_1 \times \mathbf{R}. \end{cases}$$

3. THE SYSTEM OF THE DIFFERENTIAL PROBLEMS COMPATIBLE TO PROBLEM (1) – (7)

To solve problem (1) – (7) we will consider a suitable system of differential problem (I) – (III).

Problem (I) consists in the construction of the classical solution $U^1 \in C^{2,1}(D_1) \cap C^{1,0}(\overline{D}_1)$ to the equation

(8) $$P_1 U^1(x,t) = 0, \quad (x,t) \in D_1,$$

satisfying the limit conditions

(9) $$U^1(x,0) = f_1(x), \quad x \in \mathbf{J},$$

(10) $$U^1(0,t) = h_1(t), \quad t \in (0,T],$$

(11) $$U^1(1,t) = h_2(t), \quad t \in (0,T].$$

Problem (II) consists in the construction of the classical solution $U^2 \in$ $\in C^{4,2}(D_1) \cap C^{3,1}(\overline{D}_1)$ to the equation

(12) $$P_2 P_1 U^2(x,t) = 0, \qquad (x,t) \in D_1,$$

satisfying the limit conditions

(13) $$U^2(x,0) = 0, \qquad x \in \mathbf{J},$$

(14) $$P_1 U^2(x,0) = f_2(x), \qquad x \in \mathbf{J},$$

(15) $$U^2(0,t) = 0, \qquad t \in (0,T],$$

(16) $$U^2(1,t) = 0, \qquad t \in (0,T],$$

(17) $$D_x P_1 U^2(0,t) = k_1(t), \quad t \in (0,T],$$

(18) $$D_x P_1 U^2(1,t) = k_2(t), \quad t \in (0,T].$$

Problem (III) consists in the construction of the classical solution $U^3 \in$ $\in C^{4,2}(D_1) \cap C^{3,1}(\overline{D}_1)$ to the equation

(19) $$P_2 P_1 U^3(x,t,u(x,t)) = F(x,t,u(x,t)), \qquad (x,t) \in D_1,$$

satisfying the homogeneous limit conditions

(20) $$U^3(x,0,u(x,0)) = 0, \qquad x \in \mathbf{J},$$

(21) $$P_1 U^3(x,0,u(x,0)) = 0, \qquad x \in \mathbf{J},$$

(22) $$U^3(0,t,u(0,t)) = 0, \qquad t \in (0,T],$$

(23) $$U^3(1,t,u(1,t)) = 0, \qquad t \in (0,T],$$

(24) $$D_x P_1 U^3(0,t,u(0,t)) = 0, \qquad t \in (0,T],$$

(15) $$D_x P_1 U^3(1,t,u(1,t)) = 0, \qquad t \in (0,T].$$

In the sequel we shall give the integral equation equivalent to system (I) – (III).

4. THE GREEN FUNCTION G_1

Firstly, we shall study the Green function G_1 to problem (I). For this purpose define the sequence $x_{k,i}$ $(k = 0,1,2,...; i = 1,2)$ by the formulas

(26) $$x_{0,i} = x \ (i = 1,2), \quad x_{2n,1} = x + 2n, \quad x_{2n,2} = x - 2n \quad (n = 1,2,...),$$

$$x_{2n+1,1} = -x - 2n, \quad x_{2n+1,2} = -x + 2n \quad (n = 0,1,2,...).$$

Now, let us define the Green function G_1 by the formula

$$G_1(x,t,y,s) = U_1(x,t,y,s) + \sum_{n=1}^{\infty} (1)^n [U_1(x_{n,1},t,y,s) +$$

$$+ U_1(x_{n,2},t,y,s)], \qquad (x,t,y,s) \in D_{3,}$$

where

$$D_3 := \{(x,t,y,s) : 0 \le s < t \le T, (x,y) \in \overline{\mathbf{J}}^2, \ x \ne y\}$$

and U_1 is the fundamential solution to the equation $P_1 U_1 = 0$, given by the formula

$$U_1(x,t,y,s) := A_1 (t-s)^{-1/2} \exp(B_1(t,s)(x-y)^2), \quad (x,t,y,s) \in D_3,$$

where

$$A_1 := (4C_1 \Pi)^{-1/2} \quad \text{and} \quad B_1(t,s) := -(4C_1(t-s))^{-1}.$$

