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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF THE THIRD ORDER QUASILINEAR NEUTRAL DIFFERENTIAL EQUATIONS

ABSTRACT: In the paper there are given sufficient and necessary conditions for the existence of nonoscillatory solutions of the equation

$$(r_2(t)(r_1(t)\varphi(L_0'x(t)))')' + f(t, x(g(t))) = 0$$

with specified asymptotic behaviour as $t \to \infty$.

KEY WORDS: quasilinear neutral differential equations, nonoscillatory solutions, Schauder – Tychonoff fixed point theorem

1. INTRODUCTION

We consider quasilinear neutral differential equations of the third order in the form

(E) $$(r_2(t)(r_1(t)\varphi(L_0'x(t)))')' + f(t, x(g(t))) = 0, \quad t \geq a \geq 0,$$

where

(1.1) $$L_0x(t) = x(t) - p(t)x(h(t)).$$

With regard to (E) the following conditions are always assumed to hold:

(C) (a) $r_i : [a, \infty) \to (0, \infty)$, $i = 1,2$ are continuous;

(b) $p : [a, \infty) \to [0, \lambda]$ is continuous, $0 < \lambda < 1$;

(c) $h, g : [a, \infty) \to R$ are continuous, h is strictly increasing, g is nondecreasing and $h(t) < t$, $g(t) \leq t$ for $t \geq a$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$;

(d) $\varphi : R \to R$ is continuous, strictly increasing and such that $u\varphi(u) > 0$ for $u \neq 0$, $\varphi(R) = R$;

(e) $f : [a, \infty) \times R \to R$ is continuous and $f(t, x)$ is nondecreasing in x and such that $uf(t, u) > 0$ for $u \neq 0$ and all $t \geq a$.

Let $t_1 \geq a$ be such that

$$t_0 = \min\{t_1, h(t_1), \inf_{t \geq t_1} g(t)\} \geq a.$$

Denote

$$D_1^{\varphi}x(t) = r_1(t)\varphi(L_0'x(t)),$$
(1.2)
$$D_2^{\varphi}x(t) = r_2(t)\varphi(D_1^{\varphi}x(t))', \quad t \ge t_1.$$

By a proper solution of (E) we mean a continuous function $x:[t_0,\infty)\to R$ such that $L_0x(t)$, $D_1^{\varphi}x(t)$, $D_2^{\varphi}x(t)$, are continuously differentiable on $[t_1,\infty)$ and $x(t)$ satisfies the equation (E) on $[t_1,\infty)$. The solutions which vanish for all large t will be excluded from our consideration. A proper solution $x(t)$ of (E) is nonoscillatory if there exists a $T_1 \ge t_0$ such that $x(t)\neq 0$ for all $t \ge T_1$.

The objective of this paper is to give conditions for the existence of several types of nonoscillatory proper solutions of (E) with specified asymptotic behaviour as $t\to\infty$.

The equation (E) reduces to

$$(r_1(t))(r_2(t)\varphi(x'(t)))')' + f(t,x(t)) = 0, \tag{E_1}$$

in the case when $p(t)\equiv 0$, $g(t)\equiv t$ and to

$$(r_1(t))(r_2(t)(x(t)-p(t)x(h(t)))')')' + f(t,x(g(t))) = 0, \tag{E_2}$$

in the case $\varphi(t)\equiv t$ and $p(t)\equiv 0$.

The existence of nonoscillatory solutions of the equation (E_1) has been studied in the paper [2]. A systematic study of nonoscillatory properties of quasilinear neutral differential equations of second order have been done for example in the paper [3, 4, 7, 8].

This paper is designed to extend some results obtained in the paper [2] to the equation (E).

In the next we will assume that

$$\int_a^{\infty} \frac{1}{r_2(t)}\,dt = \infty, \tag{1.3}$$

$$\int_a^{\infty} \left|\varphi^{-1}\left(\frac{k}{r_2(t)}\right)\right| dt = \infty \tag{1.4}$$

for every $k\neq 0$, where φ^{-1} is the inverse function to φ.

