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# SMOOTHING CONDITIONS AND THEIR APPLICATIONS

ABSTRACT: Let $T_s = T(x_0,...,x_s)$ and $T_k = T(x_0,...,x_k, y_{k+1},...,y_s)$ be two adjacent $s$-simplices with $T(x_0,...,x_k)$ as the common face. Suppose that $f \in C^r(T_s \cup T_k)$ with

$$f\big|_{T_s} = P(x), \quad f\big|_{T_k} = Q(x),$$

where $P$ and $Q$ are Bezier polynomials in $R^s$ with degree $n$. The conditions, which must be required for function $f$ to be in class $C^r$ across $T$, were introduced by C.K. Chui and M. Lai (see [3], [4] and Theorem 1 in this paper). The improvement of these conditions were discussed in [6]. As the aplication an algorthm for computation of polynomial coefficients for function $f$ to be in class $C^r$ across $T$ is given.

KEY WORDS: polynomial coefficients, Bezier polynomials.

## SYSTEMS OF LINEARY INDEPENDET POINTS IN $R^s$

Let $A_1$ and $A_2$ be two systems of lineary independed points in $R^s$:

$$A_1 = (x_0,...,x_s), \tag{1}$$
$$A_2 = (y_0,...,y_s),$$

where $x_i, y_i \in R^s$ for every $i = 0,...,s$. Then any point $x \in R^s$ may be expressed as:

$$x = \sum_{i=0}^{s} \alpha_i x_i, \quad \text{where} \quad \sum_{i=0}^{s} \alpha_i = 1, \tag{2}$$

or

$$x = \sum_{i=0}^{s} \beta_i y_i, \quad \text{where} \quad \sum_{i=0}^{s} \beta_i = 1. \tag{3}$$

In particular, every point $y_i$ from $A_1$ one can express as convex combination of points $x_i$ form system $A_2$, i.e.:

$$y_i = \sum_{j=0}^{s} c_{ji} x_j, \quad \text{where} \quad \sum_{j=0}^{s} c_{ji} = 1. \tag{4}$$

Replacing $y_i$ in (3) by the right-hand side of (4) we can obtain following equality:

$$x=\sum_{i=0}^{s}\beta_i y_i=\sum_{i=0}^{s}\beta_i\left(\sum_{j=0}^{s}c_{ji}x_j\right)=\sum_{j=0}^{s}x_j\left(\sum_{i=0}^{s}\beta_i c_{ji}\right). \tag{5}$$

Comparing the coefficients of $x_i$ in (2) and (5) we get

$$\alpha_j=\sum_{i=0}^{s}\beta_i c_{ji}, \tag{6}$$

for $j=0,\dots,s$. The above equality may be written using matrix notation:

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_s\end{bmatrix}=\begin{bmatrix}c_{00} & c_{01} & \cdots & c_{0s}\\ c_{10} & c_{11} & \cdots & c_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ c_{s0} & c_{s1} & \cdots & c_{ss}\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_s\end{bmatrix}. \tag{7}$$

Matrix $C=[c_{ij}]$, on the right-hand side of (7), is a transformation matrix from base $A_1$ and $A_2$, and the columns of this matrix are baricentric coordinates of points of system $A_2$ respect to $A_1$.

*Lemma 1. For any* $k=0,1,\dots,s$ *we have* $\sum_{i=0}^{s}c_{ik}=1$.

*Proof.* It suffices to observe that the columns of matrix $C=[c_{ij}]$ are a baricentric coordinates of some points.

Now, we consider two basis in $R^s$

$$B_1=(x_1-x_0,x_2-x_0,\dots,x_s-x_0), \tag{8}$$

$$B_2=(y_1-y_0,y_2-y_0,\dots,y_s-y_0).$$

Then, any vector $x\in R^s$ can be expressed as:

$$x=\sum_{i=1}^{s}a_i(x_i-x_0), \tag{9}$$

or

$$x=\sum_{i=1}^{s}b_i(y_i-y_0). \tag{10}$$

In particular, every vector $y_i-y_0$, for $i=1,\dots,s$, may be written as:

$$y_i-y_0=\sum_{j=1}^{s}d_{ji}(x_j-x_0). \tag{11}$$

Coefficients $d_{ji}$ $(i, j = 1,\ldots,s)$ form transformation matrix D from base $B_1$ to $B_2$.