Lemma 1. The Green function G_1 satisfies the following conditions:

(*i*) $P_1 G_1(x,t,y,s) = 0$ *for* $(x,t,y,s) \in D_3$,

(*ii*) $G_1(0,t,y,s) = G_1(1,t,y,s) = 0$ *for* $(0,t,y,s),(1,t,y,s) \in D_3$,

(*iii*) $G_1(x,t,0,s) = G_1(x,t,1,s) = 0$ *for* $(x,t) \in D_1$, $s \in (0,t]$,

(*iv*) $0 \le G_1(x,t,y,s) \le U_1(x,t,y,s)$ *for* $(x,t,y,s) \in D_3$,

(*v*) $\int_0^t \int_0^1 G_1(x,t,y,s)dyds \le \int_0^t \int_0^1 U_1(x,t,y,s)dyds \le 2A_1 t^{1/2}$,

(*vi*) *for every closed subset of the domain D_3 the Green function G_1 behaves as the function $(t-s)^{-1/2}$ if $s \to t$.*

Proof. See [1] and [3].

It is known (see [3], p. 475) that

$$\left| D_x^i \sum_{n=2}^{\infty} (-1)^n \left[U_1(x_{n,1},t,y,s) + U_1(x_{n,2},t,y,s)\right]\right| =$$

$$= \left| \sum_{n=2}^{\infty} (-1)^n \left[D_x^i U_1(x_{n,1},t,y,s) + D_x^i U_1(x_{n,2},t,y,s)\right]\right| \le$$

$$\le C_{i+3} \sum_{n=2}^{\infty} (n-1)^{-2} \quad (i = 0,1,2),$$

$$\left| D_t \sum_{n=2}^{\infty} (-1)^n \left[U_1(x_{n,1}, t, y, s) + U_1(x_{n,2}, t, y, s) \right] \right| \le C_6 \sum_{n=2}^{\infty} (n-1)^{-2},$$

where C_i $(i = 3,4,5,6)$ are positive constants.

5. The Green function G_2

Firstly, we shall define the Green function G_2 to problem (II) by the formula

$$G_2(x,t,y,s) = U_2(x,t,y,s) + \sum_{n=1}^{\infty} [U_2(x_{n,1}, t, y, s) +$$
$$+ U_2(x_{n,2}, t, y, s)], \qquad (x,t,y,s) \in D_3,$$

where U_2 is the fundamential solution to the equation $P_2 u = 0$, given by the formula

$$U_2(x,t,y,s) := A_2(t-s)^{-1/2} \exp(B_2(t,s)(x-y)^2), \quad (x,t,y,s) \in D_3,$$

where

$$A_2 := (4C_2\Pi)^{-1/2} \quad \text{and} \quad B_2(t,s) := -(4C_2(t-s))^{-1}.$$

By the definition of G_2 and by (26), we have

$$G_2(x,t,y,s) = U_2(x,t,y,s) + \tag{27}$$
$$+ A_2(t-s)^{-1/2} \left[\sum_{n=0}^{\infty} \exp(B(t,s)(x+2n+2-y)^2) + \right.$$
$$+ \sum_{n=0}^{\infty} \exp(B(t,s)(-x+2n+2+y)^2) + \sum_{n=0}^{\infty} \exp(B(t,s)(x+2n+y)^2) +$$
$$\left. + \sum_{n=0}^{\infty} \exp(B(t,s)(-x+2n+2-y)^2) \right], \qquad (x,t,y,s) \in D_3.$$