We denote

$$\phi_{k,T}(r_1,r_2:t) = \int_T^t \varphi^{-1}\left(\frac{1}{r_1(s)}\int_T^s \frac{k}{r_2(\tau)}\,d\tau\right)ds, \tag{1.5}$$

$$\phi_k(r_1,r_2:t) = \phi_{k,a}(r_1,r_2:t) \quad t\ge T\ge a, \quad k\neq 0.$$

Let $x(t)$ be a nonoscillatory solutions of (E) defined on $[t_0,\infty)$. From the equation (E) and the assumptions (C) it follows that the function $L_0x(t)$ has to be eventually of constant sign, so that

$$x(t)L_0x(t) > 0$$

or

$$x(t)L_0x(t) < 0$$

for all sufficientl large t. We denote by N the set of all nonoscillatory solutions of (E) and define

$$N^+ = \{x \in N : x(t)L_0x(t) > 0 \text{ for all large } t\},$$
$$N^- = \{x \in N : x(t)L_0x(t) < 0 \text{ for all large } t\}. \tag{1.6}$$

We introduce the notation:

$$\gamma(t) = \sup(s \geq a : g(s) < t),$$
$$\gamma_h(t) = \sup(s \geq a : h(s) < t).$$

$$h^{[0]}(t) \equiv t, \quad h^{[k]}(t) = h^{k-1}(h(t)), \quad k = 1,2,\ldots$$
$$P_0(t) \equiv 1, \quad P_k(t) = \prod_{i=0}^{k-1}(h^{[i]}(t)), \quad k = 1,2,\ldots \tag{1.9}$$

Let $x(t) \in N^+$ for $t \geq t_1$. Then from (1.1) with regard to the last relations we obtain

$$x(t) = \sum_{k=0}^{n(t)-1} P_k(t)L_0x(h^{[k]}(t)) + P_{n(t)}(t)x(h^{[h(t)]}(t)), \quad t \geq t_{n(t)} = \gamma_h(t_{n(t)-1}), \tag{1.10}$$

where $n(t)$ denotes the last positive integer euch that $T_0 \leq h^{[n(t)]}(t) \leq T$.

2. Existence of nonoscillatory solutions

In this section we give criteria for the existence of nonoscillatory solutions of (E) mentioned in the following.

Lemma 1. *Any nonoscillatory solutions $x(t) \in N^+$ of the equation* (E) *has one of the following three types.*

(I) $\lim_{t\to\infty} |L_0x(t)| = \infty, \quad \lim_{t\to\infty} |D_1^\varphi x(t)| = \infty, \quad \lim_{t\to\infty} |D_2^\varphi x(t)| = c_1 > 0,$

(II) $\lim_{t\to\infty} |L_0x(t)| = \infty, \quad \lim_{t\to\infty} |D_1^\varphi x(t)| = c_2 > 0, \quad \lim_{t\to\infty} D_2^\varphi x(t) = 0,$

(III) $\lim_{t\to\infty} |L_0x(t)| = c_3 \geq 0, \quad \lim_{t\to\infty} D_1^\varphi x(t) = 0, \quad \lim_{t\to\infty} D_2^\varphi x(t) = 0.$

PROOF. Let $x(t) \in N^+$. Without loss of generality we suppose that $x(g(t)) > 0$ on $[t_0, \infty)$, $t_0 \geq a$. In view of the assumptions (c) – (e) from the equation (E) we obtain that $D_2^\varphi x(t)$ is decreasing on $[t_0, \infty)$. With regard to (c) – (e), (1.3), (1.4) we obtain that $D_2^\varphi x(t) > 0$ for all $t \geq t_0$ and the limit $\lim_{t\to\infty} D_2^\varphi x(t)$ is either positive or zero.

I. Let $\lim_{t\to\infty} |D_2^\varphi x(t)| = c_1 > 0$. Then in view of (1.2)

$$D_1^\varphi x(t) \geq D_1^\varphi x(t_0) + c_1 \int_{t_0}^{t} \frac{1}{r_2(s)} ds, \qquad t \geq t_0.$$

Then with regard to (1.3) $\lim_{t\to\infty} |D_1^\varphi x(t)| = \infty$. Now for any $k > 0$ there exists $t_1 \geq t_0$ such that $\varphi(L_0' x(t)) \geq k/r_1(t)$, $t \geq t_1$. In view of (c) – (d), (1.4) the last inequality implies that $\lim_{t\to\infty} L_0 x(t) = \infty$, i.e. the solutions $x(t)$ is of the type (I).