*Lemma 2. Dependence between the coefficients* $c_{ij}$ *and* $d_{ij}$ *is given by the following formula:*

$$c_{ij} - c_{i0} = \begin{cases} d_{ij} & \text{for} \quad i = 1,2,\ldots,s, \\ -\sum_{i=1}^{s} d_{ij} & \text{for} \quad i = 0. \end{cases} \tag{12}$$

*Proof.* Let us consider the equalities (4) and (11). Replacing all $y_i$ in (11) by the right-hand side of (4) we can rewrite (4) as:

$$\sum_{j=1}^{s} (c_{ij} - c_{i0}) x_i + (c_{0j} - c_{00}) x_0 = \sum_{i=1}^{s} d_{ij} x_i + \sum_{i=1}^{s} d_{ij} x_0. \tag{13}$$

Comparing the coefficients of $x_i$ follows (12).

*Corollary 1. The coefficients* $d_{ij}$ $(i, j = 1,\ldots,s)$ *of* D *are given by the following formula:*

$$d_{ij} = c_{ij} - c_{i0}. \tag{14}$$

*Corollary 2. For the matrices* C *and* D *we have* $\det(C) = \det(D)$.

*Proof.*

$$\det(\mathbf{C}) = \det \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0s} \\ c_{10} & c_{11} & \cdots & c_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s0} & c_{s1} & \cdots & c_{ss} \end{bmatrix} = \det \begin{bmatrix} c_{00} & c_{01} - c_{00} & \cdots & c_{0s} - c_{00} \\ c_{10} & c_{11} - c_{10} & \cdots & c_{1s} - c_{10} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s0} & c_{s1} - c_{s0} & \cdots & c_{ss} - c_{s0} \end{bmatrix} =$$

$$= \det \begin{bmatrix} \sum c_{00} & \sum c_{01} - \sum c_{00} & \cdots & \sum c_{0s} - \sum c_{00} \\ c_{10} & c_{11} - c_{10} & \cdots & c_{1s} - c_{10} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s0} & c_{s1} - c_{s0} & \cdots & c_{ss} - c_{s0} \end{bmatrix},$$

where $\sum c_{ij}$ denotes $\sum_{i=0}^{s} c_{ij}$.

Now, it is sufficient to notice that $\sum_{i=0}^{s} c_{ij}$ is equal to 1 for any $j = 0,\ldots,s$ (see Lemma 1). Hence, determinant of C may be written as:

$$\det(C) = \det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_{10} & c_{11}-c_{10} & \cdots & c_{1s}-c_{10} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s0} & c_{s1}-c_{s0} & \cdots & c_{ss}-c_{s0} \end{bmatrix} =$$

$$= \det\begin{bmatrix} c_{11}-c_{10} & c_{12}-c_{10} & \cdots & c_{1s}-c_{10} \\ c_{21}-c_{20} & c_{22}-c_{20} & \cdots & c_{2s}-c_{20} \\ \vdots & \vdots & \ddots & \vdots \\ c_{s1}-c_{s0} & c_{s2}-c_{s0} & \cdots & c_{ss}-c_{s0} \end{bmatrix} = \det(D).$$

(The last equality is direct consequence of Corollary 1).

Now, we turn our attention to the base $A = (x_0, ..., x_s)$ in $R^s$ and consider a sequence of the basis $A_i = (x_0, ..., x_{i+1}, y_i, ..., y_s)$ for $i = s-1, s-2, ..., k > 0$, where $A_{s-1}$ and $A_{i-1}$ are obtained from A and $A_i$ by replacing $x_s$ and $x_i$ by $y_s$ and $y_i$, respectively.

