FASCICULI MATHEMATICI

Nr 31 2001

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CONTROLLABILITY OF NONLINEAR INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES WITH NONLOCAL CONDITIONS

ABSTRACT: In this paper, we shall establish sufficient conditions for the controllability of semilinear integrodifferential inclusions in Banach spaces, with nonlocal conditions. We shall rely of a fixed point theorem for condensing maps due to Martelli.

KEY WORDS: nonlocal condition, mild solution, evolution, controllability, fixed point.

1. INTRODUCTION

In this paper, we shall establish sufficient conditions for the controllability of semilinear integrodifferential inclusions in Banach spaces, with nonlocal initial conditions. More precisely we consider the following integrodifferential inclusions of the form

$$y' - Ay \in \int_0^t K(t,s)F(s,y)ds + (Bu)(t), \qquad t \in J = [0,b], \tag{1.1}$$

$$y(0) + f(y) = y_0, \tag{1.2}$$

where $F: J \times E \to 2^E$ is a bounded, closed, convex multivalued map, $K: \Delta \to \Re$, $\Delta = \{(t,s) \in J \times J : t \geq s\}$, $f: C(J,E) \to E$, $y_0 \in E$, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ and E a real Banach space with the norm $||\cdot||$. Also the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Finally B is a bounded linear operator from U to E.

Controllability results for nonlinear integrodifferential systems in Banach spaces, by using the Schauder fixed point theorem, was studied by Balachandran, Balasubramaniam and Dauer in [1]. Han and Park [12], by using a Banach fixed point theorem, proved boundary controllability of differential equations with nonlocal conditions. Also Han et all [13], by using the Schauder fixed point theorem, proved controllability results for functional integro-differential equations in Banach spases with delay.

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [8]. For example the nonlocal condition can be defined by the formula

(C) $$y(0)+c_1 y(t_1)+c_2 y(t_2)+\ldots+c_p y(t_p) = y_0$$

where c_i, $i=1,2,\ldots,p$ are given constants. As remarked by Byszewski the results on nonlocal conditions can be applied to kinematics to determine the evolution $t \to y(t)$ of the location of a physical object for which we do not know the positions $y(0), y(t_1), \ldots, y(t_p)$, but we know that the nonlocal condition (C) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $y(0)=y_0$. From (1.2) it is clear that when $f=0$ we have the classical initial condition.

In the few past years several papers have been devoted to study the existence of solutions for differential equations with nonlocal conditions. Among others we refer the papers by Balachandran and Chandrasekaran [3], Balachandran and Ilamaran [2], Byszewski [7], [8] and Ntouyas and Tsamatos [17].

Moreover existence results for first order semilinear integrodifferential inclusion with nonlocal conditions was recently studied by the authors in [5].

In this paper we study the controllability of system (1.1) – (1.2) relied on a fixed point theorem for condensing maps due to Martelli [16]. As an application we give in Section 4 a nonlocal controllability result for a semilinear delay integrodifferential inclusion of Sobolev type. Finally, is Section 5, a physical example is worked out to illustrate the results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper. $C(J,E)$ is the Banach space of continuous functions from J into E normed by

$$\| y \|_\infty = \sup\{\| y(t) \| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E. A measurable function $y: J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [19]).

$L^1(J,E)$ denotes the Banach space of continuous functions $y: J \to E$ which are Bochner integrable normed by

$$\| y \|_{L^1} = \int_0^b \| y(t) \| \, dt \qquad \text{for all} \quad y \in L^1(J, E).$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(D) = \bigcup_{x \in D} G(x)$ is bounded in X for any bounded set D of X (i.e. $\sup\limits_{x \in D}\{\sup\{\| y \| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set V of X containing $G(x_0)$, there exists an open neighbourhood A of x_0 such that $G(A) \subseteq V$.

G is said to be completely continuous if $G(D)$ is relatively compact for every bounded subset $D \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$). G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(X)$ is said to be measurable, if for each $x \in X$ the distance between x and $G(t)$, is a measurable function on J. For more details on multivalued maps see the books of Deimling [11] and Hu and Papageorgiou [14].