II. Let $\lim_{t\to\infty} D_2^\varphi x(t) = 0$. With regard to the monotonicity of $D_2^\varphi x(t)$ we have that $D_2^\varphi x(t) > 0$ and $D_1^\varphi x(t)$ is increasing on $[t_0, \infty)$. Then the limit $\lim_{t\to\infty} D_1^\varphi x(t)$ is either positive or zero.

i) Let $\lim_{t\to\infty} D_1^\varphi x(t) = c_2 > 0$, then there exists $t_2 \geq t_1 \geq t_0$ such that $D_1^\varphi x(t) \geq c_2/2$ for $t \geq t_2$. In view of (c) – (d), (1.4) the last relation gives that $\lim_{t\to\infty} L_0 x(t) = \infty$, i.e. the solution $x(t)$ is of the type (II).

ii) Let $\lim_{t\to\infty} D_1^\varphi x(t) = 0$. Then with regard to the monotonicity of $D_1^\varphi x(t)$ we get that $D_1^\varphi x(t) < 0$ and $\varphi(L_0' x(t)) < 0$ for all $t \geq t_2$. In view of (c) – (d) the function $L_0 x(t)$ is decreasing, i.e. the solution $x(t)$ is of the type (III).

REMARK 1. Let $x \in N$.

i) Then in view of (c) – (d) and (1.1)

(2.1) $$|x(t)| \geq |L_0 x(t)| \quad for \quad t \geq \gamma_h(a).$$

ii) Let $x(t) \in N^+$ is of the type (I) on (II). Then from (1.10) we get that

(2.2) $$|x(t)| \leq \frac{|L_0 x(t)|}{1-\lambda} + k_x,$$

for some positive constant k_x and all large t.

Then with regard to (2.1) for the solution $x(t)$ of the type (I) and (II) it holds

(2.3) $$\lim_{t\to\infty} |x(t)| = \infty.$$

iii) Let $x(t) \in N^+$ is of the type (III) with $\lim_{t\to\infty} |L_0 x(t)| > 0$. Then in view of (2.1) and (1.1) we have

(2.4) $$0 < \liminf_{t\to\infty} |x(t)|, \qquad \limsup_{t\to\infty} |x(t)| < \infty.$$

THEOREM 1. *Let the equation* (E) *has a nonoscillatory solution of the* (I). *Then*

(2.5) $$\int_{\gamma(a)}^{\infty} | f(t, c\phi_k(r_1, r_2 : g(t))) | dt < \infty$$

for some constants $k \neq 0$ *and* $c \neq 0$.

PROOF. Let $x(t)$ be a nonoscillatory solution of the type (I). Without loss of generality we suppose that $x(t)$ is positive on $[t_0, \infty)$, $t_0 \geq \gamma(a)$. Then there exist a constant $k > 0$ and $t_1 \geq t_0$ such that $D_2^{\varphi} x(t) \geq k$ for $t \geq t_1$.

Hence in view of the assuption (c) – (d) and (1.2) from the last inequality we get

(2.6) $$L_0' x(t) \geq \varphi^{-1}\left(\frac{1}{r_2(t)} \int_{t_1}^{\infty} \frac{k}{r_2(s)} ds \right), \qquad t \geq t_1.$$

Integarting (2.6) from $[t_1, t]$ and then using (1.8) we have

$$x(t) \geq L_0 x(t) \geq \phi_{k,t_1}(r_1, r_2; t), \quad t \geq t_1.$$

Then there exist a positive constant c such that $x(g(t)) \geq c\phi_k(r_1, r_2; g(t))$, $t_1 \geq \gamma(x)$. An integration of (E) on $[t_1, \infty)$ and then combined with the above inequality leads to

$$\int_{t_1}^{\infty} f(t, c\phi_k(r_1, r_2; g(t))) dt < \infty.$$

THEOREM 2. *Suppose that for each fixed* $k \neq 0$ *and* $T \geq a$

(2.7) $$\lim_{l\to 0, k>0} \frac{\phi_{l,T}(r_1, r_2; t)}{\phi_{k,T}(r_1, r_2; t)} = 0$$

uniformly on any subinterval $[T_1,\infty)\subset[T,\infty)$. *Then the equation* (E) *has a nonoscillatory solution of the* (I) *if* (2.5) *holds for some* $c\neq 0$ *and* $k\neq 0$.