Let C denotes matrix of transformation from A to $A_k$ and $C_i$ $(i = s, ..., k+1)$ denotes matrix of transformation from $A_i$ and $A_{i-1}$. Then abowe matrices can be written as:

(15) $$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & c_{0,k+1} & \cdots & c_{0,s} \\ 0 & 1 & \cdots & 0 & c_{1,k+1} & \cdots & c_{1,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & c_{k,k+1} & \cdots & c_{k,s} \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c_{s,k+1} & \cdots & c_{s,s} \end{bmatrix},$$

and

(16) $$C_i = \begin{bmatrix} 1 & 0 & \cdots & a_{0,i} & \cdots & 0 \\ 0 & 1 & \cdots & a_{1,i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{i,i} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{s,i} & \cdots & 1 \end{bmatrix}.$$

Then, it is clear that

(17) $$C = C_s C_{s-2} \ldots C_{k+1}.$$

*Lemma 3. Let* $c_{.i}$ *and* $a_{.j}$ *denote i-th column of* C *and* $\mathrm{C_i}$ $(i = s,...,k+1)$ *repectively. Then the column* $c_{.i}$ *can be expressed as:*

$$
(18) \qquad c_{.i} = \begin{bmatrix} a_{0i} + \sum_{n=i+1}^{s} c_{0n} a_{ni} \\ \vdots \\ a_{ii} + \sum_{n=i+1}^{s} c_{in} a_{ni} \\ \sum_{n=i+1}^{s} c_{i+1,n} a_{ni} \\ \vdots \\ \sum_{n=i+1}^{s} c_{sn} a_{ni} \end{bmatrix} = \begin{bmatrix} a_{0i} \\ a_{1i} \\ \vdots \\ a_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \sum_{n=i+1}^{s} c_{n} a_{ni}.
$$

*Proof.* From (17), for $i = s$, we have $\mathrm{C} = \mathrm{C_s}$. Hence it implies that

$$
c_{ss} = \begin{bmatrix} c_{0s} \\ \vdots \\ c_{ss} \end{bmatrix} = \begin{bmatrix} a_{0s} \\ \vdots \\ a_{ss} \end{bmatrix} = a_{0s}.
$$

Suppose, the formula (18) holds for $m+1$, i.e.:

$$
\mathrm{C}_s \mathrm{C}_{s-1} \ldots \mathrm{C}_{\mathrm{m+1}} = \begin{bmatrix} 1 & 0 & \cdots & c_{0m+1} & \cdots & c_{0s} \\ 0 & 1 & \cdots & c_{1m+1} & \cdots & c_{1s} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{sm+1} & \cdots & c_{ss} \end{bmatrix}.
$$

For $m$ we have:

$$
\mathrm{C_s C_{s-1} \ldots C_{m+1} C_m} = (\mathrm{C_s C}_{s-1} \ldots \mathrm{C}_{m+1})\mathrm{C_m} =
$$

$$
= \begin{bmatrix} 1 & 0 & \cdots & c_{0m+1} & \cdots & c_{0s} \\ 0 & 1 & \cdots & c_{1m+1} & \cdots & c_{1s} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{sm+1} & \cdots & c_{ss} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & c_{0m} & \cdots & 0 \\ 0 & 1 & \cdots & c_{1m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{sm} & \cdots & 1 \end{bmatrix} =
$$

$$=\begin{bmatrix} 1 & 0 & \cdots & a_{0m}+\sum_{i=m+1}^{s} a_{im}c_{0i} & c_{0,m+1} & \cdots & c_{0s} \\ 0 & 1 & \cdots & a_{1m}+\sum_{i=m+1}^{s} a_{im}c_{1i} & c_{1,m+1} & \cdots & c_{1s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ & & & a_{mm}+\sum_{i=m+1}^{s} a_{im}c_{mi} & c_{m,m+1} & \cdots & c_{ms} \\ & & & \sum_{i=m+1}^{s} a_{im}c_{m+1,i} & c_{m+1,m+1} & \cdots & c_{m+1,s} \\ & 0 & & \vdots & \vdots & \vdots & \vdots \\ & & & \sum_{i=m+1}^{s} a_{im}c_{s,i} & c_{s,m+1} & \cdots & c_{ss} \end{bmatrix}.$$

It is easy to see that the $m$-th column of above matrix is given by (18).