An upper semi-continuous map $G : X \to 2^X$ is said to be condensing if for any bounded subset $D \subseteq X$, with $\alpha(D) \neq 0$, we have $\alpha(G(D)) < \alpha(D)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel [6].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

DEFINITION 2.1. *A continuous solution $y(t)$ of the integral inclusion*

$$y(t) \in T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)(Bu)(s)\, ds +$$
$$+ \int_0^t T(t-s) \int_0^s K(s,\tau) F(\tau, y(\tau))\, d\tau \, ds$$

is called a mild solution of $(1.1)-(1.2)$ on J.

DEFINITION 2.2. *The system* (1.1) – (1.2) *is said to be nonlocally controllable on the intrval J, if for every* $y_0, y_1 \in E$, *the exists a control* $u \in L^2(J,U)$, *such that the mild solution* $y(t)$ *of* (1.1) – (1.2) *satisfies* $y(b) + f(y) = y_1$.

Let us list the following hypotheses:

(H1) A is the infinitesimal generator of a linear semigroup $T(t)$, $t \geq 0$ and there exists $M > 0$ such that $\| T(t) \| \leq M$, $t \geq 0$.

(H2) $F : J \times E \to BCC(E)$; $(t, y) \to F(t, y)$ is strongly measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J,E)$ the set

$$S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}$$

is nonempty;

(H3) for each $t \in J$, $K(t,s)$ is measurable on $[0,t$ and

$$K(t) = \text{ess sup}\{|K(t,s), \ 0 \leq s \leq t\},$$

is bounded on J;

(H4) the map $t \to K_t$ is continuous from J to $L^\infty(J, \Re)$; here $K_t(s) = K(t,s)$;

(H5) there exists a constant $L > 0$ such that $\| f(y) \| \leq L$ for each $y \in E$;

(H6) the linear operator $W : L^2(J,U) \to E$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\, ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J,U) \setminus kerW$ and there exist positive constants M_1 and M_2 such that $\|B\| \leq M_1$ and $\| W^{-1} \| \leq M_2$;

(H7) $\|F(t,y)\| := \sup\{\|v\| : v \in F(t,y)\} \leq p(t)\, \psi(\|y\|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \Re_+)$ and $\psi : \Re_+ \to (0, \infty)$ is continuous and increasing with

$$Mb \sup_{t \in J} K(t) \int_0^b p(s) ds < \int_c^\infty \frac{du}{\psi(u)},$$

where $c = M(\|y_0\| + L + M_0)$, and

$$M_0 = bM_1M_2\left[\| y_1 \| + L + M \| y_0 \| + ML + Mb \sup_{t \in J} K(t) \int_0^b p(s)\psi(\| y(s) \|)ds\right];$$

(H8) the function f is completely continuous;

(H9) for each bounded set $D \subset C(J,E)$, and $t \in J$ the set

$$\left\{ T(t)y_0 + \int_0^t T(t-s) \int_0^s K(s,\tau)g(\tau)\,d\tau\,ds \, : \, g \in S_{F,y}, \; y \in D \right\}$$

is relatively compact.

REMARK 2.3. If $dim\,E < \infty$, then for each $y \in C(J,E)\; S_{F,y} \neq 0$ (see Lasota and Opial [15]).

The following lemmas are crucial in the proof of our main theorem.

LEMMA 2.4. [15]. *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H2) *and let* Γ *be a linear continuous mapping from* $L^1(I,X)$ *to* $C(I,X)$*, then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC(C(I,X)), \qquad y \to (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

LEMMA 2.5. [16]. *Let X be Banach space and* $N : X \to BCC(X)$ *a condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \;\; for\ some\ \lambda > 1\}$$

is bounded, then N has a fixed point.

3. MAIN RESULT

Now, we are able to state and prove our main theorem.

THEOREM 3.1. *Let* $f : C(J,E) \to E$ *be a continuous function. Assume that hypotheses* (H1) – (H9) *are satisfied. Then the problem* (1.1) – (1.2) *is non-locally controllable on J.*

PROOF. Using hypothesis (H6) for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\left[y_1 - f(y) - T(b)y_0 + T(b)f(y) - \int_0^b T(b-s) \int_0^s K(s,\tau)g(\tau)d\tau ds \right](t),$$

where

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}.$$

We shall now show that, when using this control, the operator $N:C(J,E)\to 2^{C(J,E)}$ defined for each $t\in J$ by

$$Ny:=\left\{h\in C(J,E): h(t)=T(t)y_0-T(t)f(y)+\int_0^t T(t-s)(Bu_y)(s)\,ds+\right.$$

$$\left.+\int_0^t T(t-s)\int_0^s K(s,\tau)\,g(\tau)\,d\tau\,ds\;:\;g\in S_{F,y}\right\}$$

has a fixed point. This fixed point is then a solution of the system (1.1) – (1.2).