PROOF. Suppose that (2.5), (2.7) hold for some $k>0$ and $c>0$. Then in view of (2.7) there exists $l_1>0,\ l_1<k$ such that

$$\phi_{l_1}(r_1,r_2:t)\le c\phi_k(r_1,r_2:t).$$

Let $T>a,\ l>0$ such that $2l>l_1$,

$$T_0=\min\{T,h(T),\inf_{t\ge T} g(t)\}\ge a \tag{2.8}$$

and

$$\int_T^\infty f(t_1,c\phi_{2l}(r_1,r_2:g(t))/(1-\lambda))dt<l. \tag{2.9}$$

Let $C[T_0,\infty)$ be the locally convex space of all continuous functions defined on $[T_0,\infty)$ with the topology of uniform convergence on any compact subinterval of $[T_0,\infty)$. Define a closed connex subset Y of $C[T_0,\infty)$ by

$$Y=\{y\in C[T_0,\infty):\phi_{l,T}(r_1,r_2:t)\le\phi_{2l,T}(r_1,r_2;t)\quad \text{for } t\ge T;\ y(t)=0 \text{ for } t\in[T_0,T]\}. \tag{2.10}$$

With each $L_0x(t)=y(t)$ we associate the function $\tilde{x}(t)$ defined by

$$\tilde{x}(t)=\begin{cases}\sum_{k=0}^{n(t)-1}P_k(t)y(h^{[k]}(t)), & t\ge T,\\ 0, & T_0\le t\le T,\end{cases} \tag{2.11}$$

when $n(t)$ is defined as in (1.10).

Using (2.11) we can easily show that the function $\tilde{x}(t)$ satisfies for $t\ge T$ the following relation

$$\tilde{x}(t)-p(t)\tilde{x}(h(t))=y(t) \tag{2.12}$$

$$\phi_{l,T}(r_1,r_2;t)\le y(t)\le\tilde{x}(t)\le\phi_{2l,T}(r_1,r_2:t)/(1-\lambda). \tag{2.13}$$

We now define the operation $F:Y\to C[T_0,\infty)$ by

$$(Fy)(t)=\begin{cases}\int_T^t\varphi^{-1}\left(\frac{1}{r_1(t)}\int_T^\tau\frac{1}{r_2(s)}\left(l+\int_s^\infty f(r,\tilde{x}(g(r)))dr\right)ds\right)d\tau, & t\ge T,\\ 0, & T_0\le t\le T.\end{cases} \tag{2.14}$$

i) F maps Y into it self. Let $y \in Y$, then in view of (2.12), (c) – (e) for every $s \geq T$

$$\int_s^{\infty} f(r, \widetilde{x}(g(r)))dr \leq \int_T^{\infty} f(r, \phi_{2l,T}(r_1, r_2 : g(r)))/(1-\lambda)dr .$$

Using above inequality (1.5) and (2.9), from (2.14) we obtain

$$\phi_{l,T}(r_1, r_2; t) \leq (Fy)(t) \leq \phi_{2l,T}(r_1, r_2; t),$$

i.e. $(Fy)(t) \in Y$.

ii) F is continuous. Let $y_n\}$, $a = 1,2,\ldots$ be a sequence of elements of Y converging to $y \in Y$ as $n \to \infty$ in the topology of $C[T_0, \infty)$. Using the Lebesque dominated convergence theorem we can show that $(Fy_n)(t) \to (Fy)(t)$ uniformly on any compact subintervals of $[T, \infty)$, which implies convergence in $[T, \infty)$.

iii) $F(Y)$ is a relatively compact. This follows from the relation

$$0 \leq (Fy)'(t) = \varphi^{-1}\left(\frac{1}{r_1(t)} \int_T^t \frac{1}{r_2(s)}\left(l + \int_s^{\infty} f(r, \widetilde{x}(g(r)))\, dr\right) ds\right) \leq$$

$$\leq \varphi^{-1}\left(\frac{1}{r_1(t)} \int_T^t \frac{1}{r_2(s)}\left(l + \int_s^{\infty} f(r, \phi_{2l,T}(r_1, r_2; g(r)))\, dr\right) ds\right) \leq$$

$$\leq \varphi^{-1}\left(\frac{1}{r_1(t)} \int_T^t \frac{2l}{r_2(s)} ds\right), \qquad t \geq T$$

holds for all $y \in Y$.