## THE SMOOTHING CONDITIONS

In this section we assume $T_s$ and $T_k$ to be two $s$-simplices in $R^s$, i.e.:

(19)
$$T_s = T(x_0,\dots,x_s),$$
$$T_s = T(x_0,\dots,x_k,y_{k+1},\dots,y_s).$$

It is clear that the sets of vertces of $T_s$ and $T_k$ are two systems of lineary independent points in $R^s$. The common face of $T_s$ and $T_k$ is $k$-simplex with vertices $x_0,\dots,x_k$, i.e.: $T_s \cap T_k = T(x_0,\dots,x_k)$. In this section we will consider a function $f: D \to R$ $(D = T_s \cup T_k)$ such that:

(20)
$$f\,|_{T_s} = P(x) = \left(\sum_{i=0}^{s} u_i E_i\right)^n a_{0\dots0},$$
$$f\,|_{T_k} = Q(x) = \left(\sum_{i=0}^{s} v_i E_i\right)^n b_{0\dots0},$$

where $(u_0,\dots,u_s)$ and $(v_0,\dots,v_s)$ are baricentric coordinates of $x \in R^s$ with respect to simplex $T_s$ and $T_k$, respectively; and $E_i$ is a shift operator (see [1] and [2]) defined by:

(21)
$$E_i a_{\alpha_0,\dots,\alpha_s} = a_{\alpha_0,\dots,\alpha_{i-1},\alpha_i+1,\alpha_{i+1},\dots,\alpha_s}.$$

Polynomials $P(x)$ and $Q(x)$ are called Bezier polynomials.

*Theorem 1.* (due ro C.K. Chui and M.J. Lai, see [3]) *Let* $c_{i,j}$ *(for* $i=(0,...,s)$*) be baricentric coodinates of* $y_j$ $(j=k+1,...,s)$. *Then for any* $r \in Z_+$, $f \in C^{\gamma}(T_s \cup T_k)$ *the condition* (22) *holds if and only if for all* $\gamma_{k+1}+...+\gamma_s = l$, $\alpha_0+...+\alpha_k = n-l$, $l=0,...,r$:

$$(22)\qquad \Delta^{\gamma_{k+1}}_{k+1,0}\ldots\Delta^{s}_{s,0} b_{\alpha_0,...,\alpha_k,0,...,0} = \prod_{j=k+1}^{s}\left(\sum_{i=1}^{s} c_{i,j}\Delta_{i,0}\right)^{\gamma_j} a_{\alpha_0,...,\alpha_k,0,...,0}.$$

The operator $\Delta_{i0}$ is defined by:

$$(23)\qquad \Delta_{i0}\, a_{\alpha_0,...,\alpha_s} = E_i a_{\alpha_0,...,\alpha_s} - E_0 a_{\alpha_0,...,\alpha_s}.$$

*Theorem 2. The condition* (22) *in the Theorem 1 may by replaced by* (24):

$$(24)\qquad b_{\alpha_0,...,\alpha_k\gamma_{k+1},...,\gamma_s} = \prod_{j=k+1}^{s}\left(\sum_{i=0}^{s} c_{i,j}E_i\right)^{\gamma_j} a_{\alpha_0,...,\alpha_k,0,...,0}.$$

Now, we turn our attention to the case $k=s-1$. Then the condition (24) may be written as:

$$(25)\qquad b_{\alpha_0,...,\alpha_{s-1}\gamma_s} = \left(\sum_{i=0}^{s} c_{i,s}E_i\right)^{\gamma_s} a_{\alpha_0,...,\alpha_{s-1}0}$$

for all $\alpha_0+...\alpha_k = n-\gamma_s$, $\gamma_s = 0,...,r$.

By symetry in (25) the following Lemma may be formulated:

*Lemma 4. Let* $T_1 = T(x_0,...,x_s)$ *and* $T_2 = T(x_0,...,x_{k-1},y_k,x_{k+1},...,x_s)$ *be two adjacent symplices with* $(s-1)$*-symplex as the common face. Let* $f: D \to R$, $D = T_1 \cup T_2$ *be function defined by* $f|_{T_1} = P$ *and* $f|_{T_2} = Q$, *where* $P$ *and* $Q$ *are Bezier polynomials in* $R^s$ *with total degree n. Then* $f \in C^r(D)$ *if and only if*

$$(26)\qquad b_{\alpha_0...\alpha_{k-1}\gamma_k\alpha_{k+1}...\alpha_s} = \left(\sum_{i=0}^{s} c_{i,k}E_i\right)^{\gamma_k} a_{\alpha_0...\alpha_{k-1}0\alpha_{k+1}...\alpha_s},$$

*for* $\sum_{i\neq k}\alpha_i = n-\gamma_k$, $\gamma_k = 0,...,r$.