Clearly $y_1-f(y)\in(Ny)(b)$.

We shall show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps.

Step 1: *Ny is convex for each* $y\in C(J,E)$.

This is trivial, since $S_{F,y}$ is convex (because F has convex values) and therefore it is omitted.

Step 2: *N is bounded on bounded sets of* $C(J,E)$.

Indeed, it is enough to show that there exists a positive constant l such that for each $h\in Ny$, $y\in B_r=\{y\in C(J,E):\|y\|_\infty\le r\}$ one has $\|h\|_\infty\le l$. If $h\in N_y$, then there exists $g\in S_{F,y}$ such that

$$h(t)=T(t)y_0-T(t)f(y)+\int_0^t T(t-s)(Bu_y)(s)\,ds+$$

$$+\int_0^t T(t-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds,\qquad t\in J.$$

By (H3) – (H7) we have for each $t\in J$ that

$$\|h(t)\|\le\|T(t)\|\,\|y_0\|+\|T(t)\|\,\|f(y)\|+\left\|\int_0^t T(t-s)(Bu_y)(s)\,ds\right\|+$$

$$+\left\|\int_0^t T(t-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds\right\|\le$$

$$\le M\|y_0\|+ML+bMM_1M_2[\|y_1\|+L+M\|y_0\|+$$

$$+ML + Mb \sup_{t\in J} K(t)\| p\|_{L^1} \sup_{t\in J} \psi(\| y(t)\|)] +$$

$$+ M \int_0^t \int_0^s | K(s,\tau) | p(\tau)\psi(\| y(\tau)\|) d\tau ds \leq$$

$$\leq M \| y_0 \| + ML + bMM_1M_2[\| y_1 \| + L + M \| y_0 \| +$$

$$+ML + Mb \sup_{t\in J} K(t)\| p\|_{L^1} \sup_{t\in J} \psi(\| y(t)\|)] +$$

$$+Mb \sup_{t\in J} K(t)\| p\|_{L^1} \sup_{t\in J} \psi(\| y(t)\|)] = l.$$

Step 3: *N sends bounded sets of* $C(J,E)$ *into equicontinuous sets.*

Let $t_1, t_2 \in J$, $t_1 < t_2$ and B_r be a bounded set of $C(J,E)$. For each $y \in B_r$ and $h \in Ny$, there exists $g \in S_{F,y}$ such that

$$h(t) = T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)(Bu_y)(s)ds +$$

$$+ \int_0^t T(t-s) \int_0^s K(s,\tau)g(\tau)\, d\tau\, ds, \qquad t \in J.$$

Thus

$$\| h(t_2) \| - h(t_1) \| \leq \| (T(t_2) - T(t_1))y_0 \| + L \| T(t_2) - T(t_1) \| +$$

$$+ \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)] BW^{-1} \Big[y_1 - f(y) - T(b)y_0 + \right.$$

$$\left. + T(b)f(y) + \int_0^b T(b-s) \int_0^s K(s,\tau)g(\tau)d\tau ds \Big](\eta)\, d\eta \right\| +$$

$$+ \left\| \int_{t_1}^{t_2} T(t_1 - s) BW^{-1} \Big[y_1 - f(y) - T(b)y_0 + T(b)f(y) + \right.$$

$$\left. + \int_0^b T(b-s) \int_0^s K(s,\tau)g(\tau)\, d\tau\, ds \Big](\eta)\, d\eta \right\| +$$

$$+ \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)] \int_0^s K(s,\tau)g(\tau)\, d\tau\, ds \right\| +$$

$$+\left\|\int_{t_1}^{t_2} T(t_1-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds\right\|\le$$