For $D_x G_2(x,t,y,s)$ we obtain the formula

$$D_x G_2(x,t,y,s) = -A_2(2C_2)^{-1}(t-s)^{-3/2}(x-y)\exp(B(t,s)(x-y)^2) - \tag{28}$$
$$- A_2(2C_2)^{-1}(t-s)^{-3/2} \left[\sum_{n=0}^{\infty} (x+2n+2-y)\exp(B(t,s)(x+2n+2-y)^2) + \right.$$

$$+\sum_{n=0}^{\infty}(-x+2n+2+y)\exp(B(t,s)(-x+2n+2+y)^2)-$$

$$-\sum_{n=0}^{\infty}(x+2n+y)\exp(B(t,s)(x+2n+y)^2)+$$

$$\left.+\sum_{n=0}^{\infty}(-x+2n+2-y)\exp(B(t,s)(-x+2n+2-y)^2)\right],\quad (x,t,y,s)\in D_3.$$

Lemma 2. If $z \geq 0$ and k is a positive constant then

(29) $$z^k\exp(-z^2) \leq C_7,$$

where C_7 is a positive constant.

We omit the simple proof.
By formulas (27) – (29), we obtain

Lemma 3. The Green function G_2 satisfies the following conditions:

(*i*) $P_2G_2(x,t,y,s)=0$ *for* $(x,t,y,s)\in D_3$,

(*ii*) $D_xG_2(0,t,y,s)=D_xG_2(1,t,y,s)=0$ *for* $(0,t,y,s),(1,t,y,s)\in D_3$,

(*iii*) $0\leq G_2(x,t,y,s)\leq U_2(x,t,y,s)+C_8\sum_{n=2}^{\infty}\frac{1}{(n-1)^2}$ *for* $(x,t,y,s)\in D_3$,
where C_8 is a positive constant,

(*iv*) $G_2(x,t,y,s)\to 0$ *uniformly when* $t\to 0$.

6. The Green function $G_{1,2}$

Let us define the function $G_{1,2}$ by the formula

(30) $$G_{1,2}(x,t,y,s):=\int_0^t\int_0^1 G_1(x,t,y_1,s_1)\,G_2(y_1,s_1,y,s)dy_1\,ds_1.$$

Lemma 4. The function $G_{1,2}$ satisfies the conditions:

(*i*) $P_1G_{1,2}(x,t,y,s)=G_2(x,t,y,s)$ *for* $(x,t,y,s)\in D_3$,

(*ii*) $P_2P_1G_{1,2}(x,t,y,s)=0$ *for* $(x,t,y,s)\in D_3$,

(iii) $G_{1,2}(0,t,y,s)=G_{1,2}(1,t,y,s)=0$ *for* $(0,t,y,s),(1,t,y,s)\in D_3$,

(iv) $D_x P_1 G_{1,2}(0,t,y,s)=D_x P_1 G_{1,2}(1,t,y,s)=0$ *for* $(0,t,y,s),(1,t,y,s)\in D_3$,

(v) $\int_0^t \int_0^1 G_{1,2}(x,t,y,s)dyds \le 4A_1 A_2 t$ *for* $(x,t)\in D_1$,

(vi) *if* $F\in\mathcal{K}_1$ *and* $I(x,t,u(x,t)) := \int_0^t \int_0^1 F(y,s,u(y,s))\Big[\int_0^{s_1}\int_0^1 G_1(x,t,y_1,s_1)\times$ $\times G_2(y_1,s_1,y,s)dy_1 ds_1\Big]dyds$ *then* $P_2P_1I(x,t,u(x,t))=F(x,t,u(x,t))$ *for* $(x,t)\in D_1$,

(vii) $I(0,t,u(0,t))=I(1,t,u(1,t))=0$ *for* $t\in(0,T]$,

$(viii)$ $D_x P_1 I(0,t,u(0,t))=D_x P_1 I(1,t,u(1,t))=0$ *for* $t\in(0,T]$.