## MAIN RESULT

For the pair of simplices (19) consider sequence symplices $T_l$ $(l = s, s-1, \dots, k)$ defined by (27):

$$T_l = T(x_0, \dots, x_l, y_{l+1}, \dots, y_s). \tag{27}$$

i.e., each simplex $T_l$ is obtained from $T_{l+1}$ by replacing vertex $x_{l+1}$ by $y_{l+1}$. Without loss of generality we can assume each $T_l$ $(l = s, s-1, \dots, k)$ is $s$-simplex. Let $P_l(x)$ denote Bezier polynomial of degree $n$ defined by:

$$P_l(x) = \left( \sum_{i=0}^{s} z_{l,i} E_i \right)^n a^l_{0\dots0}, \tag{28}$$

where $z_{l,i}$ $(i = 0, \dots, s)$ are baricentric coordinates of $x$ respect to $T_l$ and $a^l_{\alpha_0 \dots \alpha_s}$ are Bezier coefficients of $P_l(x)$.

*Theorem 3. Let* $T_s, T_{s-1}, \dots, T_k$ *be a sequence of simplices as defined in* (27) *and* $P = P_s, P_{s-1}, \dots, P_k = Q$ *be sequence of Bezier polynomials given by* (20) *and* (28). *Suppose that for every pair of polynomials* $P_{l+1}, P_l$ $(l = s-1, \dots, k)$ *the following conditions hold*:

$$b^l_{\alpha_0 \dots \alpha_l, \gamma_{l+1}, \alpha_{l+2} \dots \alpha_s} = \left( \sum_{i=0}^{s} d_{i,l+1} E_i \right)^{\gamma_{l+1}} a^{l+1}_{\alpha_0 \dots \alpha_l, 0, \alpha_{l+2} \dots \alpha_s}, \tag{29}$$

*for all* $\sum_{i \neq 1} \alpha_i = n - \gamma_{l+1}$, *where* $\gamma_{l+1} = 0, \dots, r$ *and* $\sum_{i=0}^{k} a_i \geq n - r$. *Then the coefficients of polynomials P i Q satisfy* (24).

*Proof.* 1. For the case $l = s-1$ the formula (29) may be written as:

$$a^{s-1}_{\alpha_0 \dots \alpha_{s-1}, \gamma_s} = \left( \sum_{i=0}^{s} d_{i,s} E_i \right)^{\gamma_s} a^s_{\alpha_0 \dots \alpha_s, 0},$$

for all $\sum_{i=1}^{s-1} a_i = n - \gamma_s$, and $\gamma_s = 0, \dots, r$. From this it follows (25) is particular case of condition (24).

2. Assume that (24) holds for the case $l = k+1$ i.e.:

$$a^{k+1}_{\alpha_0 \dots \alpha_{k+1}, \gamma_{k+2} \dots \gamma_s} = \prod_{j=k+2}^{s} \left( \sum_{i=0}^{s} c_{i,j} E_i \right)^{\gamma_j} a_{\alpha_0 \dots \alpha_{k+1}, 0 \dots 0}, \tag{30}$$

for all $\gamma_{k+1}+...+\gamma_s=l$, $a_0+...+a_{k+1}=n-l$, $l=0,...,r$, and $a_0+...+a_k\ge n-r$. Suppose that conditions (29) hold for $l=k$, i.e.:

$$a^k_{\alpha_0...\alpha_k,\gamma_{k+1},\alpha_{k+2}...\alpha_s}=\left(\sum_{i=0}^{s}d_{i,k+1}E_i\right)^{\gamma_{k+1}}a^{l+1}_{\alpha_0...\alpha_k,0,\alpha_{k+2}...\alpha_s}, \tag{31}$$

for $\sum\limits_{\substack{i=0\\ i\neq k+1}}^{s}a_i=n-\gamma_{k+1}$, $\gamma_{k+1}=0,...,r$, $\sum\limits_{i=0}^{k}a_i\ge n-r$.