$$\le \|T(t_2)-T(t_1)\|\,\|y_0\|+L\|T(t_2)-T(t_1)\|+$$

$$+\int_0^{t_2}\|T(t_2-s)-T(t_1-s)\|M_1M_2\Bigg[\|y_1\|+L+M\|y_0\|+$$

$$+ML+Mb\sup_{t\in J}K(t)\int_0^b p(s)\psi(\|y(s)\|)\,ds\Bigg](\eta)\,d\eta+$$

$$+\int_{t_1}^{t_2}\|T(t_1-s)\|M_1M_2\Bigg[\|y_1\|+L+M\|y_0\|+ML+$$

$$+Mb\sup_{t\in J}K(t)\int_0^b p(s)\psi(\|y(s)\|)\,ds\Bigg](\eta)\,d\eta+$$

$$+\sup_{t\in J}K(t)\left\|\int_0^{t_2}[T(t_2-s)-T(t_1-s)]\int_0^s g(\tau)d\tau\,ds\right\|+$$

$$+M\sup_{t\in J}K(t)(t_2-t_1)\int_0^b\|g(s)\|\,ds.$$

As $t_2\to t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and hypotheses (H8) and (H9), together with the Ascoli-Arzela theorem, we can conclude that N is completely continuous.

Step 4: *N has a closed graph.*

Let $y_n\to y_*$, $h_n\in N(y_n)$, and $h_n\to h_*$. We shall prove that $h_*\in N(y_*)$. $h_n\in N(y_n)$ means that there exists $g_n\in S_{F,y_n}$ such that

$$h_n(t)=T(t)y_0-T(t)f(y_n)+\int_0^t T(t-s)(Bu_{y_n})(s)ds+$$

$$+\int_0^t T(t-s)\int_0^s K(s,\tau)g_n(\tau)\,d\tau\,ds,\qquad t\in J,$$

where

$$u_{y_n}(t) = W^{-1}\Big[y_1 - f(y_n) - T(b)y_0 + T(b)f(y_n) -$$

$$- \int_0^b T(b-s)\int_0^s K(s,\tau)g_n(\tau)\,d\tau\,ds\Big](t).$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t)y_0 - T(t)f(y_*) + \int_0^t T(t-s)(Bu_{y_*})(s)ds +$$

$$+ \int_0^t T(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau\,ds, \qquad t \in J$$

where

$$u_{y_*}(t) = W^{-1}\Big[y_1 - f(y_*) - T(b)y_0 + T(b)f(y_*) -$$

$$- \int_0^b T(b-s)\int_0^s K(s,\tau)g_*(\tau)\,d\tau\,ds\Big](t).$$

Set

$$\overline{u}_y(t) = W^{-1}\left[y_1 - f(y_n) - T(b)y_0 + T(b)f(y_n)\right].$$

Since f, W^{-1} are continuous, then $\overline{u}_{y_n}(t) \to \overline{u}_{y_*}(t)$ for $t \in J$. Clearly we have that

$$\Bigg\|\left(h_n - T(t)y_0 + T(t)f(y_n) - \int_0^t T(t-s)(B\overline{u}_{y_n})(s)ds\right) -$$

$$-\left(h_* - T(t)y_0 + T(t)f(y_*) - \int_0^t T(t-s)(B\overline{u}_{y_*})(s)ds\right)\Bigg\|_\infty \to 0, \quad \text{as} \quad n \to \infty.$$

Consider the operator

$$\Gamma : L^1(J,E) \to C(J,E)$$

$$g \to \Gamma(g)(t) = \int_0^t T(t-s)\left[BW^{-1}\left(\int_0^b T(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds\right)(t)\right]ds +$$

$$+ \int_0^t T(t-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds.$$

Clearly, Γ is linear and continuous. Indeed one has

$$\| \Gamma g \|_\infty \leq bM \sup_{t\in J} K(t)(bMM_1M_2+1)\| g \|_{L^1}.$$

From Lemma 2.4, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - T(t)y_0 + T(t)f(y_n) - \int_0^t T(t-s)(B\bar{u}_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.4 that

$$h_*(t) - T(t)y_0 + T(t)f(y_*) - \int_0^t T(t-s)(B\bar{u}_{y_*})(s)ds =$$

$$= \int_0^t T(t-s)\left[BW^{-1}\left(\int_0^b T(b-s)\int_0^s K(s,\tau)g_*(\tau)d\tau\, ds\right)(t)\right]ds +$$

$$+ \int_0^t T(t-s)\int_0^s K(s,\tau)g_*(\tau)\,d\tau\, ds$$

for some $g_* \in S_{F,y_*}$.

Consequently the operator N is completely continuous with bounded, closed convex values and it is upper semicontinuous. In order to apply Lemma 2.5 we need one more step.