Proof. (i): By (30) and by [3], we obtain

$$P_1G_{1,2}(x,t,y,s)=P_1\int_0^t\int_0^1 G_1(x,t,y_1,s_1)G_2(y_1,s_1,y,s)dy_1\,ds_1=$$
$$=\int_0^t\int_0^1 P_1G_1(x,t,y_1,s_1)G_2(y_1,s_1,y,s)dy_1\,ds_1=$$
$$=G_2(x,t,y,s)\quad\text{for}\quad(x,t,y,s)\in D_3.$$

(ii):

$$P_2P_1G_{1,2}(x,t,y,s)=P_2G_2(x,t,y,s)=0\quad\text{for}\ (x,t,y,s)\in D_3.$$

$(iii)-(v)$: By properties of functions G_1 and G_2, we have $(iii)-(v)$. Observe that

$$\int_0^t\int_0^1 G_{1,2}(x,t,y,s)dyds\le A_1\,A_2\int_0^t\frac{ds_2}{(t-s_2)^{1/2}}\int_0^t\frac{ds_1}{(t-s_1)^{1/2}}\le$$
$$\le 4A_1A_2t\quad\text{for}\ (x,t)\in D_1.$$

(vi): By the Poisson theorem (see [3]), we obtain

$$P_2P_1I(x,t,u(x,t))=P_2P_1\int_0^t\int_{-\infty}^{+\infty}G_1(x,t,y_1,s_1)\times$$
$$\times\left[\int_0^{s_1}\int_{-\infty}^{+\infty}\widetilde{F}(y,s,u(y,s))G_2(y_1,s_1,y,s)dyds\right]dy_1\,ds_1=$$

$$= P_2 \int_0^t \int_{-\infty}^{+\infty} G_2(x,t,y,s)\widetilde{F}(y,s,u(y,s))dyds =$$

$$= \tilde{F}(x,t,u(x,t)) = F(x,t,u(x,t)), \quad (x,t) \in D_1, \quad u \in \mathcal{K}_1.$$

(vii): Since the majorant of the integral I is of the form $4A_1A_2Mt^2$ then, by the local uniform convergence of the integral I, we get

$$I(0,t,u(0,t)) = \int_0^t \int_{-\infty}^{+\infty} \widetilde{F}(y,s,u(y,s)) \left[\int_0^{s_1} \int_{-\infty}^{+\infty} \lim_{x\to 0} G_1(x,t,y_1,s_1) \times \right.$$

$$\left. \times G_2(y_1,s_1,y,s)dy_1 ds_1 \right] dyds = 0 \qquad \text{for} \quad t \in (0,T].$$

Similarly, we obtain that $I(1,t,u(1,t)) = 0$ for $t \in (0,T]$.

$(viii)$: Since

$$P_1 I(x,t,u(x,t)) = \int_0^t \int_{-\infty}^{+\infty} \widetilde{F}(y,s,u(y,s)) G_2(x,t,y,s)dyds$$

then, as in (vi), we get

$$D_x P_1 I(0,t,u(0,t)) = \int_0^t \int_{-\infty}^{+\infty} \widetilde{F}(y,s,u(y,s)) \lim_{x\to 0} D_x G_2(x,t,y,s)dyds = 0$$

for $t \in (0,T]$ and

$$D_x P_1 I(1,t,u(1,t)) = \int_0^t \int_{-\infty}^{+\infty} \widetilde{F}(y,s,u(y,s)) \lim_{x\to 1} D_x G_2(x,t,y,s)dyds = 0$$

for $t \in (0,T]$.