Consider the equality (31):

$$a^k_{\alpha_0...\alpha_k,\gamma_{k+1},\alpha_{k+2}...\alpha_s}=\left(\sum_{i=0}^{s}d_{i,k+1}E_i\right)^{\gamma_{k+1}}a^{l+1}_{\alpha_0...\alpha_k,0,\alpha_{k+2}...\alpha_s}=$$

$$=\left(\sum_{j_0+...+j_s=\gamma_{k+1}}\binom{\gamma_{k+1}}{j_0...j_s}d^{j_0}_{0,k+1}...d^{j_s}_{s,k+1}E_0^{j_0}...E_s^{j_s}\right)a^{k+1}_{\alpha_0...\alpha_k0\gamma_{k+2}...\gamma_k}.$$

Replacing coefficients $a^{k+1}_{\alpha_o...\alpha_k0\gamma_{k+2}...\gamma_k}$ of polynomial $P_{k+1}$ by the right-hand side of (30) we obtain the following formula:

$$\sum_{j_0+...+j_s=\gamma_{k+1}}\binom{\gamma_{k+1}}{j_0...j_s}d^{j_0}_{0,k+1}...d^{j_s}_{s,k+1}\prod_{l=k+2}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{\gamma_l+j_l}a^{k+1}_{\alpha_0+j_0...\alpha_k+j_k,j_{k+1},0...0}=$$

$$=\sum_{j_0+...+j_s=\gamma_{k+1}}\binom{\gamma_{k+1}}{j_0...j_s}d^{j_0}_{0,k+1}...d^{j_s}_{s,k+1}E_0^{j_0}...E_{k+1}^{j_{k+1}}\prod_{l=k+2}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{\gamma_l+j_l}a^{k+1}_{\alpha_0...\alpha_k,0...0}=$$

$$=A\prod_{l=k+2}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{\gamma_l}a_{\alpha_0...\alpha_k,0...0},$$

where

$$A=\sum_{j_0+...+j_s=\gamma_{k+1}}\binom{\gamma_{k+1}}{j_0...j_s}d^{j_0}_{0,k+1}...d^{j_s}_{s,k+1}E_0^{j_0}...E_{k+1}^{j_{k+1}}\prod_{l=k+2}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{j_l}=$$

$$=\left(\sum_{m=0}^{k+1}d_{m,k+1}E_m+\sum_{m=k+2}^{s}\left(d_{m,k+1}\sum_{i=0}^{s}c_{i,m}E_i\right)\right)^{\gamma_{k+1}}=$$

$$=\left(\sum_{m=0}^{k+1}d_{m,k+1}+\sum_{i=0}^{s}\left(\sum_{m=k+2}^{s}d_{m,k+1}c_{i,m}\right)E_i\right)^{\gamma_{k+1}}=$$

$$=\left(\sum_{i=0}^{k+1}\left(d_{i,k+1}+\sum_{m=k+2}^{s}d_{m,k+1}c_{i,m}\right)E_i+\sum_{i=k+2}^{s}\left(\sum_{m=k+2}^{s}d_{m,k+1}c_{i,m}\right)E_i\right)^{\gamma_{k+1}}.$$

Now, it suffices to observe (see Lemma 3) that: $B_i=c_{i,k+1}$, for $i=0,...,k+1$ and $C_i=c_{i,k+1}$, for $i=k+2,...,s$. From this we have:

$$A=\left(\sum_{i=0}^{k+1}B_iE_i+\sum_{i=k+2}^{s}C_iE_i\right)^{\gamma_{k+1}}=\left(\sum_{i=0}^{s}c_{i,k+1}E_i\right)^{\gamma_{k+1}},$$

and

$$a^k_{\alpha_0...\alpha_k,\gamma_{k+1},\alpha_{k+2}...\alpha_s}=\left(\sum_{i=0}^{s}c_{i,k+1}E_i\right)^{\gamma_{k+1}}\prod_{l=k+2}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{\gamma_j}a_{\alpha_0...\alpha_k0...0}=$$

$$=\prod_{l=k+1}^{s}\left(\sum_{i=0}^{s}c_{i,l}E_i\right)^{\gamma_l}a_{\alpha_0...\alpha_k0...0},$$

which completes the proof.