Step 5: *The set*

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y),\ \textit{for some}\ \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)y_0 - \lambda^{-1}T(t)f(y) +$$

$$+ \lambda^{-1}\int_0^t T(t-s)BW^{-1}\Big[y_1 - f(y) - T(b)y_0 + T(b)f(y) -$$

$$- \int_0^b T(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\, ds\Big](\eta)d\eta +$$

$$+ \lambda^{-1}\int_0^t T(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds.$$

This implies by (H3) – (H7) that for each $t \in J$ we have

$$\|y(t)\| \le M\|y_0\| + ML +$$

$$+ bMM_1 M_2 \Big[\|y_1\| + L + M\|y_0\| + ML +$$

$$+ bM \sup_{t\in J} K(t) \int_0^b p(s)\psi(\|y(s)\|)ds \Big] +$$

$$+ M \left\| \int_0^t \int_0^s K(s,\tau)g(\tau)\,d\tau\,ds \right\| \le$$

$$\le M\|y_0\| + ML + bMM_1 M_2 \Big[\|y_1\| + L + M\|y_0\| + ML +$$

$$+ bM \sup_{t\in J} K(t) \int_0^b p(s)\,\psi(\|y(s)\|)ds \Big] +$$

$$+ bM \sup_{t\in J} K(t) \int_0^b p(s)\psi(\|y(s)\|)\,ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M\|y_0\| + ML + MM_0, \qquad \|y(t)\| \le v(t)$$

and

$$v'(t) = Mb \sup_{t\in J} K(t)p(t)\,\psi(\|y(t)\|), \qquad t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \le Mb \sup_{t\in J} K(t)p(t)\,\psi(v(t)), \qquad t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le Mb \sup_{t\in J} K(t) \int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant d such that $v(t) \le d$, $t \in J$, and hence $\|y\|_\infty \le d$ where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J,E)$. As a consequence of Lemma 2.5 we deduce that N has a fixed point and thus the system (1.1) – (1.2) is nonlocally controllable on J.

4. AN APPLICATION

As an application of the above results we study the nonlocal controllability of a semilinear delay integrodifferential inclusion of Sobolev type

$$(4.1)\qquad (Qy(t))' - Ay \in \int_0^t K(t,s)\, F(s, y(\sigma(s)))\, ds + (Bu)(t), \qquad t \in J := [0,b],$$

$$(4.2)\qquad y(0) + f(y) = y_0$$

where F, K, f, y_0 are as in problem (1.1) – (1.2), A and Q are linear operator and $\sigma : J \to J$ a continuous function satisfying $\sigma(t) \leq t$ for $t \in J$.

This type of equations arises in various applications such as in the flow of fluid through fissured rocks [4], thermodynamics [10] and shear in second order fluids [18].

In order to prove our controllability result we need here the following additional assumptions.

(H10) A and Q are closed linear operators with domains contained in Banach space E and ranges contained in Banach space Y, $D(Q) \subset D(A)$, B is bijective and $Q^{-1} : Y \to D(Q)$ is continuous.

REMARK 4.1. The above assumption and the closed graph theorem imply the boundendness of the linear operator $AQ^{-1} : Y \to Y$ and $-AQ^{-1}$ generates a uniform continuous semigroup $T(t)$, $t \geq 0$ of bounded linear operators from Y into Y, satisfying

$$\| T(t) \| \leq M_0 e^{\omega t}$$

for some $M_0 \geq 1$ and $\omega \in \Re$.

(H11) $F : J \times E \to BCC(E)$; $(t, y) \to F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \{ g \in L^1(J, E) : g(t) \in F(t, y(\sigma(t))) \quad \text{for a.e. } t \in J \}$$

is nonempty;

(H12) $\sigma : J \to J$ is a continuous function such that $\sigma(t) \leq t$ for $t \in J$;

(H13) the linear operator $W : L^2(J, U) \to E$, defined by

$$Wu = \int_0^b Q^{-1}T(b-s)\,B\,u(s)\,ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J,U)\setminus kerW$ and there exist positive constants M_1 and M_2 such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