7. The fundamental formulas to the solution of system (I) – (III)

It is known (see [3]) that, under some assumptions, for problem (8) – (11), the following integral equation can be considered:

(31) $$U^1(x,t) = \int_0^t U^1(0,s) D_y G_1(x,t,0,s)ds -$$

$$-\int_0^t U^1(1,s)D_yG_1(x,t,1,s)ds+\int_0^1 U^1(y,0)G_1(x,t,y,0)dy=$$

$$=\int_0^1 f_1(y)G_1(x,t,y,0)dy+\int_0^t h_1(s)D_yG_1(x,t,0,s)ds-$$

$$-\int_0^t h_2(s)D_yG_1(x,t,1,s)ds.$$

As a consequence of the potential theory, we have:

Lemma 5. If $f_1\in\mathcal{K}_2$ *and* $h_1\in\mathcal{K}_3$ $(i=1,2)$ *then the function* U^1, *given by formula* (31), *satisfies the conditions:*

(*i*) $P_1U^1(x,t)=0$ *for* $(x,t)\in D_1$,

(*ii*) $U^1(x,0)=f_1(x)$ *for* $x\in\mathbf{J}$,

(*iii*) $U^1(0,t)=h_1(t)$ *for* $t\in(0,T]$,

(*iv*) $U^1(1,t)=h_2(t)$ *for* $t\in(0,T]$.

Conversely, if function U^1 *satisfies conditions* (*i*) – (*iv*) *then* U^1 *satisfies the integral equation* (31) *with* $U^1(y,0)=f_1(y)$, $U^1(0,t)=h_1(t)$ *and* $U^1(1,t)=$ $=h_2(t)$.

Observe that for problem (12) – (18) we obtain the following integral equation:

$$P_1U^1(x,t)=\int_0^t P_1U^2(y,0)G_2(x,t,y,0)dy- \tag{32}$$

$$-\int_0^t D_yP_1U^2(0,s)G_2(x,t,0,s)ds+$$

$$+\int_0^1 D_yP_1U^2(1,s)G_2(x,t,1,s)ds=$$

$$=\int_0^1 f_2(y)G_2(x,t,y,0)dy-\int_0^t k_1(t)G_2(x,t,0,s)ds+$$

$$+\int_0^t k_2(s)G_2(x,t,1,s)ds=:W_1(x,t)\quad\text{for}\quad(x,t)\in D_2.$$

By the last formula, we get

$$U^2(x,t) = \int_0^t \int_0^1 W_1(y,s)G_1(x,t,y,s)dyds \quad \text{for} \quad (x,t) \in D_2. \tag{33}$$

Lemma 6. If $f_2 \in \mathcal{K}_2$ *and* $k_i \in \mathcal{K}_3$ $(i = 1,2)$ *then function* U^2*, given by formula* (33)*, satisfies the conditions:*

(i) $P_2 P_1 U^2(x,t) = 0$ *for* $(x,t) \in D_1$,

(ii) $U^2(0,t) = U^2(1,t) = 0,\ U^2(x,0) = 0$ *for* $x \in \mathbf{J},\ t \in (0,T]$,

(iii) $P_1 U^2(x,0) = f_2(x),\ D_x P_1 U^2(0,t) = k_1(t)$ *for* $x \in \mathbf{J},\ t \in (0,T]$,
$D_x P_1 U^2(1,t) = k_2(t)$ *for* $t \in (0,T]$.

(iv) *Conversely if function* U^2 *satisfies conditions* (i) – (iii) *then* U^2 *satisfies the integral equation* (33) *with*

$$U^2(0,t) = U^2(1,t) = 0, \quad U^2(y,0) = 0, \quad P_1 U^2(y,0) = f_2(y),$$

$$D_x P_1 U^2(1,t) = k_2(t) \textit{ and } D_x P_1 U^2(0,t) = k_1(t).$$

Proof. (i): By Poisson theorem, we obtain

$$P_1 U^2(x,t) = W_1(x,t) \quad \text{for} \quad (x,t) \in D_1$$

and

$$P_2 P_1 U^2(x,t) = P_2 W_1(x,t) = 0 \quad \text{for} \quad (x,t) \in D_1.$$

(ii): From properties of the Green function G_1, we have

$$U^2(0,t) = \int_0^t \int_0^1 G_1(0,t,y,s)W_1(y,s)dyds = 0, \qquad t \in (0,T]$$

and

$$U^2(1,t) = \int_0^t \int_0^1 G_1(1,t,y,s)W_1(y,s)dyds = 0, \qquad t \in (0,T].$$

By formula (33) we get $U^2(x,0) = 0$ for $x \in \mathbf{J}$.