Now we consider a Bezier polynomial $P(x)$ defined as:

(32) $$P(x)=\left(\sum_{i=0}^{s}\lambda_iE_i\right)^{n}a_{0...0},$$

where $\lambda_0,...,\lambda_s$ are baricentric coordinates of $x$. The right-hand side of equality (32) may be evaluated by a recursion formula:

(33) $$\left(\sum_{i=0}^{s}\lambda_iE_i\right)^{n}a_{0...0}=\sum_{j=0}^{s}\left(\lambda_j\left(\sum_{i=0}^{s}\lambda_iE_i\right)^{n-1}E_ja_{0...0}\right).$$

The recursion formula (33) provides de Casteljau algorithm for evaluation $P(x)$ at given point $x$:

(34) $$a^0_{\alpha_0...\alpha_s}=a_{\alpha_0...\alpha_s},$$

$$a^{\gamma}_{\alpha_0...\alpha_s}=\sum_{i=0}^{s}\lambda_iE_ia^{\gamma-1}_{\alpha_0...\alpha_s}\quad\text{for}\quad\gamma=1,...,n,$$

where $\lambda_0,...,\lambda_s$ are baricentric coordinates of $x$ and $a^{\gamma}_{\alpha_0...\alpha_s}$ are auxiliary points with $\sum_{i=0}^{s}\alpha_i=n-\gamma$. It is easy to show that $P(x)=a^n_{0...0}$ (see [5]). We now turn to the case of two polynomial $P_{l+1}$ and $P_l$ from Theorem 3. Applying the algorthm (34) to the polynomial $P_{l+1}$ and $y_{l+1}$ we obtain a procedure for evaluating the coefficients of $P_l$ to satisfying the conditions (29). Since $d_{i,l+1}$ $(i=0,...,s)$ be baricentric coordinates of $y_{l+1}$ with respect to the simplex $T_{l+1}$, then the algorithm de Casteliau may be written as:

(35)
$$b^0_{\alpha_0...\alpha_s} = a^{l+1}_{\alpha_0...\alpha_s},$$
$$b^\gamma_{\alpha_0...\alpha_s} = \sum_{i=0}^{s} d_{i,l+1} E_i b^{\gamma-1}_{\alpha_0...\alpha_s} \quad \text{for } \gamma = 1,...,n.$$

Now, it is sufficient to note that $a^l_{\alpha_0...\alpha_l,\gamma,\alpha_{l+1},...,\alpha_s} = b^\gamma_{\alpha_0...\alpha_l,0,\alpha_{l+1},...,\alpha_s}$ $(\gamma=0,...,r)$. Since the conditions (29) are defined for all $\sum_{i\neq 1}\alpha_i = n-\gamma_{l+1}$, where $\gamma_{l+1}=0,...,r$ and $\sum_{i=0}^k a_i \geq n-r$, then algorithm (35) may be expressed as:

(36)
$$b^0_{\alpha_0...\alpha_s} = a^{l+1}_{\alpha_0...\alpha_s} \quad \text{for all } \sum_{i=0}^{k}\alpha_i \geq m-r,$$
$$b^\gamma_{\alpha_0...\alpha_s} = \sum_{i=0}^{s} d_{i,l+1} E_i b^{\gamma-1}_{\alpha_0...\alpha_s} \quad \text{for } \gamma = 1,...,r.$$

For the general case of two simplices (19) and two polynomials (20) we can obtain the coefficients $b_{\alpha_0...\alpha_s}$ of $Q$ from the coefficients $a_{\alpha_0...\alpha_s}$ of $P$ for all $\sum_{i=0}^k a_i \geq n-r$ in two steps:

1. We need to construct of $n-k$ auxiliary simplices and $n-k$ corresponding polynomials (as it is shown in Theorem 3).
2. Then, it suffices to calculate their coefficients by using the algorithm (36) for each pair $l+1$ and $l$ $(l=s-1,...,k)$.

The implementation of the above algorithms will be considered in the next paper.

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