Then by the Schauser – Tychonov fixed point theorem there exists a fixed element y_0 of F such that $Fy_0 = y_0$, which satisfies

$$\text{(2.15)} \qquad y_0(t) = \int_T^t \varphi^{-1}\left(\frac{1}{r_1(\tau)} \int_T^{\tau} \frac{1}{r_2(s)}\left(l + \int_s^{\infty} f(r, \widetilde{x}_0(g(r)))dr\right) ds\right) d\tau, \qquad t \geq T$$

$$y_0(t) = 0, \quad T_0 \leq t \leq T,$$

where $y_0(t) = \widetilde{x}_0(t) - \rho(t)\widetilde{x}_0(h(t))$, $t \geq T$.

Differentiating (2.15) we can see that $\widetilde{x}_0(t)$ is a positive solution of (E) of the type (I).

Combining Theorem 1 am Theorem 2 we have

THEOREM 3. *The equation* (E) *has a nonoscillatory solution of the type* (I) *if and only if* (2.5) *and* (2.7) *hold.*

THEOREM 4. *The equation* (E) *has a nonoscillatory solution of the type* (III) *with a nonzero limit for $t \to \infty$ if and only if*

$$\int_a^\infty \varphi^{-1}\left(\frac{1}{r_1(t)}\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty f(r,c)drds\right)dt < \infty \tag{2.16}$$

for some constant $c \neq 0$.

PROOF. i) (The "only if" part). Let $x(t)$ be a nonoscillatory solution of (E) of the type (III) with the limit $\lim_{t\to\infty} L_0x(t) \neq 0$. Without loss of generality we suppose that $x(g(t)) > 0$ for $t \geq t_0$. Let $\lim_{t\to\infty} L_0x(t) = d > 0$. Then by (2.4) there exists $c > 0$, and $t_1 \geq t_0$.

$$c \leq x(g(t)), \quad t \geq t_1. \tag{2.17}$$

Integrating the equation (E) from t ($\geq t_1$) to ∞ we get

$$\int_t^\infty f(r, x(g(r)))dr = D_2^\varphi x(t), \quad t \geq t_1.$$

Hence

$$\frac{1}{r_1(t)}\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty f(r, x(g(r)))drds \leq -\varphi(L_0'x(t)), \quad t \geq t_1.$$

In view of (2.17) the last inequality proves the truth of (2.15).

ii) "The if part". Suppose that (2.15) holds for some constant $c > 0$. The case $c < 0$ can be treated similarly. Let $d = c(1-x)/2$ and take $T \geq a$ so large that (2.8) holds and

$$\int_T^\infty \varphi^{-1}\left(\frac{1}{r_1(t)}\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty f(r,c)drds\right)dt < \frac{d}{2}. \tag{2.8}$$

Let $C[T_0,\infty)$ be the locally convex space of all continuous functions defined on $[T_0,\infty)$ with the topology of unifrom convergence on any compact subintervals of $[T_0,\infty)$. Define a closed convex subset Y of $C[T_0,\infty)$ by

$$Y = \{y \in C[T_0,\infty) : d \leq y(t) \leq 2d \text{ on } [T,\infty),\ y(t) = y(T) \text{ on } [T_0,T]\}. \tag{2.19}$$

With each $y \in Y$ we associate the function $\widetilde{x}:[T_0,\infty) \to R$ defined by

$$(2.20) \qquad \widetilde{x}(t) = \begin{cases} \sum_{k=0}^{n(t)-1} P_k(t) y(h^{[k]}(t)) + P_n(t) \dfrac{y(T)}{1-p(T)}, & t \geq T, \\ \dfrac{y(T)}{1-p(T)}, & T_0 \leq t \leq T, \end{cases}$$

where $n(t)$ is defined as in (1.10).

It is easy to show that $\widetilde{x}(t) \in C[T_0,\infty)$ and satisfies (2.12). In view of (2.2) and (2.12) we obtain that

$$d \leq \widetilde{x}(t) \leq \frac{2d}{1-\lambda} = c, \qquad t \geq T.$$

We now define an operator $F: Y \to C[T_0,\infty)$ by

$$(Fy)(t) = \begin{cases} \dfrac{3d}{2} - \displaystyle\int_t^\infty \varphi^{-1}\left(\frac{1}{r_1(t)} \int_\tau^\infty \frac{1}{r_2(s)} l_t \int_s^\infty f(r, \widetilde{x}(g(r))) dr ds \right) d\tau, & t \geq T, \\ (Fy)(T), \qquad T_0 \leq t \leq T. \end{cases}$$