(iii): $P_1 U^2(x,0) = W_1(x,0) = f_2(x)$ for $x \in \mathbf{J}$.

$D_x P_1 U^2(0,t) = D_x W_1(0,t) = k_1(t)$ for $t \in (0,T]$,

$D_x P_1 U^2(1,t) = D_x W_1(1,t) = k_2(t)$ for $t \in (0,T]$.

(iv): By the fundamental formula (32), we get (iv).

Remark. Applying the fundamental formula, we obtain to problem (19) – (25) the equivalent integral equation:

$$(34) \qquad U^3(x,t) = \int_0^t \int_0^1 F(y,s,u(y,s))G_{1,2}(x,t,y,s)dyds, \qquad (x,t) \in D_1.$$

Namely, by the fundamential formula, we get

$$\int_0^t \int_0^1 \big[P_2 P_1 U^3(y,s,u)G_{1,2}(x,t,y,s) -$$

$$-U^3(y,s,u)\, P_2\, P_1 G_{1,2}(x,t,y,s)\big]dyds = U^3(x,t,u).$$

Since $P_2 P_1 U^3(x,t,u) = F(x,t,u)$ then, by (35), we have (34).

8. THE INTEGRAL EQUATION TO PROBLEM (1) – (7)

Let us consider the integral equation

$$(36) \qquad u(x,t) = U^1(x,t) + U^2(x,t) + U^3(x,t,u(x,t)), \qquad (x,t) \in D_1,$$

with an unknown function u.

To solve the integral equation (36) we shall apply the Banach fixed point theorem and the method of the succesive approximations. We shall also werify that the solution of the integral equation (36) is also the solution of the differential problem (1) – (7).

9. THE BANACH SPACES OF THE CONTINUOUS FUNCTIONS

Let

$$K_1 := \max\left\{ \sup_{y\in\bar{J}} |f_1(y)|, \sup_{s\in[0,T]} |h_1(s)|, \sup_{s\in[0,T]} |h_2(s)| \right\},$$

$$K_2 := \max\left\{ \sup_{y\in\bar{J}} |f_2(y)|, \sup_{s\in[0,T]} |k_1(s)|, \sup_{s\in[0,T]} |k_2(s)| \right\},$$

$$q := \frac{Lt}{C_1 C_2 \Pi}$$

and

$$T < \frac{C_1 C_2 \Pi}{L}.$$

It is easy to see that $q \in (0,1)$ for $t \in (0,T]$.

Let us consider the following Banach space B_1 of the cotinuous functions:

$$B_1 := \{Z : Z(x,t) = U^1(x,t) + U^2(x,t),\ \ (x,t) \in D_1\},$$

with the norm

$$\|Z\|_{B_1} := \sup_{(x,t)\in \overline{D}_1} \left|U^1(x,t) + U^2(x,t)\right| \le$$

$$\le \sup_{(x,t)\in \overline{D}_1} \left|U^1(x,t)\right| + \sup_{(x,t)\in \overline{D}_1} \left|U^2(x,t)\right| \le$$

$$\le K_1(1 + C_9 t + C_{10} t^{1/2}) + K_2(1 + C_{11} t + C_{12} t^{1/2}) \int_0^t \int_0^1 G_1(x,t,y,s)dyds \le$$

$$\le K_1(1 + C_9 t + C_{10} t^{1/2}) + K_2(1 + C_{11} t + C_{12} t^{1/2}) 2 A_1 A_2 t^{1/2} := L_1(t),$$

where C_i $(i = 9,10,11,12)$ are positive constants.