(H14) $\| F(t,y)\| := \sup\{\| v\| : v \in F(t,y)\} \le p(t)\psi(\| y\|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \Re_+)$ and $\psi : \Re_+ \to (0,\infty)$ is continuous and increasing with

$$\| Q^{-1} \| Mb \sup_{t\in J} K(t) \int_0^b p(s)ds < \int_{\bar{c}}^{\infty} \frac{du}{\psi(u)},$$

where $\bar{c} = \| Q^{-1} \| M \| Q \| (\| y_0 \| + L) + M\overline{M}_0,\ M = \sup\{\|T(t,s)\|;(t,s) \in \gamma\}$ and

$$\overline{M}_0 = bM_1M_2 \| Q^{-1} \| \Bigg[\| y_1 \| + L + \| Q^{-1} \| M(\| y_0 \| + L) + $$

$$+ \| Q^{-1} \| Mb \sup_{t\in J} K(t) \int_0^b p(s)\psi(\| y(s) \|)\,ds \Bigg];$$

(H15) for each bounded set $D \subset C(J,E)$, and $t \in J$ the set

$$\left\{ Q^{-1}T(t)Qy_0 + \int_0^t Q^{-1}T(t-s) \int_0^s K(s,u)g(u)duds : g \in S_{F,y},\ y \in D \right\}$$

is relatively compact.

DEFINITION 4.2. *A continuous solution* $y(t)$ *of the integral inclusion*

$$y(t) \in Q^{-1}T(t)Q\,y_0 - Q^{-1}T(t)Qf(y) + \int_0^t Q^{-1}T(t-s)(Bu)(s)ds +$$

$$+ \int_0^t Q^{-1}T(t-s) \int_0^s K(s,u)F(u, y(\sigma(u)))\,du\,ds$$

is called a mild solution of (4.1) – (4.2) *on J.*

DEFINITION 4.3. *The system* (4.1) – (4.2) *is said to be nonlocally controllable on the interval J, if for every* $y_0, y_1 \in E$, *there exists a control*

$u \in L^2(J,U)$, such that the mild solution $y(t)$ of (4.1) – (4.2) satisfies $y(b) + f(y) = y_1$.

THEOREM 4.4. *Let $f : C(J,E) \to E$ be a continuous function. Assume that hypotheses (H3) – (H5), (H8) and (H10) – (H15) are satisfied. Then the problem (4.1) – (4.2) is nonlocally controllable on J.*

PROOF. Using hypothesis (H13) for an arbitrary function $y(\cdot)$ define the control

$$u_y(t) = W^{-1}\Bigg[y_1 - f(y) - Q^{-1}T(b)Qy_0 + Q^{-1}T(b)Qf(y) - \\ - \int_0^b Q^{-1}T(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau\,ds\Bigg](t),$$

where

$$S_{F,y} := \{g \in L^1 J,E) : g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J\}.$$

We shall now show that, when using this control, the operator $N : C(J,E) \to 2^{C(J,E)}$ defined for each $t \in J$ by

$$Ny := \Bigg\{h \in C(J,E) : h(t) = Q^{-1}T(t)Qy_0 - Q^{-1}T(t)Qf(y) + \\ + \int_0^t T(t-s)(Bu_y)(s)ds + \int_0^t Q^{-1}T(t-s)\int_0^s K(s,u)g(u)duds : g \in S_{F,y}\Bigg\}$$

has a fixed point. This fixed point is then a solution of the system (4.1) – (4.2).

Clearly $y_1 - f(y) \in (Ny)(b)$.

As in Theorem 3.1 we can show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. We repeat here the step 5, i.e. we show that the set

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}Q^{-1}T(t)Q(y_0 - f(y)) + \int_0^t Q^{-1}T(t-s)(Bu_y)(s)ds +$$

$$+\lambda^{-1}\int_0^t Q^{-1}T(t-s)\int_0^s K(s,u)g(u)duds, \qquad t\in J.$$

Then for each $t\in J$ we have

$$\|y(t)\| \le \|Q^{-1}\|M\|\|Q\|(\|y_0\|+L)+\left\|\int_0^t\int_0^s Q^{-1}T(t-s)(Bu_y)(s)ds\right\|+$$