Proceeding similar as in the proof of Theorem 2 we obtain that the operator F is continuous mapping the convex subset Y into a relatively compact subset Y. Therefore the Schauder – Tychonoff fixed point theorem ensures the existence of an element $y_0(t) \in Y$ such that

$$(2.21) \qquad y_0(t) = \frac{3d}{2} - \int_t^\infty \varphi^{-1}\left(\frac{1}{r_1(\tau)} \int_\tau^\infty \frac{1}{r_2(s)} \int_s^\infty f(r, \widetilde{x}_0(g(r))) dr ds \right) d\tau$$

where $y_0(t) = \widetilde{x}_0(t) - p(t)\widetilde{x}_0(h(t))$.

Differentiating (2.21) and using the last relation we get that $y_0(t)$ is a nonoscillatory solution of (E) of the type (III) with $\lim_{t\to\infty} y_0(t) = 3d/2$.

THEOREM 5. *The equation* (E) *has a nonoscillatory solution of the type* (II) *if* (2.5), (2.7) *hold for some constant* $k \neq 0$ *and*

$$(2.22) \qquad \int_a^\infty \left| \varphi^{-1}\left(\frac{1}{r_1(t)} \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty f(s,d) dr ds \right) \right| dt = \infty$$

for any constant $d \neq 0$.

PROOF. It suffices to consider the case $k>0$, $d>0$ and $c>0$. Let c_0 be such that $0<c_0<c$. According to (2.7) there exists $l>0$ and $T\geq a$ such that (2.8),

$$c_0+\phi_l(r_1,r_2;t)\leq c\phi_k(r_1,r_2;t), \qquad t\geq T \tag{2.23}$$

and

$$\int_t^\infty f(s,(c_0+\phi_l(r_1,r_2:g(s)))/(1-\lambda))ds\leq l. \tag{2.24}$$

Define the set $Y\subset[T_0,\infty)$ and the mapping $F:Y\to C[T_0,\infty)$ as follows:

$$Y=\{y\in C[T_0,\infty):c_0\leq y(t)\leq c_{0t}\phi_l(r_1,r_2;t) \text{ on } [T,\infty) \text{ and } y(t)=y(T) \text{ on } [T_0,T]\},$$

$$(Fy)(t)=\begin{cases} c_0+\int_T^t\varphi^{-1}\left(\frac{1}{r_1(t)}\int_\tau^\infty\frac{1}{r_2(s)}l_t\int_s^\infty f(r,\tilde{x}(g(r)))drds\right)d\tau, & t\geq T,\\ (Fy)(T) \quad \text{on } [T_0,\infty), \end{cases}$$

where $\tilde{x}(t)$ denotes a function associated with $y(t)$ via (2.20). In view of (2.20) and (2.23) we have that

$$c_0\leq\tilde{x}(t)\leq(c_0+\phi_l(r_1,r_2:t))/(1-\lambda),\quad t\geq T. \tag{2.25}$$

We can verify that all assumptions of the Schauder – Tychonoff fixed point theorem are fulfilled. Therefore there exists a function $y_0\in Y$ such that

$$y_0(t)=c_0+\int_T^t\varphi^{-1}\left(\frac{1}{r_1(\tau)}\int_\tau^\infty\frac{1}{r_2(s)}\int_s^\infty f(r,\tilde{x}_0(g(r)))drds\right)d\tau \tag{2.26}$$

where $y_0(t)=\tilde{x}_0(t)-p(t)\tilde{x}_0(g(t))$, $t\geq T$ and $x_0(t)$ is a solution of the equation (E). Because $\tilde{x}_0(t)\geq c_0>0$, $t\geq T$. Then in view of the monotonicity of f, (2.22) we obtain that $\lim_{t\to\infty} y_0(t)=\infty$.

After differentiating (2.26) and certain modification we get

$$D_2^\varphi y_0(t)=r_2(t)(r_1(t)\varphi(y_0'(t)))'=-\int_t^\infty f(r,\tilde{x}_0(g(r)))dr.$$

Then with respect to (2.25), (2.23) and (2.5) we have that

$$\lim_{t\to\infty} D_2^\varphi y_0(t)=0,$$

which proves that $\tilde{x}_0(t)$ is a nonoscillatory solution of (E) of the type (II).

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