Moreover, let us define the following Banach space B_2 of the continuous functions:

$$B_2 := \{\mathcal{U} : \mathcal{U}(x,t,u(x,t)) = U^3(x,t,u(x,t)),\ (x,t) \in D_1,\ u \in \mathcal{K}_1\}.$$

Observe that

$$U^3(x,t,u(x,t)) = \int_0^t \int_0^1 F(y,s,u(y,s)) G_{1,2}(x,t,y,s)dyds.$$

Consequently, by (v) from Lemma 4, we have

$$\left|U^3(x,t,u(x,t)\right| \le 4 M A_1 A_2 t$$

and

$$\left\|U^3\right\|_{B_2} =: \sup_{(x,t)\in \overline{D}_2} \left|U^3(x,t,u(x,t)\right| \le 4 M A_1 A_2 t := L_2(t).$$

Let

$$E := B_1 \times B_2 = \{(Z, U^3) : Z \in B_1,\ U^3 \in B_2\}.$$

We consider space E together with the norm

$$\|u\|_E = \left\|(Z, U^3)\right\|_E =: \max\left\{\|Z\|_{B_1}, \left\|U^3\right\|_{B_2}\right\} \le \|Z\|_{B_1} + \left\|U^3\right\|_{B_2} =$$

$$= L_1(t) + L_2(t) \qquad \text{for} \quad u \in E,$$

where $t \in (0,T]$.

Let $L_1(t)+L_2(t)\le R$ for $t\in(0,T]$ and let 0 denote the function equal identically zero in $\overline{D}_1$.

By $K(0,R)$ we denote the ball of the functions $u\in E$ such that $\|u\|_E\le R$. Let $K(0,qR)$ denote the ball of the functions $Z\in B_1$ for which

$$\|Z\|_{B_1}\le qR$$

and let $K(0,(1-q)R)$ denote the ball of the functions $U^3\in B_2$ for which

$$\left\|U^3\right\|_{B_2}\le(1-q)R.$$

10. The transformation to the Banach method

Let $u\in K(0,R)$, $Z\in K(0,qR)$ and $U^3\in K(0,(1-q)R)$. Consider the transformation

(37) $$S:E\ni u\to S(u):=Z+U^3(\cdot,\cdot,u(\cdot,\cdot)).$$

Theorem 1. If $q\in(0,1)$, $Z\in K(0,qR)$, $U^3\in K(0,(1-q)R)$ and $F\in\mathcal{K}_4$ then
(i) *the transformation S is the contraction together with the constant q,*
(ii) *S transforms the ball $K(0,R)$ into itself.*

Proof. (i): Let $u_1,u_2\in K(0,R)$. Consequently,

$$\|S(u_1)-S(u_2)\|_E=\left\|U^3(\cdot,\cdot,u_1(\cdot,\cdot))-U^3(\cdot,\cdot,u_2(\cdot,\cdot))\right\|_E\le q\|u_1-u_2\|_E.$$

(ii): Let $u\in E$ and let $u=Z+U^3\in K(0,R)$. Then, by (37), we obtain

$$\|u\|_E=\left\|Z+U^3\right\|_E\le\|Z\|_{B_1}+\left\|U^3\right\|_{B_2}\le qR+(1-q)R=R.$$

By Theorem 1 and by the Banach fixed point theorem, we obtain

Lemma 7. If $q\in(0,1)$, $F\in\mathcal{K}_4$ and $Z\in K(0,qR)$ then, in the class of functions belonging to $\mathcal{K}_4$, there exists the fixed point V to transformation (37) *such that*