$$+\|Q^{-1}\|M\left\|\int_0^t\int_0^s K(s,u)g(u)\,du\,ds\right\|\le$$

$$\le\|Q^{-1}\|M\|\|Q\|(\|y_0\|+L)+$$

$$+bM\|Q^{-1}\|M_1M_2\Big[\|y_1\|+L+\|Q^{-1}\|M(\|y_0\|+L)+$$

$$+\|Q^{-1}\|bM\sup_{t\in J}K(t)\int_0^b p(s)\psi(\|y(\sigma(s))\|)ds\Big]+$$

$$+\|Q^{-1}\|M\int_0^t\int_0^s |K(s,u)|p(u)\psi(\|y(\sigma(u))\|)\,du\,ds\le$$

$$\le\|Q^{-1}\|M\|\|Q\|(\|y_0\|+L)+$$

$$+bM\|Q^{-1}\|M_1M_2\Big[\|y_1\|+L+\|Q^{-1}\|M(\|y_0\|+L)+$$

$$+\|Q^{-1}\|bM\sup_{t\in J}K(t)\int_0^b p(s)\psi(\|y(\sigma(s))\|)ds\Big]+$$

$$+\|Q^{-1}\|Mb\sup_{t\in J}K(t)\int_0^t p(s)\psi(\|y(\sigma(s))\|)\,ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0)=\|Q^{-1}\|M\|Q\|(\|y_0\|+L)+M\overline{M}_0, \qquad \|y(t)\|\le v(t), \quad t\in J,$$

and

$$v'(t)=\|Q^{-1}\|Mb\sup_{t\in J}K(t)p(t)\,\psi(\|y(\sigma(t))\|)\le$$

$$\le\|Q^{-1}\|Mb\sup_{t\in J}K(t)p(t)\,\psi(v(\sigma(t))), \qquad t\in J.$$

Since ψ and σ are increasing and $\sigma(t) \le t$ for $t \in J$, the last inequality yields

$$v'(t) \le \|Q^{-1}\| Mb \sup_{t\in J} K(t) p(t)\, \psi(v(t)), \qquad t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \| Q^{-1} \| \, Mb \sup_{t\in J} K(t) \int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies taht there exists a constant d such that $v(t) \le d,\ t \in J$, and hence $\|y\|_\infty \le d$ where d depends only on b and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J,E)$. As a consequence of Lemma 2.5 we deduce that N has a fixed point and thus the system (4.1) – (4.2) is nonlocally controllable on J.

5. An Example

Consider the following partial integrodifferential equation of the from

(5.1) $$\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) = \int_0^t K(t,s) q(s, z(s,x))ds + Bu(t),$$

$$0 \le x \le \pi, \ t \in J$$

(5.2) $$z(t,0) = z(t,\pi), \qquad z(0,x) + z(1,x) = z_0(x)$$

where $q : J \times E \to E$, is continuous.

Take $X = Y = L^2[0,\pi]$ and define $A : E \to E$ by $Aw = w''$ with domain

$D(A) = \{w \in E,\ w,\ w'$areabsolutely continuous, $w'' \in E,\ w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \qquad w \in D(A)$$

where $w_n(s) = \sqrt{2/\pi}\, \sin ns,\ n = 1,2,\ldots$ is the orthogonal set of eigenvectors in A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t,s),\ t \ge 0$ in X and is given by

$$T(t,s)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E.$$

Since the analytic semigroup $T(t,s)$ is compact there exist a contant M such that

$$\| T(t) \| \le M.$$

Assume that the operator $B: U \to Y$, $U \subset J$, is a bounded linear operator and the operator

$$Wu = \int_0^b T(b-s)Bu(s)ds$$

has a bounded invertible operator W^{-1} which takes values in $L^2(J,U) \setminus kerW$. Examples with $W: L^2(J,U) \to E$ such that W^{-1} exists and is bounded are discussed in [9].

Assume that there exists an integrable function $p: J \to [0,\infty)$ such that

$$\| q(t, w(t)) \| \le p(t)\psi(\| w \|)$$

where $\psi: [0,\infty) \to (0,\infty)$ is continuous and nondecreasing with

$$Mb \sup_{t \in J} K(t) \int_0^b p(t)dt < \int_c^\infty \frac{ds}{\psi(s)}$$

where c is a constant.

Then the problem $(1.1)-(1.2)$ is an abstract formulation of the problem $(5.1)-(5.2)$. Since all the conditions Theorem 3.1 are satisfied, the problem $(5.1)-(5.2)$ is nonlocally controllable on J.

ACKNOWLEDGEMENT. *The authors thanks the referee for his/her comments and suggestions. Also the authors are thankful to Proffessor J. Werbowski for his kind help.*

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Received on 2000.01.28 and, in revised form, on 2000.04.27.