(38) $$V(x,t)=Z(x,t)+\int_0^t\int_0^1 G_{1,2}(x,t,y,s)F(y,s,V(y,s))dyds.$$

11. The construction of the solution to the method of the succesive approximations

To solve equation (38) we shall apply the method of the succesive approximations. For this purpose, let us consider the sequence $\{V_i\}_{i=1}^{\infty}$ such that

$$V_0(x,t)=Z(x,t),$$

$$V_1(x,t)=Z(x,t)+\int_0^t\int_0^1 G_{1,2}(x,t,y,s)F(y,s,V_0(x,t))dyds,$$

....,

$$V_n(x,t)=Z(x,t)+\int_0^t\int_0^1 G_{1,2}(x,t,y,s)F(y,s,V_{n-1}(x,t))dyds,$$

where V_0 is an arbitrary function of the ball $K(0,qR)$. Applying the iteration argument form [7], we obtain the estimation

$$\left\|V_{n+p}-V_n\right\|_E \le q^n(1-q)^{-1}\left\|V_1-V_0\right\|_E,$$

where p is a positive integer. Since the sequence $\{V_n\}_{n=1}^{\infty}$ satisfies the Cauchy condition and E is a complete space thus there exists

$$\lim_{n\to\infty} V_n(x,t) = \lim_{n\to\infty} V_{n-1}(x,t) =: V(x,t), \qquad (x,t)\in D_1,$$

and, consequently, V is the unique solution to equation (38). By the foregoing lemmas, function $V=u$ satisfies equation (36) and the suitable limit conditions (2) – (7).

12. The relation between the integral equation and the differential problem

Theorem 2. If $f_i\in\mathcal{K}_2$, $k_i,h_i\in\mathcal{K}_3$ $(i=1,2)$, $F\in\mathcal{K}_4$, $T<(C_1C_2\Pi)/L$ *and* V, *belonging to* $\mathcal{K}_4$, *is a solution to the integral equation* (36) *then function* V *is a solution to the differential problem:*

(*i*) $P_2P_1V(x,t)=F(x,t,V(x,t))$ *for* $(x,t)\in D_1$,

(*ii*) $V(x,0)=f_1(x)$ *for* $x\in\mathbf{J}$,

(*iii*) $V(0,t)=h_1(t)$ *for* $t\in(0,T]$,

(*iv*) $V(1,t)=h_2(t)$ *for* $t\in(0,T]$,

(*v*) $P_1V(x,0)=f_2(x)$ *for* $x\in\mathbf{J}$,

$$(vi) \quad D_x P_1 V(0,t) = k_1(t) \quad \textit{for} \quad t \in (0,T],$$
$$(vii) \quad D_x P_1 V(1,t) = k_2(t) \quad \textit{for} \quad t \in (0,T].$$

Proof. By the Poisson theorem, we obtain

$$P_2 P_1 V(x,t) = P_2 P_1 Z(x,t) + \int_0^t \int_0^1 F(y,s,V(y,s)) G_{1,2}(x,t,y,s) dy ds =$$
$$= P_2 P_1 + \int_0^t \int_0^1 F(y,s,V(y,s)) G_{1,2}(x,t,y,s) dy ds =$$
$$= F(x,t,V(x,t)) \quad \text{for} \quad (x,t) \in D_1.$$

The boundary-value conditions (ii) – (vii) follow from Lemmas 1 – 6 and from Remark.

13. The uniqueness theorem

Theorem 3. If $F \in \mathcal{K}_4$, $f_i \in \mathcal{K}_2$, $k_i, h_i \in \mathcal{K}_3$ $(i = 1,2)$ *and* $T < (C_1 C_2 \Pi)/L$ *then the differential problem* (1) – (7) *has a unique classical solution in class* $\mathcal{K}_1$.

Proof. The solution of the integral equation (36) is equivalent to the solution of problem (1) – (7) and the integral problem has a unique solution. If we suppose that problem (1) – (7) has two solutions then the equation has two solutions, but it is not possible.

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